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Research Article

Ranking of Efficiency in Context-dependent Data Envelopment Analysis with Non-discretionary Data

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Abstract

In this paper, we use the context-dependent method for ranking decision making units with non-discretionary data in data envelopment analysis . This methodology partitions a set DMUs into the several levels of efficient frontier, then obtains the efficiency of each DMU against of every calculate efficient frontier. We calculate a ranking score for each DMU by this efficiency indexes. Therefore we can rank DMUs by this score. In the end, we use our proposed method in the numerical example.

Keywords : Data Envelopment Analysis, Decision Making Units, Context-dependent, Non-discretionary data, Ranking, Efficiency.

1 Introduction

Data envelopment analysis (DEA) is a methodology for measuring the relative efficiencies of a set of decision-making units that use multiple inputs to produce multiple outputs. The standard models assume that all inputs and outputs are crisp and can be changed at the discretion of management. While crisp input and output data are fundamentally indispensable in the standard evaluation process, input and output data in real-world problems are often imprecise or ambiguous. In addition, real-world problems may also include non-discretionary factors that are beyond the control of a DMU's management.

The percentage of controllability, or the degree to which the data are discretionary, in the modified Russell model of not only calculates the relative efficiency of homogeneous decision making units , but also determines the factors contributing to inefficiency and the measure of inefficiency in inefficient DMUs. The results obtained by this model help the management to improve inefficient units with the aim of reaching the efficient units. However, these efficiency improvement methods must be possible, i.e., the management must be able to effect the expected changes, such as decreasing the inputs or increasing the outputs, within his/her area of responsibility. For instance, in evaluating bank branches, the important factors include branch area, number of staff, resources, interest received, and costs. Branch area is an almost non-discretionary input, since it cannot be decreased at will. Also, the resources of a bank branch are an almost non-discretionary output type. In order to improve an inefficient bank

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branch, the management has to increase the resources. However, it is known that the overall resources of banks are fixed; that is, the overall amount of cash available in the country is almost constant. Therefore, a bank manager cannot increase the resources of the branch at will to improve the performance of his branch, since this requires a decrease in the resources of other branches, which is only to some degree possible. Thus, the resources of a branch constitute an output that can be changed to a certain degree, and not infinitely. Therefore, it is necessary for a more accurate evaluation to precisely recognize the sphere of authority of a manager in effecting the changes and consider it in the modeling, so that the pattern presented to the inefficient units will be possible to reach.

Farzipoor Saen [8] presented a methodology for treating non-discretionary in slacks-based measure (SBM) formulation. We believe that this method is one of the best method to calculating the relative efficiency of DMUs with non-discretionary factors. Therefore, we will compare the efficiency results of our model with the efficiency results in [8].

Azizi et al. [2] proposed a pair of data envelopment analysis model to measure of relative efficiencies of decision-making units in the presence of non-discretionary factors and imprecise data. Compared to traditional , the proposed interval approach measures the efficiency of each DMU relative to the inefficiency frontier, also called the input frontier, and is called the worst relative efficiency or pessimistic efficiency. The pair of proposed interval DEA models takes into account the crisp, ordinal, and interval data, as well as non-discretionary factors, simultaneously for measurement of relative efficiencies of DMUs.

Saati et al. [15] show that considering bounded factors in DEA models results in the desertion of the concept of relative efficiency since the efficiency of the DMUs are calculated by comparing the DMUs with their lower and/or upper bounds. In addition, they present a fuzzy DEA model with discretionary and non-discretionary factors in both the input and output-oriented CCR models.

It is often necessary in real performance assessment practice to rank a group of decision making units (DMUs) in terms of their efficiencies. Data envelopment analysis developed by charnes et al. [6] has been universally recognized as a useful tool of performance assessment, but very often more than one DMU is evaluated as DEA efficient, which makes DEA efficient units unable to be compared or ranked.

To rank DEA efficient units, quite a lot of research has been done and many ranking methodologies have been suggested in the DEA literature. For example, Andersen and Petersen [1] proposed a procedure that was later referred to as the super-efficiency method for ranking DEA efficient units. Super-efficiency refers to the DEA efficiency measured by excluding the DMU under evaluation from the constraints of DEA models and has been deeply researched in the literature [3, 7, 12, 13, 17]. Farzipoor Saen [9] developed an innovative algorithm to rank technology suppliers in the presence of non-discretionary factors from suppliers perspective Unlike the current work, ranking of DMUs with some non-discretionary factors have not studied before.

The context-dependent DEA [17, 20] is introduced to measure the relative attractiveness of a particular DMU when compared to others. Relative attractiveness depends on the evaluation context constructed from alternative DMUs. The original DEA method evaluates each DMU against a set of efficient DMUs and cannot identify which efficient DMU is a better option with respect to the inefficient DMU. This is because all efficient DMUs have an efficiency score of one. Although one can use the super-efficiency DEA model [1, 17, 19] to rank the performance of efficient DMUs, the evaluation context changes in each evaluation and the efficient DMUs are not evaluated against the same reference set.

