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On Inclusion Relations Between Generalized Wiener Classes

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Abstract

In this paper we aim to study inclusion relations between the generalized Wiener classes $\Lambda\text{BV}^{(p_n\uparrow p)}$. In particular, we give a sufficient condition for the inclusion $\Lambda\text{BV}^{(p_n\uparrow p)} \subseteq \Gamma\text{BV}^{(q_n\uparrow q)}$ which leads us to new results for other classes of functions previously considered. We also obtain a necessary and sufficient condition for equality of two distinct classes of this type. Furthermore, we extend and unify a number of results in the literature including an important theorem of Avdispahić about Waterman spaces.

Keywords : Generalized bounded variation; Modulus of variation; Generalized Wiener class; Waterman class.

1 Introduction

We commence this paper by recalling a generalization of the classical concept of bounded variation which is central to our work here. A nondecreasing sequence $\Lambda = \{\lambda_j\}$ of positive reals is said to be a Λ-sequence if $\sum_{j=1}^{\infty} \frac{1}{\lambda_j} = \infty$.

Definition 1.1 *Let Λ be a Λ-sequence and $\{p_n\}$ be a sequence of positive reals such that $1 \leq p_n \uparrow p \leq \infty$. A real function f on an interval $[a,b] \subseteq \mathbb{R}$ is said to be of p_n-Λ-bounded variation if :*

$$V_\Lambda(f; p_n \uparrow p) := \sup_{n\geq 1} \sup_{\{I_j\}} \left(\sum_{j=1}^{s} \frac{|f(I_j)|^{p_n}}{\lambda_j} \right)^{\frac{1}{p_n}} < \infty$$

where the $\{I_j\}_{j=1}^{s}$ are collections of nonoverlapping subintervals of $[a,b]$ such that $\inf_j |I_j| \geq \frac{b-a}{2^n}$.

The symbol $\Lambda\text{BV}^{(p_n\uparrow p)}$ stands for the linear space of functions of p_n-Λ-bounded variation. This class was introduced by Vyas in [11] where, among other things, it is shown that $\Lambda\text{BV}^{(p_n\uparrow p)}$ with pointwise operations and a suitable norm turns into a Banach algebra.

When $\lambda_j = 1$ for all j, we obtain the class $\text{BV}^{(p_n\uparrow p)}$—introduced by Kita and Yoneda [6]—which is a generalization of the well-known Wiener class BV_p. On the other hand, taking $p_n = p$ for all n, we obtain the class $\Lambda\text{BV}^{(p)}$ [10]. If further we take $p = 1$, the Waterman class ΛBV is obtained. In the sequel, we suppose that $[a,b] = [0,1]$.

The main purpose of this paper is extending and unifying a number of inclusion theorems in the literature. More specifically, in Section 3 we give a necessary and sufficient condition for

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equality of two distinct $\Lambda\mathrm{BV}^{(p_n\uparrow p)}$ classes, which extends the main result of [6]. We shall also present a sufficient condition for the inclusion $\Lambda\mathrm{BV}^{(p_n\uparrow p)}\subseteq\Gamma\mathrm{BV}^{(q_n\uparrow q)}$ for arbitrary Λ, Γ, p_n and q_n; this condition is new even for the special case of $p_n=q_n=1$. In Section 4, a unifying condition is introduced that yields three inclusion results already known. We note that a main ingredient in our proofs is an inequality which is discussed in the next section.

2 Preliminaries

In [4], the authors established an inequality for positive monotonic sequences which can be used to relate norms of certain function spaces:

$$\begin{aligned}&\Big(\sum_{j=1}^{n}x_j^q z_j\Big)^{\frac{1}{q}}\\ &\le\sum_{j=1}^{n}x_jy_j\max_{1\le k\le n}\Big(\sum_{j=1}^{k}z_j\Big)^{\frac{1}{q}}\Big(\sum_{j=1}^{k}y_j\Big)^{-1}\end{aligned}\tag{2.1}$$

where $1\le q<\infty$, and $\{x_j\}$, $\{y_j\}$ and $\{z_j\}$ are positive nonincreasing sequences.

The following lemma supplements this inequality and will be used in our proofs.

Lemma 2.1 *If $0<q<1$, then (2.1) holds whenever the sequence*

$$\Big\{\sum_{i=1}^{k}z_i\Big/\sum_{i=1}^{k}y_i\Big\}_k$$

is nondecreasing.

