Available online at http://ijim.srbiau.ac.ir/
Int. J. Industrial Mathematics (ISSN 2008-5621)
Vol. 12, No. 2, 2020 Article ID IJIM-1129, 11 pages
Research Article

n-fold Obstinate Filters in Pseudo-Hoop Algebras

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Received Date: 2017-10-09 Revised Date: 2018-02-11 Accepted Date: 2019-11-25

Abstract

In this paper, we introduce the concepts of n-fold obstinate pseudo-hoop and n-fold obstinate filter in pseudo-hoops. Then we investigated these notions and proved some properties of them. Also, we discussed the relationship between n-fold obstinate pseudo-hoop and n-fold obstinate filter and other types of n-fold pseudo-hoops and n-fold filters such as n-fold(positive) implicative filter and n-fold fantastic filter in pseudo-hoops. For example, we proved that any n-fold obstinate filter is a maximal filter. Finally, we obtain a characterization of n-fold obstinate filters in terms of congruences and we show that any n-fold obstinate pseudo-hoop is an n-fold fantastic, n-fold positive implicative, n-fold implicative pseudo-hoop and simple pseudo-hoop.

Keywords : Pseudo-hoop algebra; Filter; n-fold obstinate pseudo-hoop; n-fold obstinate filter.

1 Introduction

NAturally ordered commutative residuated integral monoids (hoop) introduced by B. Bosbach in [5, 6], then studied by J. R. Büchi et al. in [7], a paper never published. Also G. Georgescu, L. Leustean et al. study the pseudo-hoops in [8]. It is well-known that in various logical systems, filters play a fundamental role, filters correspond to sets of provable formulas closed with respect to Modus Ponnen. In [10, 12, 14, 16, 17] the authors investigated the notation folding theory to residuated lattices, n-folding fantastic filters and obstinate filters in BL-algebras, generalization of integral filters and n-fold integral BL-algebras and n-fold filters of MTL-algebras. In [2], R. A. Borzooei et al., survey the notion of n-fold(implicative, positive implicative and fantastic filters) of pseudo-hoops. They show that if F is an n-fold(implicative, positive implicative and fantastic)filter, then A/F is an n-fold (implicative, positive implicative and fantastic)pseudo-hoops. Also in [15], A. Namdar et al., proposed the obstinate filter in hoops.
In this disquisition, we define and study the notion of n-fold obstinate pseudo-hoop and n-fold obstinate filters in pseudo-hoops and generalization of the corresponding notion in the crisp case. Several properties of n-fold obstinate pseudo-hoop and n-fold obstinate filters are given. We show that F is an n-fold obstinate filter of A if and only if A/F is an n-fold obstinate pseudo-hoop. On the other hands if F is an n-fold obstinate filter of A, then A/F is a local and simple pseudo-hoop. Also, we show that F is an n-fold

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obstinate filter if and only if F is a maximal and n-fold positive implicative filter.

2 Preliminaries

In this section, we recollect some definitions and results which will be used in this paper.

Definition 2.1 *[8] A* pseudo-hoop algebra *or* pseudo-hoop *is an algebra* $(A, \odot, \to, \rightsquigarrow, 1)$ *of type* $(2, 2, 2, 0)$ *such that, for all* $x, y, z \in A$*:*
$(PH1)$ $x \odot 1 = 1 \odot x = x$,
$(PH2)$ $x \to x = x \rightsquigarrow x = 1$,
$(PH3)$ $(x \odot y) \to z = x \to (y \to z)$,
$(PH4)$ $(x \odot y) \rightsquigarrow z = y \rightsquigarrow (x \rightsquigarrow z)$,
$(PH5)$ $(x \to y) \odot x = (y \to x) \odot y = x \odot (x \rightsquigarrow y) = y \odot (y \rightsquigarrow x)$.

On pseudo-hoop A, we define $x \leq y$ if and only if $x \to y = x \rightsquigarrow y = 1$. It is easy to see that $\leq$ is a partial order relation on A. If $\odot$ is commutative(or equivalently $\to = \rightsquigarrow$), then A is said to be a hoop. A pseudo-hoop A is *bounded* if there is an element $0 \in A$ such that $0 \leq x$, for all $x \in A$. For any $x \in A$, we consider $x^- = x \to 0$ and $x^\sim = x \rightsquigarrow 0$. An element $x \in A$ is called *atom* if it is a minimal among elements in bounded hoop $A \backslash \{0\}$. Also, element $x \in A$ is called *idempotent* if $x^2 = x$. The *order* of $1 \neq x \in A$, in symbols $ord(x)$ is the smallest $n \in \mathbb{N}$ such that $x^n = 0$. If no such n exists, then $ord(x) = \infty$. (See [8])

Definition 2.2 *[8] For pseudo-hoop A and for any $x, y \in A$, we define $x \vee y = ((x \to y) \rightsquigarrow y) \wedge ((y \to x) \rightsquigarrow x) = ((x \rightsquigarrow y) \to y) \wedge ((y \rightsquigarrow x) \to x)$. If $\vee$ is the join operation on A, then A is called a* pseudo $\vee$-hoop.