The context-dependent DEA approach starts with clustering the DMUs and obtaining several performance levels. For this purpose, an algorithm is developed to remove the best-practice frontier to allow the remaining (inefficient) DMUs to form a new second-level best-practice frontier. If this new second frontier is removed, a third- level best-practice frontier is formed, and so on, until no DMU is left [20]. Each evaluation level represents an efficient frontier composed by DMUs in a specific performance level.

The next step of the context-dependent DEA approach after clustering the DMUs to several levels is the calculation of attractiveness and progress scores to produce a ranking at every performance level [14, 16]. When DMUs in a specific level are viewed as having equal performance, the attractiveness measure allows us to differentiate the "equal performance" based upon the same specific evaluation context. A combined use of attractiveness and progress measures can further characterize the performance of DMUs.

The objective of the present paper is to introduce a model of data envelopment analysis to measure of relative efficiencies of decision-making units (DMUs) in the presence of non-discretionary factors and a ranking methodology for DMUs is proposed. This methodology ranks DMUs by the slack-based context-dependent model with non-discretionary data.

The paper is organized as follows: in section 2, we present a slack-based context-dependent model on DEA efficiency and give a description of non-discretionary model with slacks-based measure. In section 3, we measure efficiency in context-dependent DEA with non-discretionary data. Section 4 proposes a method for ranking DMUs. Numerical examples are provided in section 5 to illustrate the proposed efficiency and ranking methodology and also the efficiency results of our proposed model is compared with model (2.5) (Farzipoor Saens model [8]). The paper is concluded in section 6.

2 Background

2.1. Slack-based context-dependent DEA

Presume that there are n $DMUs$, each DMU_o ($o \in J = \{1, \cdots, n\}$) producing a vector of outputs $Y_o = (y_{1o}, \cdots, y_{so})$ by using a vector of inputs $X_o = (x_{1o}, \cdots, x_{mo})$. A slack-based measure (SBM) of efficiency [15] is introduced to evaluate the efficiency together with the slack value. The following index ρ:

$$\rho = \frac{1 - \frac{1}{m}\sum_{i=1}^{m}\frac{s_i^-}{x_{io}}}{1 + \frac{1}{s}\sum_{r=1}^{s}\frac{s_r^+}{y_{ro}}} \tag{2.1}$$

is defined in terms of the amount of slack, and has the value between 0 and 1. The SBM efficiency score is obtained from the following program:

$$\begin{aligned} Min \quad & \rho = \frac{1 - \frac{1}{m}\sum_{i=1}^{m}\frac{s_i^-}{x_{io}}}{1 + \frac{1}{s}\sum_{r=1}^{s}\frac{s_r^+}{y_{ro}}} \\ s.t. \quad & \sum_{j=1}^{n}\lambda_j x_{ij} + s_i^- = x_{io}, \quad i = 1, \dots, m, \\ & \sum_{j=1}^{n}\lambda_j y_{rj} - s_r^+ = y_{ro}, \quad r = 1, \dots, s, \\ & \lambda_j, s_i^-, s_r^+ \geq 0, \quad j = 1, \cdots, n, i = 1, \dots, m, \\ & r = 1, \dots, s. \end{aligned} \tag{2.2}$$

The SBM efficiency score is normalized between 0 and 1. DMU_o is efficient iff $\rho^* = 1$, because $\rho^* = 1$ implies that all slacks are zero and the DMU lies on the strong efficient frontier. Note that throughout this paper, when a DMU is efficient, it is dominated by no real or virtual DMUs. We defined the set of all DMUs as J^1 and the set of efficient DMUs in J^1 as E^1. Then the sequences of J^l and E^l are defined interactively as $J^{l+1} = J^l - E^l$. The set of E^l can be found as the DMUs with the optimal value ρ_o^l of 1 in the following fractional programming problem:

$$\begin{aligned} Min \quad & \rho_o^l \frac{1 - \frac{1}{m}\sum_{i=1}^{m}\frac{s_i^-}{x_{io}}}{1 + \frac{1}{s}\sum_{r=1}^{s}\frac{s_r^+}{y_{ro}}} \\ s.t. \quad & \sum_{j \in J^l}\lambda_j x_{ij} + s_i^- = x_{io}, \quad i = 1, \dots, m, \\ & \sum_{j \in J^l}\lambda_j y_{rj} - s_r^+ = y_{ro}, \quad r = 1, \dots, s, \\ & \lambda_j, s_i^-, s_r^+ \geq 0, \quad j \in J^l, i = 1, \dots, m, \\ & r = 1, \dots, s. \end{aligned} \tag{2.3}$$

Example 2.1 *Suppose we have 8 DMUs with two inputs and one single output of one (see Table 1).*

We define $J^1 = \{DMU_1, \cdots, DMU_8\}$ and obtain $E^1 = \{DMU_3, DMU_4\}$ by solving (2.3). As a result the DMUs on the first level are DMU_3 and DMU_4. Thereafter, we obtain $J^2 = J^1 - E^1 =$

Table 1: Data and ρ_o^l of DMUs in sample.