Proof. First, we apply (2.1) with $q=1$ to obtain

$$\sum_{j=1}^{n}x_jz_j\le\sum_{j=1}^{n}x_jy_j\max_{1\le k\le n}\Big(\sum_{i=1}^{k}z_i\Big)\Big(\sum_{i=1}^{k}y_i\Big)^{-1}\tag{2.2}$$

Then an application of the Hölder inequality yields

$$\begin{aligned}\sum_{j=1}^{n}x_j^qz_j&=\sum_{j=1}^{n}(x_jz_j)^qz_j^{1-q}\\ &\le\Big(\sum_{j=1}^{n}x_jz_j\Big)^q\Big(\sum_{j=1}^{n}z_j\Big)^{1-q}\\ &\le\Big(\sum_{j=1}^{n}x_jy_j\Big)^q\Big(\sum_{j=1}^{n}z_j\Big)^{1-q}\max_{1\le k\le n}\Big(\sum_{i=1}^{k}z_i\Big)^q\Big(\sum_{i=1}^{k}y_i\Big)^{-q}\\ &\le\Big(\sum_{j=1}^{n}x_jy_j\Big)^q\max_{1\le k\le n}\Big(\sum_{i=1}^{k}z_i\Big)\Big(\sum_{i=1}^{k}y_i\Big)^{-q}\end{aligned}$$

where the last two inequalities are due, respectively, to (2.2) and the fact that

$$\Big\{\sum_{i=1}^{k}z_i\Big/\sum_{i=1}^{k}y_i\Big\}_k$$

is nondecreasing.

We will also need the following inequality which is sometimes called the equimonotonic sequences inequality [5, Theorem 368]:

$$\sum_{j=1}^{n}\bar{x}_j\bar{y}_{n+1-j}\le\sum_{j=1}^{n}x_jy_j\le\sum_{j=1}^{n}\bar{x}_j\bar{y}_j\tag{2.3}$$

where $\{x_j\}$, $\{y_j\}$ are sequences of real numbers, and $\{\bar{x}_j\}$, $\{\bar{y}_j\}$ are their descending rearrangements, respectively.

3 Relationships between the generalized Wiener classes $\Lambda\mathrm{BV}^{(p_n\uparrow p)}$

Now we turn our attention to the mutual relations between generalized Wiener classes $\Lambda\mathrm{BV}^{(p_n\uparrow p)}$. The following result is a nontrivial extension of [6, Theorem 3.1]. For the necessity part of the proof we use a refinement of the method in [6].

Let $1\le p\le q\le\infty$, $1\le p_n\uparrow p$ and $1\le q_n\uparrow q$. Then $\Lambda BV^{(p_n\uparrow p)}=\Lambda BV^{(q_n\uparrow q)}$ if and only if

$$\limsup_{n\to\infty}\Big(\sum_{j=1}^{2^n}\frac{1}{\lambda_j}\Big)^{\left|\frac{1}{p_n}-\frac{1}{q_n}\right|}<\infty\tag{3.4}$$

Proof. *Sufficiency.* Assume that $f \in \Lambda BV^{(p_n \uparrow p)}$. For an arbitrary but fixed n, let $\{I_j\}_{j=1}^s$ be a nonoverlapping collection of subintervals of $[0,1]$ with $\inf|I_j| \geq \frac{1}{2^n}$, and put $q = q_n/p_n$, $x_j = |f(I_j)|^{p_n}$, $y_j = z_j = 1/\lambda_j$. Note that with inequality (2.3) in mind, we may assume that the x_j are in descending order.

If $p_n \leq q_n$, applying inequality (2.1) we obtain:

$$\sum_{j=1}^{s} \frac{(|f(I_j)|^{p_n})^{\frac{q_n}{p_n}}}{\lambda_j} \leq \Big(\sum_{j=1}^{s} \frac{|f(I_j)|^{p_n}}{\lambda_j}\Big)^{\frac{q_n}{p_n}} \max_{1\leq k\leq s} \Big(\sum_{j=1}^{k} \frac{1}{\lambda_j}\Big)^{1-\frac{q_n}{p_n}}$$

hence:

$$\begin{aligned} &\Big(\sum_{j=1}^{s} \frac{|f(I_j)|^{q_n}}{\lambda_j}\Big)^{\frac{1}{q_n}} \\ &\leq \Big(\sum_{j=1}^{s} \frac{|f(I_j)|^{p_n}}{\lambda_j}\Big)^{\frac{1}{p_n}} \max_{1\leq k\leq s} \Big(\sum_{j=1}^{k} \frac{1}{\lambda_j}\Big)^{\frac{1}{q_n}-\frac{1}{p_n}} \\ &\leq \Big(\sum_{j=1}^{s} \frac{|f(I_j)|^{p_n}}{\lambda_j}\Big)^{\frac{1}{p_n}} \max_{1\leq k\leq 2^n} \Big(\sum_{j=1}^{k} \frac{1}{\lambda_j}\Big)^{|\frac{1}{q_n}-\frac{1}{p_n}|} \\ &\leq \Big(\sum_{j=1}^{s} \frac{|f(I_j)|^{p_n}}{\lambda_j}\Big)^{\frac{1}{p_n}} \Big(\sum_{j=1}^{2^n} \frac{1}{\lambda_j}\Big)^{|\frac{1}{q_n}-\frac{1}{p_n}|} \end{aligned}$$

If $q_n < p_n$, using Lemma 2.1 we obtain:

$$\sum_{j=1}^{s} \frac{|f(I_j)|^{q_n}}{\lambda_j} \leq \Big(\sum_{j=1}^{s} \frac{|f(I_j)|^{p_n}}{\lambda_j}\Big)^{\frac{q_n}{p_n}} \Big(\sum_{j=1}^{2^n} \frac{1}{\lambda_j}\Big)^{1-\frac{q_n}{p_n}}$$

Thus in any event, we have shown that:

$$\Big(\sum_{j=1}^{s} \frac{|f(I_j)|^{q_n}}{\lambda_j}\Big)^{\frac{1}{q_n}} \leq \Big(\sum_{j=1}^{s} \frac{|f(I_j)|^{p_n}}{\lambda_j}\Big)^{\frac{1}{p_n}} \Big(\sum_{j=1}^{2^n} \frac{1}{\lambda_j}\Big)^{|\frac{1}{q_n}-\frac{1}{p_n}|}$$

Taking suprema over all collections $\{I_j\}_{j=1}^s$ as above, and over all n yields:

$$\begin{aligned} &V_\Lambda(f; q_n \uparrow q) \\ &\leq V_\Lambda(f; p_n \uparrow p). \sup_n \Big(\sum_{j=1}^{2^n} \frac{1}{\lambda_j}\Big)^{|\frac{1}{q_n}-\frac{1}{p_n}|} < \infty \end{aligned}$$

which means that $f \in \Lambda BV^{(q_n \uparrow q)}$. Repeating a similar argument, we obtain the reverse inclusion as well.

Necessity. To proceed by contraposition, suppose that (3.4) does not hold. We may, without loss of generality, assume that:

$$\limsup_{n\to\infty} \Big(\sum_{j=1}^{2^n} \frac{1}{\lambda_j}\Big)^{\frac{1}{q_n}-\frac{1}{p_n}} = \infty$$

and

$$d_n := \Big(\sum_{j=1}^{2^n} \frac{1}{\lambda_j}\Big)^{\frac{-1}{p_n}} \downarrow 0$$

Then we define a sequence of functions $\{f_n\}_{n=1}^\infty$ on the interval $[0,1]$ inductively. Let f_0 be the identically zero function on $[0,1]$. When f_{n-1} is defined, f_n will be defined to be the function whose graph on the interval

$$\Big[\frac{j-1}{2^{n-1}}, \frac{j}{2^{n-1}}\Big]; \quad j = 1, ..., 2^{n-1}$$

consists of the two consecutive line segments connecting the points:

$$\Big(\frac{j-1}{2^{n-1}}, f_{n-1}\Big(\frac{j-1}{2^{n-1}}\Big)\Big), \Big(\frac{2j-1}{2^n}, f_n\Big(\frac{2j-1}{2^n}\Big)\Big)$$

and

$$\Big(\frac{j}{2^{n-1}}, f_{n-1}\Big(\frac{j}{2^{n-1}}\Big)\Big)$$

where :

$$f_n\Big(\frac{2j-1}{2^n}\Big) := \begin{cases} \min\Big\{f_{n-1}\Big(\frac{j-1}{2^{n-1}}\Big), f_{n-1}\Big(\frac{j}{2^{n-1}}\Big)\Big\} + d_n\ , \\ (n \text{ is odd}) \\ \max\Big\{f_{n-1}\Big(\frac{j-1}{2^{n-1}}\Big), f_{n-1}\Big(\frac{j}{2^{n-1}}\Big)\Big\} - d_n\ , \\ (n \text{ is even}) \end{cases}$$