Proposition 2.1 [8] *In any pseudo-hoop A, the following properties hold, for all $x, y, z \in A$:*

(i) $(A, \leq)$ is a meet-semilattice with $x \wedge y = (x \to y) \odot x = x \odot (x \rightsquigarrow y)$,
(ii) $1 \to x = x$, $1 \rightsquigarrow x = x$, $x \rightsquigarrow x = 1$,
(iii) $y \leq x \to y$ and $y \leq x \rightsquigarrow y$,
(iv) if $x \leq y$, then $y \rightsquigarrow z \leq x \rightsquigarrow z$ and $y \to z \leq x \to z$,
(v) $x \odot y \leq x, y$ and $x^n \leq x$, for any $n \in \mathbb{N}$,
(vi) if $\vee$ exists, then $(x \vee y) \rightsquigarrow z = (x \rightsquigarrow z) \wedge (y \rightsquigarrow z)$, $(x \vee y) \to z = (x \to z) \wedge (y \to z)$.

Proposition 2.2 [8] *Let A be a bounded pseudo-hoop. Then the following properties hold, for all $x, y, z \in A$:*

(i) if $x \leq y$, then $y^\sim \leq x^\sim$ and $y^- \leq x^-$,
(ii) $(x^n)^- \leq (x^{n+1})^-$ and $(x^n)^\sim \leq (x^{n+1})^\sim$,
(iii) $0^- = 0^\sim = 1$ and $1^- = 1^\sim = 0$,
(iv) $x \leq (x^-)^\sim$ and $x \leq (x^\sim)^-$,
(v) $x \odot x^- = x \odot x^\sim = 0$,
(vi) $x^- \leq x \to y$ and $x^\sim \leq x \rightsquigarrow y$.

Definition 2.3 *[8] Let A be a pseudo-hoop. A non-empty subset F of A is called a* filter *of A if,*

$(F1)$ $x \in F$ and $x \leq y$, then $y \in F$, for any $x, y \in A$,
$(F2)$ $x \odot y \in F$, for any $x, y \in F$.
Clearly, $1 \in F$, for all filters of A. A filter F of A is called a *proper filter* if $F \neq A$. It is easy to see that, if A is a bounded pseudo-hoop, then a filter is proper if and only if it is not containing 0. The set of all filters of A denoted by $\mathcal{F}(A)$.

Proposition 2.3 [8] *Let A be a pseudo-hoop. If F is a non-empty subset of pseudo-hoop A such that $1 \in F$, then the following statements are equivalent, for any $x, y \in A$:*

(i) F is a filter,
(ii) if $x, x \to y \in F$, then $y \in F$,
(iii) if $x, x \rightsquigarrow y \in F$, then $y \in F$.
Notation: It is easy to see that the intersection of all filters of pseudo-hoop A is a filter. Hence, for any $B \subseteq A$, $\bigcap_{B \subseteq F \in \mathcal{F}(A)} F$ is a filter and denoted by $[B)$ and we called *generated filter by B*.

Theorem 2.1 [8] *Let $x \in A$. Then $[x) = \{a \in A \mid x^n \leq a$, for some $n \geq 1\}$, $F(x) = [F \cup \{x\}) = \{t \mid t \geq f \odot x^n$ for $f \in F, n \in \mathbb{N}\}$ and $[F \cup G) = \{a \in A \mid a \geq f \odot g$ for $f \in F$, $g \in G\}$, for any $F, G \in \mathcal{F}(A)$.*

Definition 2.4 *[8] A filter F of pseudo-hoop A is called a* normal filter *if $x \to y \in F$ if and only if $x \rightsquigarrow y \in F$, for all $x, y \in A$.*

Definition 2.5 *[8] A proper filter F of a pseudo $\vee$-hoop A is called a* prime filter *of A if $x \vee y \in F$, then $x \in F$ or $y \in F$, for any $x, y \in A$.*

A *maximal filter* of pseudo-hoop A is a proper filter M of A that is not included in any other proper filters of A. $Max(A)$ is the set of all maximal filters of A.

Proposition 2.4 [8] *Let A be a pseudo-hoop and F be a non-empty subset of pseudo-hoop A. Then the following conditions are equivalent, for any $x \in A$:*

(i) F is a maximal filter,
(ii) $x \notin F$ if and only if $(x^n)^-, (x^n)^\sim \in F$, for some $n \in \mathbb{N}$.