DMU_j	1	2	3	4	5	6	7	8
Input 1	4	5	3	1	3	6	6	2
Input 2	4	2	1	3	3	3	4	4
ρ_o^1	0.500	0.550	1.000	1.000	0.667	0.417	0.375	0.625
ρ_o^2	0.750	1.000	-	-	1.000	0.750	0.625	1.000
ρ_o^3	1.000	-	-	-	-	1.000	0.833	-
Levels	3	2	1	1	2	3	4	2

$\{DMU_1, DMU_2, DMU_5, DMU_6, DMU_7, DMU_8\}$ and by solving (2.3) obtain $E^2 = \{DMU_2, DMU_5, DMU_8\}$ then, the DMUs on the second level are DMU_2, DMU_5 and DMU_8. Similarly we obtain $J^3 = J^2 - E^2 = \{DMU_1, DMU_6, DMU_7\}$, $E^3 = \{DMU_1, DMU_6\}$ and $J^4 = E^4 = \{DMU_7\}$. Figure 1. plots these four levels of the efficient frontiers of 8 $DMUs$.

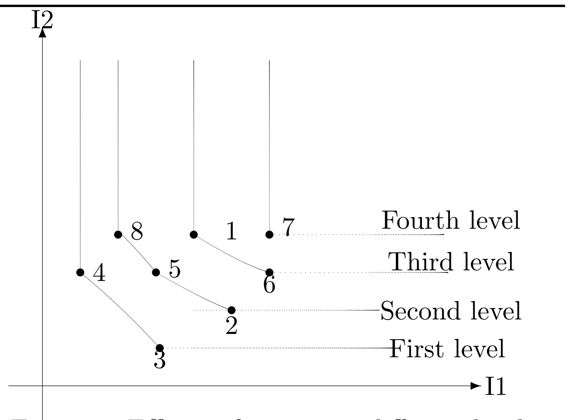

Figure 1. Efficient frontiers in different levels.

According to Morita et al. [12], the attractiveness based on the evaluation context E^l is measured with respect to the DMUs in the subset J^l. For example, the attractiveness for DMU_o based on the evaluation context E^l is obtained from the following programming problem.

$$\begin{aligned} Min \quad & \delta_o^l = \frac{\frac{1}{m}\sum_{i=1}^{m} \overline{x_i}/x_{io}}{\frac{1}{s}\sum_{r=1}^{s} \overline{y_r}/y_{ro}} \\ s.tv \quad & \sum_{j\in J^l} \lambda_j x_{ij} \le \overline{x_i}, \quad i=1,\cdots,m, \\ & \sum_{j\in J^l} \lambda_j y_{rj} \ge \overline{y_r}, \quad r=1,\ldots,s, \\ & \lambda_j \ge 0, \quad j \in J^l, \\ & 0 \le \overline{x_i} \le x_{io}, \overline{y_r} \ge y_{ro}, \quad i=1,\ldots,m, \\ & r=1,\ldots,s. \end{aligned} \tag{2.4}$$

2.2. Non-discretionary model with slacks-based measure

Suppose that the input and output factors may each be partitioned into two subsets: discretionary (D) factors, i.e., they are controllable, and non-discretionary (ND) factors, i.e., they are not controllable. Thus, $I = \{1,2,\cdots,m\} = I_D \cup I_{ND}$, $I_D \cap I_{ND} = \emptyset$ and $O = \{1,2,\cdots,s\} = O_D \cup O_{ND}$, $O_D \cap O_{ND} = \emptyset$ Banker and Morey [3] provided the first DEA model to evaluate efficiency in the presence of input non-discretionary factors. Farzipoor Saen [8] proposed a model for non-discretionary SBM, which is presented as follows:

$$\begin{aligned} Min \quad & \gamma = t - \frac{1}{m}\sum_{i=1}^{m} \frac{s_i^-}{x_{io}} \\ s.t. \quad & 1 = t + \frac{1}{s}\sum_{r=1}^{s} \frac{s_r^+}{y_{ro}}, \\ & tx_o = X\Lambda + S^-, \\ & ty_o = Y\Lambda + S^+, \\ & s_i^- \le \beta_i x_{io}, \quad i=1,\cdots,m, \\ & s_r^+ \le \mu_r y_{ro}, \quad r=1,\cdots,m, \\ & \Lambda \ge 0, S^- \ge 0, S^+ \ge 0, t > 0, \end{aligned} \tag{2.5}$$

where β_i, μ_r represent parameters. Assigning values from 0 to 1, accords different degrees of discretionariness to input i with $\beta_i = 0$ characterizing this input as non-discretionary and $\beta_i = 1$ changing the characterization to completely discretionary. Similarly, setting $\mu_r = 0$ consigns output r to a fixed (non-discretionary) value while allowing $\mu_r \longrightarrow \infty$, or, equivalently, removing this constraint on s_r^+ allows its value to vary in a freely discretionary manner.