Since $\{p_n\}$ is increasing,

$$\overline{\left(\sum_{j=1}^{2^n}\frac{1}{\lambda_j}\right)}$$

1p˙n

$$\leq \left(\sum_{j=1}^{2^{n+1}}\frac{1}{\lambda_j}\right)$$

1p˙n$\overline{\left(\sum_{j=1}^{2^n}\frac{1}{\lambda_j}\right)}$

1p˙n$\leq 2^{\frac{1}{p_n}} \leq 2$

1p˙n+1$\dfrac{d_n}{d_{n+1}} = \left(\sum_{j=1}^{2^{n+1}}\frac{1}{\lambda_j}\right)$

$$\overline{\left(\sum_{j=1}^{2^n}\frac{1}{\lambda_j}\right)}$$

1p˙n

$$\leq \left(\sum_{j=1}^{2^{n+1}}\frac{1}{\lambda_j}\right)$$

1p˙n$\overline{\left(\sum_{j=1}^{2^n}\frac{1}{\lambda_j}\right)}$

1p˙n$\leq 2^{\frac{1}{p_n}} \leq 2$

that is,

$$d_n \leq 2d_{n+1} \quad \text{for all n}$$

This, along with the fact that $d_n \downarrow 0$, implies:

$$|f_n(x) - f_{n+m}(x)| \leq d_n,$$

for all $n, m \geq 1,\ x \in [0,1]$.

Therefore $\{f_n\}_{n=1}^{\infty}$ is a Cauchy sequence, hence there exists a function f such that $f_n(x) \to f(x)$ as $n \to \infty$ and

$$|f_n(x) - f(x)| \leq d_n, \text{ for all } \ n \geq 1,\ x \in [0,1]. \tag{3.5}$$

We claim that $f \in \Lambda BV^{(p_n\uparrow p)}$ but $f \notin \Lambda BV^{(q_n\uparrow q)}$. To see this, let $\{I_j\}_{j=1}^{s}$ be a collection of nonoverlapping subintervals of $[0,1]$ such that $\inf|I_j| \geq \frac{1}{2^n}$ and set

$$\sigma_k = 1 \leq j \leq s : \left(\sum_{i=1}^{2^{k+1}}\frac{1}{\lambda_i}\right)^{-1} \leq |I_j| < \left(\sum_{i=1}^{2^k}\frac{1}{\lambda_i}\right)^{-1}, \quad k = 1, 2, \ldots .$$

Note that for $j \in \sigma_k$, if $a_j = \inf I_j$ and $b_j = \sup I_j$, using (3.5) we get

$$\begin{aligned}|f(I_j)| &\leq |f(a_j) - f_k(a_j)| \\ &\quad + |f_k(a_j) - f_k(b_j)| + |f_k(b_j) - f(b_j)| \\ &\leq d_k + |f_k(a_j) - f_k(b_j)| + d_k \leq 4d_k.\end{aligned}$$

Therefore, if we define t_n to be the smallest positive integer with $2^n \leq \sum_{j=1}^{2^{t_n}}\frac{1}{\lambda_j}$, it follows that

$$\begin{aligned}\sum_{j=1}^{s}\frac{|f(I_j)|^{p_n}}{\lambda_j} &= \sum_{k=0}^{t_n-1}\sum_{j\in\sigma_k}\frac{|f(I_j)|^{p_n}}{\lambda_j} \\ &\leq \sum_{k=0}^{t_n-1}(4d_k)^{p_n}\sum_{j\in\sigma_k}\frac{1}{\lambda_j} \\ &= 4^{p_n}\sum_{k=0}^{t_n-1}\left(\sum_{j=1}^{2^k}\frac{1}{\lambda_j}\right)^{-\frac{p_n}{p_k}}\sum_{j\in\sigma_k}\frac{1}{\lambda_j} \\ &\leq 4^{p_n}\sum_{k=0}^{t_n-1}\left(\sum_{j=1}^{2^k}\frac{1}{\lambda_j}\right)^{-1}\sum_{j\in\sigma_k}\frac{1}{\lambda_j} \\ &\leq 2.4^{p_n}\sum_{k=0}^{t_n-1}\left(\sum_{j=1}^{2^{k+1}}\frac{1}{\lambda_j}\right)^{-1}\sum_{j\in\sigma_k}\frac{1}{\lambda_j} \\ &\leq 2.4^{p_n}\sum_{k=0}^{t_n-1}\sum_{j\in\sigma_k}|I_j| \leq 8^{p_n}\end{aligned}$$

where in the third inequality we have used the fact that :

$$\left(\sum_{j=1}^{2^k}\frac{1}{\lambda_j}\right)^{-1} \leq 2\left(\sum_{j=1}^{2^{k+1}}\frac{1}{\lambda_j}\right)^{-1}$$

Consequently we have

$$\left(\sum_{j=1}^{s}\frac{|f(I_j)|^{p_n}}{\lambda_j}\right)^{\frac{1}{p_n}} \leq 8 \quad \text{for all} \quad n \geq 1$$

which means that $f \in \Lambda BV^{(p_n\uparrow p)}$.

Finally we show that $f \notin \Lambda BV^{(q_n\uparrow q)}$. Too see this, note that from the definition of f we have

$$|f(j/2^n) - f((j-1)/2^n)| = d_n \quad \text{for} \quad n \geq 1$$

Hence it follows that :

$$\begin{aligned}\left(\sum_{j=1}^{2^n}\frac{|f(I_j)|^{q_n}}{\lambda_j}\right)^{\frac{1}{q_n}} &= \left(d_n^{q_n}\sum_{j=1}^{2^n}\frac{1}{\lambda_j}\right)^{\frac{1}{q_n}} \\ &= d_n\left(\sum_{j=1}^{2^n}\frac{1}{\lambda_j}\right)^{\frac{1}{q_n}} = \left(\sum_{j=1}^{2^n}\frac{1}{\lambda_j}\right)^{\frac{1}{q_n}-\frac{1}{p_n}}\end{aligned}$$

where $I_j := [(j-1)/2^n, j/2^n]$.

Note that we have earlier assumed that $\limsup_n\left(\sum_{j=1}^{2^n}\frac{1}{\lambda_j}\right)^{\frac{1}{q_n}-\frac{1}{p_n}} = \infty$. As a result, we see that $f \notin \Lambda BV^{(q_n\uparrow q)}$, as desired.

The mutual relationship between the generalized Wiener classes $\Lambda BV^{(p_n\uparrow p)}$ is rather chaotic even in the special case where $p_n = q_n = 1$ for all n. It is worth mentioning that in order to determine when $\Lambda BV^{(p)} \subseteq \Gamma BV^{(q_n\uparrow q)}$ ([4, Theorem 1.4]), it has been assumed that $p \leq q$, or in Theorem 3 we have assumed that $\Lambda = \Gamma$. So, it would be highly desirable to find a condition that implies the general inclusion $\Lambda BV^{(p_n\uparrow p)} \subseteq \Gamma BV^{(q_n\uparrow q)}$ without any additional restrictions on the p_n, q_n, Λ and Γ. The following theorem provides such a condition which is new even for special cases; see the corollaries to this result. (Note, of course, that we deal with the case of interest where Γ is unbounded.)

The inclusion $\Lambda BV^{(p_n\uparrow p)} \subseteq \Gamma BV^{(q_n\uparrow q)}$ holds whenever:

$$\sup_{1\leq n<\infty} \sum_{k=1}^{\infty} \Delta\Big(\frac{1}{\gamma_k}\Big) \max_{1\leq m\leq k} m\Big(\sum_{j=1}^{m}\frac{1}{\lambda_j}\Big)^{-\frac{q_n}{p_n}} < \infty$$