3 Context-dependent efficiency with non-discretionary data

In this section, we apply the method used in the previous section as model (2.4). So, we have the following model:

$$
\begin{aligned}
Min \quad & \rho = \frac{\frac{1}{m}\sum_{i=1}^{m}\frac{\overline{x_i}}{x_{io}}}{\frac{1}{s}\sum_{r=1}^{s}\frac{\overline{y_r}}{y_{ro}}} \\
s.t. \quad & \sum_{j\in J^l}\lambda_j x_{ij} \le \overline{x_i}, \quad i=1,\dots,m, \\
& \sum_{j\in J^l}\lambda_j y_{rj} \ge \overline{y_r}, \quad r=1,\cdots,s, \\
& (1-\beta_i)x_{io} \le \overline{x_i} \le x_{io}, \quad i=1,\dots,m, \\
& y_{ro} \le \overline{y_r} \le (1+\mu_r)y_{ro}, \quad r=1,\dots,s, \\
& \lambda_j \ge 0, \quad j\in J^l.
\end{aligned} \tag{3.6}
$$

where β_i, μ_r represent parameters whose values vary in [0,1). If $\beta_i = 0$ or $\mu_r = 0$ then input i or output r is completely non-discretionary. Similarly, if $\beta_i = 1$ or $\mu_r \longrightarrow \infty$ then input i or output r is completely discretionary. If β_i or μ_r vary from 0 to 1 ($\beta_i, \mu_r \in (0,1)$) then different degrees of discretion are assigned to input i or output r. Note that this fractional programming problem can be transformed into a linear programming problem using the Charnes-Cooper transformation [4] as

$$
\begin{aligned}
Min \quad & \tau = 1/m\sum_{i=1}^{m}\widetilde{x_i}/x_{io} \\
s.t. \quad & \sum_{r=1}^{s}\widetilde{y_r}/y_{ro} = s, \\
& \sum_{j\in J^l}\Lambda_j x_{ij} \le \widetilde{x_i}, \quad i=1,\dots,m, \\
& \sum_{j\in J^l}\Lambda_j y_{rj} \ge \widetilde{y_r}, \quad r=1,.,.,s, \\
& (1-\beta_i)tx_{io} \le \widetilde{x_i} \le tx_{io}, \quad i=1,\dots,m, \\
& ty_{ro} \le \widetilde{y_r} \le (1+\mu_r)ty_{ro}, \quad r=1,\dots,s, \\
& t\ge 1, \Lambda_j \ge 0, \quad j\in J^l.
\end{aligned} \tag{3.7}
$$

Let an optimal solution of (3.7) be $(\tau^*, \widetilde{x^*}, \widetilde{y^*}, \Lambda^*, t^*)$. Then we have an optimal solution of (3.6) as:

$$
\rho^* = \tau^*, \lambda^* = \Lambda^*/t^*, \overline{x}^* = \widetilde{x}^*/t^*, \overline{y}^* = \widetilde{y}^*/t. \tag{3.8}
$$

When obtaining the super-efficiency of DMU_o against the PPS by model (3.7), if DMU_o is out of the PPS, the following constraints imply that the DMU moves away from the PPS and does not reach level t, and model (3.7) will, therefore, be infeasible (see Figure 2).
$(1-\beta_i)tx_{io} \le \widetilde{x_i} \le tx_{io}, \quad (i=1,\cdots,m),$
$ty_{ro} \le \widetilde{y_r} \le (1+\mu_r)ty_r, \quad (r=1,\cdots,s).$

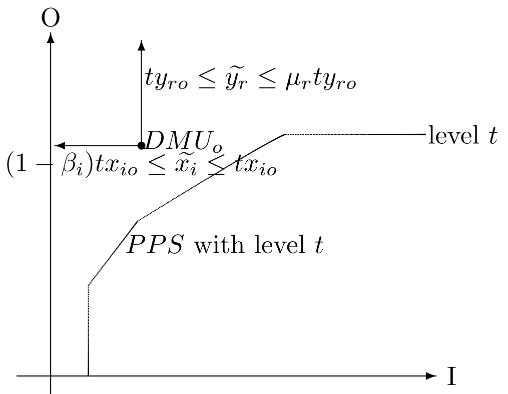

Figure 2. DMU_o against to PPS with level t.

To avoid this, these constraints are transformed to the following constraints:
$tx_{io} \le \widetilde{x_i} \le (1+\beta_i^{'})tx_{io} \quad (i=1,\cdots,m),$
$(1-\mu_r^{'})ty_{ro} \le \widetilde{y_r} \le ty_{ro} \quad (r=1,\cdots,s),$
where β_i, μ_r represent parameters whose values vary in [0,1). If $\beta_i = 0$ or $\mu_r = 0$ then input i or output r is completely non-discretionary. Similarly, if $\beta_i \longrightarrow \infty$ or $\mu_r = 1$ then input i or output r is completely discretionary. If β_i or μ_r vary from 0 to 1 ($\beta_i, \mu_r \in (0,1)$) then different degrees of discretion are assigned to input i or output r.

4 Proposed method for ranking all DMUs

For ranking DMUs, first we apply the slack-based context-dependent DEA model (3.7) to all DMUs. Second, the corresponding efficiency level is calculated by using the context-dependent DEA model. We assume that DMUs are divided to L levels. The optimal weight of each level is determined by the management, which is represented by β_l ($l = 1,\cdots,L$). If DMU_o belongs to level k ($k \in \{1,\cdots,L\}$), then we obtain the efficiency of DMU_o against level l ($l = 1,\cdots,k-1$). Thus, the optimal soulution of the model is calculated as follows:

$$
\begin{aligned}
\theta_{lo} = Min \quad & 1/m\sum_{i=1}^{m}\widetilde{x_i}/x_{io} \\
s.t. \quad & \sum_{r=1}^{s}\widetilde{y_r}/y_{ro} = s, \\
& \sum_{j\in E^l}\Lambda_j x_{ij} \le \widetilde{x_i}, \quad i=1,\dots,m, \\
& \sum_{j\in E^l}\Lambda_j y_{rj} \ge \widetilde{y_r}, \quad r=1,\dots,s, \\
& (1-\beta_i)tx_{io} \le \widetilde{x_i} \le tx_{io}, \quad i=1,\dots,m, \\
& ty_{ro} \le \widetilde{y_r} \le (1+\mu_r)ty_{ro}, \quad r=1,\dots,s, \\
& 0 < t \le 1, 0 \le \Lambda_j, \quad j\in E^l,
\end{aligned} \tag{4.9}
$$

where $1 \le l \le k-1$.

Note: It is clear that θ_{lo} $(o \in \{1, \cdots, n\})$ calculate the relative efficiency of decision making units in the presence of non-discretionary factors. We profess that the results of this model are better than the other methods. We will discover the advantage of our model by comparing its results with Farzipoor Saens efficiency model [8] in the numerical example section.

Then we obtain the super-efficiency of DMU_o against level l $(l = k+1, \cdots, L)$. So the optimal soulution of the model is calculated as follows:

$$
\begin{aligned}
\phi_{lo} = Min \quad & 1/m \sum_{i=1}^{m} \widetilde{x}_i / x_{io} \\
s.t. \quad & \sum_{r=1}^{s} \widetilde{y}_r / y_{ro} = s, \\
& \sum_{j \in E^l} \Lambda_j x_{ij} \leq \widetilde{x}_i, \quad i = 1, \ldots, m, \\
& \sum_{j \in E^l} \Lambda_j y_{rj} \geq \widetilde{y}_r, \quad r = 1, \ldots, s, \\
& t x_{io} \leq \widetilde{x}_i \leq (1+\beta_i^{'}) t x_{io}, \quad i = 1, \ldots, m, \\
& (1-\mu_r^{'}) t y_{ro} \leq \widetilde{y}_r \leq t y_{ro}, \quad r = 1, \ldots, s, \\
& t \geq 1, \Lambda_j \geq 0, \quad j \in E^l,
\end{aligned}
\tag{4.10}
$$

where $k+1 \leq l \leq L$.

Now, we show by the following simple example that model (4.10) may be infeasible.

Example 4.1 *Suppose that we have three DMUs with two inputs and one output, where input 1 is discretionary and input 2 is non-discretionary. The data of the three DMUs are shown in Table 2.*

The Farell frontier of these three DMUs is drawn in Figure 3, below:

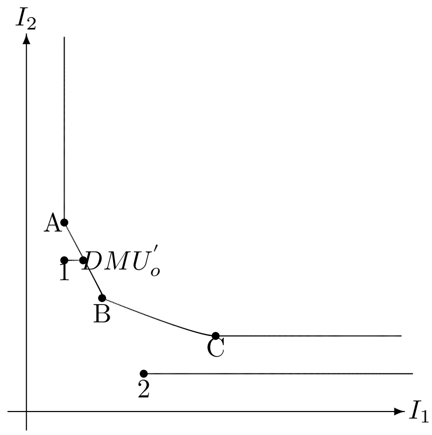

Figure.3: Farell frontier of DMUs.