Proof. Assume that $f \in \Lambda \mathrm{BV}^{(p_n\uparrow p)}$. For an arbitrary but fixed n, let $\{I_j\}_{j=1}^{s}$ be a nonoverlapping collection of subintervals of $[0,1]$ with $\inf|I_j| \geq \frac{1}{2^n}$, and put $q = q_n/p_n$, $x_j = |f(I_j)|^{p_n}$, $y_j = 1/\lambda_j$, $z_j = 1/\gamma_j$. Using inequality 2.3, we may also assume that the x_j are arranged in descending order. Now, by Abel's transformation and applying (2.1) together with Lemma 2.1 we obtain:

$$\begin{aligned}
&\sum_{k=1}^{s}\frac{|f(I_k)|^{q_n}}{\gamma_k}\\
&= \sum_{k=1}^{s-1}\Delta\Big(\frac{1}{\gamma_k}\Big)\sum_{j=1}^{k}|f(I_j)|^{q_n} + \frac{1}{\gamma_s}\sum_{j=1}^{s}|f(I_j)|^{q_n}\\
&\leq \sum_{k=1}^{s-1}\Delta\Big(\frac{1}{\gamma_k}\Big)\Big(\sum_{j=1}^{k}\frac{|f(I_j)|^{p_n}}{\lambda_j}\Big)^{\frac{q_n}{p_n}} \max_{1\leq m\leq k} m\Big(\sum_{j=1}^{m}\frac{1}{\lambda_j}\Big)^{-\frac{q_n}{p_n}}\\
&\quad + \frac{1}{\gamma_s}\Big(\sum_{j=1}^{s}\frac{|f(I_j)|^{p_n}}{\lambda_j}\Big)^{\frac{q_n}{p_n}} \max_{1\leq m\leq s} m\Big(\sum_{j=1}^{m}\frac{1}{\lambda_j}\Big)^{-\frac{q_n}{p_n}}\\
&\leq \sum_{k=1}^{s-1}\Delta\Big(\frac{1}{\gamma_k}\Big)V_\Lambda(f;p_n\uparrow p)^{q_n} \max_{1\leq m\leq k} m\Big(\sum_{j=1}^{m}\frac{1}{\lambda_j}\Big)^{-\frac{q_n}{p_n}}\\
&\quad + \frac{1}{\gamma_s}V_\Lambda(f;p_n\uparrow p)^{q_n} \max_{1\leq m\leq s} m\Big(\sum_{j=1}^{m}\frac{1}{\lambda_j}\Big)^{-\frac{q_n}{p_n}}\\
&\leq \sum_{k=1}^{s-1}\Delta\Big(\frac{1}{\gamma_k}\Big)V_\Lambda(f;p_n\uparrow p)^{q_n} \max_{1\leq m\leq k} m\Big(\sum_{j=1}^{m}\frac{1}{\lambda_j}\Big)^{-\frac{q_n}{p_n}}\\
&\quad + \sum_{k=s}^{\infty}\Delta\Big(\frac{1}{\gamma_k}\Big)V_\Lambda(f;p_n\uparrow p)^{q_n} \max_{1\leq m\leq k} m\Big(\sum_{j=1}^{m}\frac{1}{\lambda_j}\Big)^{-\frac{q_n}{p_n}}\\
&= V_\Lambda(f;p_n\uparrow p)^{q_n}\sum_{k=1}^{\infty}\Delta\Big(\frac{1}{\gamma_k}\Big) \max_{1\leq m\leq k} m\Big(\sum_{j=1}^{m}\frac{1}{\lambda_j}\Big)^{-\frac{q_n}{p_n}}\\
&< \infty
\end{aligned}$$

where we have used the fact that:

$$\frac{1}{\gamma_s}\max_{1\leq m\leq s} m\Big(\sum_{j=1}^{m}\frac{1}{\lambda_j}\Big)^{-\frac{q_n}{p_n}} \leq \sum_{k=s}^{\infty}\Delta\Big(\frac{1}{\gamma_k}\Big)\max_{1\leq m\leq k} m\Big(\sum_{j=1}^{m}\frac{1}{\lambda_j}\Big)^{-\frac{q_n}{p_n}}$$

Taking suprema over all collections $\{I_j\}_{j=1}^{s}$ as above, and over all n yields $V_\Gamma(f;q_n\uparrow q) < \infty$. That is, $f \in \Gamma BV^{(q_n\uparrow q)}$.