For calculating the super-efficiency of DMU_o in position 1, we should obtain the non-radial projection of this DMU on the Farell frontier by increasing inputs. But, because input 2 is non-discretionary, we cannot increase input 2. Therefore, the projection of DMU_o on the frontier is $DMU_o^{'}$. So, if we run model (4.10) for DMU_o, this model is feasible and $\varphi_{to}^{*} > 1$. Now, if DMU_o is in position 2, then it cannot reach the Farell frontier by increasing input 1. So, if we run model (4.10) for DMU_o in position 2, then this model will become infeasible. In such cases, we say that DMU_o does not indicate super-efficiency in terms of input saving, since DMU_o cannot be moved onto the frontier constructed from the other DMUs via input increases. In fact DMU_o does not have input super-efficiency.

Similarly, model (4.10) may be infeasible, indicating that DMU_o does not have super-efficiency in its output.

Therefore, the case of infeasibility of model (4.10) results from the existence of the inputs and outputs discretion restrictions. In other words, the following restrictions in model (4.10) lead to infeasibility:

$$
\begin{aligned}
& \widetilde{x}_i \leq (1+\beta_i^{'}) t x_{io}, \quad i = 1, \cdots, m, \\
& (1-\mu_r^{'}) t y_{ro} \leq \widetilde{y}_r, \quad r = 1, \cdots, s.
\end{aligned}
\tag{4.11}
$$

To remove infeasibility, we introduce the deviation variable for them similar to the method proposed by Jahanshahloo et al. [11]. So, we will have:

$$\widetilde{x}_i + n_i - p_i = (1+\beta_i^{'}) t x_{io}, \quad i = 1, \cdots, m,$$

$$\widetilde{y}_r + n_r^{\prime} - p_r^{\prime} = (1-\mu_r^{'}) t y_{ro}, \quad r = 1, \cdots, s,$$

where n_i, p_i $(i = 1, \cdots, m)$ and $n_r^{\prime}, p_r^{\prime}$ $(r = 1, \cdots, s)$ are the deviation weights corresponding to the weight restrictions in (4.11), and $(1+\beta_i^{'}) t x_{io}$, $(i = 1, \cdots, m)$, $(1-\mu_r^{'}) t y_{ro}$, $(r = 1, \cdots, s)$ are goals of weight restrictions in (4.11). If model (4.10) becomes infeasible, then a penalty should be imposed in order to have minimized deviation from the considered goals. Toward this end, if input i (or output r) is non-discretionary ($i \in I_{ND}$ (or $r \in O_{ND}$)), then constrain i (or constrain r) will be equality in (4.11); therefore we should minimize n_i, p_i (or $n_r^{\prime}, p_r^{\prime}$) in (4.11) simultaneously. But if input i (or output r) is the percentage of controllability ($i \in \%I_{ND}$ (or $r \in \%O_{ND}$)), then constrain i (or constrain r) will be strict inequality in (4.11); therefore we should minimize p_i (or $n_r^{\prime}$) in (4.11). Thus, model (4.10) must be modified in the following form:

$$
\phi_{lo} = Min \quad 1/m \sum_{i=1}^{m} \widetilde{x}_i / x_{io} + M\Big(\sum_{i \in I_{ND}} (n_i + p_i) + \sum_{i \in \%I_{ND}} (p_i) + \sum_{r \in O_{ND}} (n_r^{\prime} + p_r^{\prime}) + \sum_{r \in \%O_{ND}} (n_r^{\prime}) \Big)
$$

Table 2: Inputs and Output.

DMU	input 1	input 2	Output
1	1	5	1
2	2	3	1
3	5	2	1

$$
\begin{aligned}
s.t. \quad & \sum_{r=1}^{s} \widetilde{y}_r / y_{ro} = s, \\
& \sum_{j \in E^l} \Lambda_j x_{ij} \leq \widetilde{x}_i, \quad i = 1, \ldots, m, \\
& \sum_{j \in E^l} \Lambda_j y_{rj} \geq \widetilde{y}_r, \quad r = 1, \ldots, s, \\
& t x_{io} \leq \widetilde{x}_i, \quad i = 1, \ldots, m, \\
& \widetilde{y}_r \leq t y_{ro}, \quad r = 1, \ldots, s, \\
& \widetilde{x}_i + n_i - p_i = (1 + \beta_i^{'}) t x_{io}, \ i \in I_{ND} \cup \% I_{ND}, \\
& \widetilde{y}_r + n_r^{\prime} - p_r^{\prime} = (1 - \mu_r^{'}) t y_{ro}, \ r \in O_{ND} \cup \% O_{ND}, \\
& t \geq 1, \Lambda_j \geq 0, \quad j \in E^l,
\end{aligned} \tag{4.12}
$$

where M is a very large positive number. Model (4.12) is always feasible (see [11]).

Theorem 4.1 *If model (4.10) is feasible, then the optimal solution of model (4.12) equal to the optimal solution of model (4.10).*

Proof. see [11].