The inclusion $\Lambda BV^{(p)} \subseteq \Gamma BV^{(q)}$ holds whenever:

$$\sum_{n=1}^{\infty}\Delta\Big(\frac{1}{\gamma_n}\Big)\max_{1\leq k\leq n} k\Big(\sum_{j=1}^{k}\frac{1}{\lambda_j}\Big)^{-\frac{q}{p}} < \infty$$

The inclusion $\Lambda BV \subseteq \Gamma BV$ holds whenever:

$$\sum_{n=1}^{\infty}\Delta\Big(\frac{1}{\gamma_n}\Big)n\Big(\sum_{j=1}^{n}\frac{1}{\lambda_j}\Big)^{-1} < \infty$$

4 Relationships between the classes $\Lambda V[h]^{(p)}$

A sequence h of positive reals is said to be a modulus of variation if it is nondecreasing and concave, i.e., $h(tn+(1-t)m) \geq th(n)+(1-t)h(m)$ whenever h is defined at n, m and $tn+(1-t)m$. Let $1 \leq p < \infty$ and Λ be a Λ-sequence.

Definition 4.1 *For a bounded real function f on $[a,b]$, the sequence*

$$\nu_{p,\Lambda}(f;n) := \sup \sum_{j=1}^{n} \frac{|f(I_j)|^p}{\lambda_j}$$

is the modulus of variation associated to f, where the supremum is taken over all collections $\{I_j\}_{j=1}^n$ of nonoverlapping subintervals of $[a,b]$. The class of all functions f for which $\nu_{p,\Lambda}(f;n) = O(h(n))$ as $n \to \infty$, is denoted by $\Lambda V[h]^{(p)}$.

This class was first introduced in a more genral context and was studied in connection with the Riemann–Stieltjes integration theory [7]. Note that many of the classes previously considered may be obtained by making special choices of Λ, p and h. This provides us with a general setting to unify a number of inclusion theorems (see Theorem 4 and its corollaries). For instance if $p=1$, then taking $\Lambda=\{1\}$ we get the Chanturiya class $V[\nu]$, and taking $h=\{1\}$ we obtain the Waterman class ΛBV.

Let $1 \leq p \leq q < \infty$. Suppose that either the sequence

$$\Big\{ \Big(\sum_{j=1}^{n} \frac{1}{\gamma_j} \Big)^{\frac{1}{q}} \Big/ \Big(\sum_{j=1}^{n} \frac{1}{\lambda_j} \Big)^{\frac{1}{p}} \Big\}_n$$

or

$$\Big\{ h_2(n)^{\frac{1}{q}} / h_1(n)^{\frac{1}{p}} \Big\}_n$$

is nondecreasing. Then the inclusion $\Lambda V[h_1]^{(p)} \subseteq \Gamma V[h_2]^{(q)}$ holds whenever:

$$\sup_{1 \leq n < \infty} \left(\sum_{j=1}^{n} \frac{1}{\gamma_j} \Big/ h_2(n) \right)^{\frac{1}{q}} \left(h_1(n) \Big/ \sum_{j=1}^{n} \frac{1}{\lambda_j} \right)^{\frac{1}{p}} < \infty \quad (4.6)$$

Proof. Let $f \in \Lambda V[h_1]^{(p)}$ and consider a fixed n. Let $\{I_j\}_{j=1}^n$ be a nonoverlapping collection of subintervals of $[0,1]$. Set $x_j := |f(I_j)|^p$, $y_j := 1/\lambda_j$ and $z_j := 1/\gamma_j$. In view of the equi-monotonic sequences inequality (2.3) we can, and do, assume that the x_j are arranged in descending order. Now, applying (2.1) with $q/p \geq 1$ in place of q we obtain:

$$\Big(\sum_{j=1}^{n} \frac{|f(I_j)|^q}{\gamma_j} \Big)^{\frac{p}{q}} \leq \sum_{j=1}^{n} \frac{|f(I_j)|^p}{\lambda_j} \max_{1 \leq k \leq n} \Big(\sum_{j=1}^{k} \frac{1}{\gamma_j} \Big)^{\frac{p}{q}} \Big(\sum_{j=1}^{k} \frac{1}{\lambda_j} \Big)^{-1}$$

Therefore, we get

$$\begin{aligned}
\Big(\sum_{j=1}^{n} \frac{|f(I_j)|^q}{\gamma_j} \Big)^{\frac{1}{q}}
&\leq \Big(\sum_{j=1}^{n} \frac{|f(I_j)|^p}{\lambda_j} \Big)^{\frac{1}{p}} \max_{1 \leq k \leq n} \Big(\sum_{j=1}^{k} \frac{1}{\gamma_j} \Big)^{\frac{1}{q}} \Big(\sum_{j=1}^{k} \frac{1}{\lambda_j} \Big)^{-\frac{1}{p}} \\
&\leq \Big(\nu_{p,\Lambda}(f;n) \Big)^{\frac{1}{p}} \max_{1 \leq k \leq n} \Big(\sum_{j=1}^{k} \frac{1}{\gamma_j} \Big)^{\frac{1}{q}} \Big(\sum_{j=1}^{k} \frac{1}{\lambda_j} \Big)^{-\frac{1}{p}} \\
&\leq C.h_1(n)^{\frac{1}{p}}.\frac{h_2(n)^{\frac{1}{q}}}{h_1(n)^{\frac{1}{p}}} = C.h_2(n)^{\frac{1}{q}}
\end{aligned}$$

for some positive constant C, depending only on f. As a result, taking supremum over all collections $\{I_j\}_{j=1}^n$ as above, it follows that

$$\nu_{q,\Gamma}(f;n) \leq C^q.h_2(n),$$

which means that $f \in \Gamma V[h_2]^{(q)}$.

Making suitable choices of Λ, Γ, p, q, h_1 and h_2 to invoke the preceding theorem, we obtain the following corollaries.

([1]) The following inclusion holds:

$$\Lambda BV \subseteq V\Big[n \Big/ \sum_{j=1}^{n} 1/\lambda_j \Big]$$

([4]) Let $1 \leq p \leq q < \infty$. Then the inclusion $\Lambda BV^{(p)} \subseteq \Gamma BV^{(q)}$ holds whenever

$$\sup_{1 \leq n < \infty} \Big(\sum_{j=1}^{n} \frac{1}{\gamma_j} \Big)^{\frac{1}{q}} \Big(\sum_{j=1}^{n} \frac{1}{\lambda_j} \Big)^{-\frac{1}{p}} < \infty$$

([2]) The inclusion $V[h_1] \subseteq V[h_2]$ holds whenever

$$\sup_{1 \leq n < \infty} \frac{h_1(n)}{h_2(n)} < \infty$$

We end this paper with a characterization of when $\Lambda V[h]^{(p)}$ coincides with $B[0,1]$, the space of all bounded functions on $[0,1]$.

The equality $\Lambda V[h]^{(p)} = B[0,1]$ holds if and only if

$$\sup_{1\leq n<\infty} h(n)^{-1}\sum_{j=1}^{n}\frac{1}{\lambda_j} < \infty \tag{4.7}$$

Proof. Assume (4.7) and let $f \in B[0,1]$. Then there exists some positive constant C such that $\sum_{j=1}^{n}\frac{1}{\lambda_j} \leq Ch(n)$ for all n. So if $\{I_j\}_{j=1}^{n}$ is a collection of nonoverlapping subintervals of $[0,1]$, we have:

$$\sum_{j=1}^{n}\frac{|f(I_j)|^p}{\lambda_j} \leq 2^p\|f\|_\infty^p\sum_{j=1}^{n}\frac{1}{\lambda_j} \leq 2^pC\|f\|_\infty^p h(n)$$

where $\|f\|_\infty = \sup\{|f(x)|: x\in[0,1]\}$. Thus, $f \in \Lambda V[h]^{(p)}$.

Conversely, let $\Lambda V[h]^{(p)} = B[0,1]$. To show that (4.7) holds, we construct an $f \in B[0,1]$ as follows:

$$f(x) := \begin{cases} 1 & \text{if } x \in (\frac{1}{2^{k+1}} + \frac{1}{2^k}, \frac{1}{2^{k-1}}); k = 1,2,... \\ 0 & \text{otherwise.}\end{cases}$$

Now consider the collection

$$I_k = [\frac{1}{2^k}, \frac{1}{2^k} + \frac{2}{3}.\frac{1}{2^k}] \quad ; \quad k = 1,2,...,n$$

Then we have:

$$\sum_{k=1}^{n}\frac{1}{\lambda_k} = \sum_{k=1}^{n}\frac{|f(I_k)|^p}{\lambda_k} \leq \nu_{p,\Lambda}(f;n) \leq h(n)\sup_{1\leq r<\infty}\frac{\nu_{p,\Lambda}(f;r)}{h(r)}$$

where the last inequality is a result of our assumption that $f \in \Lambda V[h]^{(p)}$. The proof is complete.

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