After solving model (4.9), we observe that all of the DMUs on level t (E^t) have better ranks than all of the DMUs in $E^{t+1} \cup \cdots \cup E^L$ and have worse ranks than all of the DMUs in $E^1 \cup \cdots \cup E^{t-1}$.
For each arbitrary level t ($t = 1, \cdots, L$), either of the following two cases may happen:

Case 1: $card(E^t) = 1$
It is clear that DMUs on E^t have ranks of $card(E^1 \cup \cdots \cup E^{t-1}) + 1$.

Case 2: $card(E^t) > 1$
After solving models (4.9) and (4.10), one of the following three cases may happen:

Case a: If $t = 1$, then we will calculate the following ranking index:

$$
R_o = \sum_{r=2}^{L} \beta_r \phi_{ro} \tag{4.13}
$$

Case b: If $t = L$, then we will calculate the following ranking index:

$$
R_o = \frac{1}{\sum_{i=1}^{L-1} \beta_i (1 - \theta_{io})} \tag{4.14}
$$

Case c: If $1 < t < L$ then we will calculate following ranking index:

$$
R_o = \frac{\sum_{r=k+1}^{L} \beta_r \phi_{ro}}{\sum_{i=1}^{k-1} \beta_i (1 - \theta_{io})} \tag{4.15}
$$

Definition 4.1 *If DMU_i and DMU_j are on level t, then the preference score of DMU_i is better than that of DMU_j if $R_i > R_j$.*

In fact, in the proposed ranking method, a DMU has a better rank if its relation score of the weighted sum of distances from the lower levels to the weighted distance sum of the higher levels is greater than that of the others DMUs. Owing to the use of the weighted distance, the probability of DMUs having the same rank in this raking method is very low. Therefore, it can be observed that the ranking index score obtained from the method proposed in this paper is unique for each DMU.
The proposed ranking method has important advantages, for instance, this method ranks all DMUs, e.i., it ranks extreme and non-extreme DMUs. Moreover ranking DMUs with non-discretionary inputs and outputs is possible by this method. For further illustration, we provide a simple example in the next section.

5 Numerical Example

We use the data of Farzipoor Saens numerical example [8], then we compare our results of efficiency model (4.9) with Farzipoor Saens efficiency model [8]. So we assume that there are 15 DMUs which require two inputs to produce two outputs. The data are

Table 3: Inputs and Outputs and efficiency of Farzipoor Saens mode.

DMU	input 1	input 2	Output 1	Output 2	Saens efficiency
1	4	3	5	2	1
2	7	3	4	5	0.722
3	8	1	3	8	1
4	4	2	2	1	0.303
5	2	4	3	4	1
6	10	1	2	1	1
7	12	1	3	2	1
8	10	1.5	5	2	1
9	15	25	4	5	0.412
10	25	15	5	2	0.259
11	30	20	2	1	1
12	12	12	8	3	0.267
13	5	20	1	5	1
14	42	45	3	1	0.45
15	11	12	2	3	0.583

Table 4: Results of model (4.9) and 5 levels and their ranks.

DMU_j	θ_{1j}	θ_{2j}	θ_{3j}	θ_{4j}	θ_{5j}	level	Rank
1	1.000000	-	-	-	-	1	4
2	0.721868	1.000000	-	-	-	2	5
3	1.000000	-	-	-	-	1	1
4	0.303030	0.464286	1.00000	-	-	3	10
5	1.000000	-	-	-	-	1	2
6	0.210526	0.613757	1.00000	-	-	3	9
7	0.400000	1.000000	-	-	-	2	6
8	1.000000	-	-	-	-	1	3
9	0.234948	0.327138	1.00000	-	-	3	11
10	0.085094	0.161290	0.379965	1.000000	-	4	13
11	0.037461	0.071032	0.190217	0.300000	1.0	5	14
12	0.266871	1.000000	-	-	-	2	7
13	0.381944	1.000000	-	-	-	2	8
14	0.025553	0.044164	0.120219	0.154440	1.0	5	15
15	0.207686	0.322492	0.648148	1.000000	-	4	12

Table 5: Super-efficiency of *DMU*s at the first level and their rank.

DMU_j at the first level	ϕ_{2j}	ϕ_{3j}	ϕ_{4j}	ϕ_{5j}	$R_j = \sum_{r=2}^{L} \beta_r \phi_{rj}$	Rank
DMU_1	1.3402	2.2222	3.5526	16.6667	R_1 =43.3823	4
DMU_3	1.9866	2.8872	9.4565	44.6667	R_3 =91.0213	1
DMU_5	2.7097	4.3636	7.7647	32.7273	R_5 =87.3887	2
DMU_8	1.2314	1.8713	5.0350	18.1481	R_8 =45.5545	3

presented in Table 3. The suppositions are:

(a): Input 1 is not controllable.

(b): Input 2 is controllable.

(c): Output 1 is 70% controllable.

(d): Output 2 is controllable.

We run model (4.9) and obtain 5 different levels. The results are shown in Table 4.

Table 6: Super efficiency of DMUs at the second level and their rank.

DMU_j at the second level	ϕ_{3j}	ϕ_{4j}	ϕ_{5j}	$R_j = \frac{\sum_{r=k+1}^{L} \beta_r \phi_{rj}}{\sum_{i=1}^{k-1} \beta_i (1-\theta_{ij})}$	Rank
DMU_2	1.7687	5.0649	22.1053	R_2 =8.8345	1
DMU_7	1.5714	6.1388	32.3333	R_7 =5.3017	2
DMU_{12}	1.1429	3.2258	8.5714	R_{12} =1.6705	3
DMU_{13}	1.5000	1.6923	5.2500	R_{13} =1.4798	4

Table 7: Super-efficiency of DMUs at the third level and their rank.

DMU_j at the third level	ϕ_{4j}	ϕ_{5j}	$R_j = \frac{\sum_{r=k+1}^{L} \beta_r \phi_{rj}}{\sum_{i=1}^{k-1} \beta_i (1-\theta_{ij})}$	Rank
DMU_4	2.2414	8.7500	R_4 =0.8616	2
DMU_6	3.7500	11.500	R_6 =1.2085	1
DMU_9	1.3333	5.8947	R_9 =0.4858	3

Table 8: Super-efficiency of DMUs at the fourth level and their rank.

DMU_j at the fourth level	ϕ_{5j}	$R_j = \frac{\sum_{r=k+1}^{L} \beta_r \phi_{rj}}{\sum_{i=1}^{k-1} \beta_i (1-\theta_{ij})}$	Rank
DMU_{10}	2.8147	R_{10} =0.1179	2
DMU_{15}	4.0909	R_{15} =0.2097	1

Table 9: Rank of them in the fifth level..

DMU_j in fifth level	$R_j = \frac{1}{\sum_{i=1}^{L-1} \beta_i (1-\theta_{ij})}$	Rank
DMU_{11}	R_{11} =0.0364	1
DMU_{14}	R_{14} =0.0352	2

After finding the different efficiency levels, Table 4 shows the level on which a DMU lies. Let DMU_o ($o \in \{1, \cdots, 15\}$) be on level k ($k = \{1, \cdots, 5\}$). Then, we specify which main threecases DMU_o belong to. By solving models (4.9) and (4.10), we obtain θ_{io} ($i = \{1, \cdots, k-1\}$) and ϕ_{ro} ($r = \{k+1, \cdots, 5\}$). Also we assume that the importance of each level is twise as much as that of its next level, i.e. the importance of the first level is twice of the importance of the second level. In mathematical terms:

$$\beta_1 = 16, \beta_2 = 8, \beta_3 = 4, \beta_4 = 2, \beta_5 = 1.$$

Table 4 shows that DMU_1, DMU_3, DMU_5 and DMU_8 are on the first level, so we use ranking index of case a for ranking them. Also DMU_{11} and DMU_{14} is on the last level, therefore it has the lowest rank; we use ranking index of case b. The remaining DMUs are in the case c, so we use the ranking index of case c. The results of these calculations are shown in Tables 5-9.

Comparison of our efficiency model with Farzipoor Saens efficiency model:

1: In the Farzipoor Saens efficiency model, eight out of fifteen DMUs become efficient. In other words, more than half of the DMUs are efficient. However, in our proposed method, in the first stage, only four DMUs become efficient.

2: In the Farzipoor Saens efficiency model, DMU_{11} is an efficient DMU. While in our proposed method, this DMU ranked one in the last. This example shows that our model results are closer to reality because if we look at the inputs and outputs of DMU_{11}, DMU_{14} that are in the last rank. We find that have the most inputs and the lowest outputs, so, they should have the worst rank but in the Farzipoor Saens efficiency model, DMU_{11} is a member of eight efficient DMUs.

6 Conclusion

Until now, interesting methods have been developed for calculating efficiency with non-discretionary data. This paper uses the slacks-based measure used in

Farzipoor's model [8] and obtain different efficiency levels with a slack-based measure of efficiency in the context-dependent model proposed by Morita et al. [12]. After that DMUs are divided to three cases:
a: DMUs at the first level.
b: DMUs at the last level.
c: DMUs between the first and the last levels.
So, we define a special ranking index for each group. The advantage of our proposed method over the previous methods are:
i) this method can rank DMUs with non-discretionary data,
ii) this method ranks extreme and non-extreme DMUs,
iii) due to the use of weighted distances in the ranking index, the probability of DMUs with the same rank is very low.
But the proposed model ranks DMUs with constant returns to scale. If we want to use this ranking method for DMUs with variable returns to scale, then some of the super-efficiency models(ϕ_{to}) become infeasible. So, we cannot use this method for ranking such DMUs. Our purpose in the coming paper is to obtain a ranking model for DMUs with non-discretionary data in the case of variable returns to scale.

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