Proposition 2.5 [3] *Let A be a bounded $\vee$-hoop. Then every maximal filter of A is a prime filter.*

Definition 2.6 *[2] Let F be a subset of A such that $1 \in F$. Then for any $x, y, z \in A$:*

(i) F is called an *n-fold positive implicative filter* of A, if $x^n \to (y \to z) \in F$ and $x^n \rightsquigarrow y \in F$, then $x^n \to z \in F$. Also, if $x^n \rightsquigarrow (y \rightsquigarrow z) \in F$ and $x^n \to y \in F$, then $x^n \rightsquigarrow z \in F$.
(ii) F is called an *n-fold implicative filter* of A, if $x \to ((y^n \to z) \rightsquigarrow y) \in F$ and $x \in F$, then $y \in F$. Also, if $x \rightsquigarrow ((y^n \rightsquigarrow z) \to y) \in F$ and $x \in F$, then $y \in F$.
(iii) F is called an *n-fold fantastic filter* of A, if $z \to (y \to x) \in F$ and $z \in F$, then $((x^n \to y) \rightsquigarrow y) \to x \in F$. Also, if $z \rightsquigarrow (y \rightsquigarrow x) \in F$ and $z \in F$, then $((x^n \rightsquigarrow y) \to y) \rightsquigarrow x \in F$.

Definition 2.7 *[8] Let A and B be two bounded pseudo-hoops. A map $f : A \to B$ is called a* pseudo-hoop homomorphism *if and only if for all $x, y \in A$, $f(0) = 0$, $f(1) = 1$, $f(x \odot y) = f(x) \odot f(y)$, $f(x \to y) = f(x) \to f(y)$ and $f(x \rightsquigarrow y) = f(x) \rightsquigarrow f(y)$.*

The set of all pseudo-hoop homomorphism from A to B is shown by $Hom(A, B)$.

Definition 2.8 *[8] Let A be a pseudo-hoop. Then A is called:*

(i) *n-fold positive implicative pseudo-hoop*, if $x^{n+1} = x^n$, for all $x \in A$.
(ii) *n-fold implicative pseudo-hoop*, if $(x^n \to 0) \rightsquigarrow x = x$ and $(x^n \rightsquigarrow 0) \to x = x$, for all $x \in A$.
(iii) *n-fold fantastic pseudo-hoop*, if $((x^n \to y) \rightsquigarrow y) \to x = y \to x$ and $((x^n \rightsquigarrow y) \to y) \rightsquigarrow x = y \rightsquigarrow x$, for all $x, y \in A$.
(iv) *local pseudo-hoop*, if $ord(x) < \infty$ or $ord(x^-) < \infty$ or $ord(x^\sim) < \infty$, for all $x \in A$.
(v) *simple pseudo-hoop*, if A is non-trivial and $\{1\}$ is its only proper filter.
(vi) *cancellative pseudo-hoop*, if the monoid $(A, \odot, 1)$ is cancellative if and only if $b \to (a \odot b) = a$ and $b \rightsquigarrow (a \odot b) = a$ if and only if $c \odot a = c \odot b$, then $a = b$, for any $a, b, c \in A$.

Notation: From now one, we let $(A, \odot, \to, \rightsquigarrow, 0, 1)$ or A be a bounded pseudo-hoop, unless otherwise state.

3 n-fold obstinate pseudo-hoops and n-fold obstinate filters in pseudo-hoops

In this section, we introduce the notion of n-fold obstinate pseudo-hoop and n-fold obstinate filter in pseudo-hoop and investigate some properties of them.

Definition 3.1 *A is called an n-*fold obstinate pseudo-hoop *if, for all $x \neq 1$, $x^n = 0$.*

Example 3.1 (i) *Let $(A = \{0, a, b, 1\}, \leq)$ be a chain that is $0 < a < b < 1$. Define the operations $\odot, \to$ and $\rightsquigarrow$ on A as follows:*

$\to, \rightsquigarrow$	0	a	b	1
0	1	1	1	1
a	b	1	1	1
b	a	b	1	1
1	0	a	b	1

$\odot$	0	a	b	1
0	0	0	0	0
a	0	0	0	a
b	0	0	a	b
1	0	a	b	1

Then $(A, \odot, \to, \rightsquigarrow, 1, 0)$ is a bounded pseudo-hoop and A is an 3-fold obstinate pseudo-hoop.

But it is not an 2-fold obstinate pseudo-hoop, because $b^2 \neq 0$.
(*ii*) [9]. Let $NS[0,1]$, (non-standard interval $[0,1]$) be the ordered set whose elements are pairs (a,b) such that $a=0$ and $0 \leq b$ or $0<a<1$ and b arbitrary or $a=1$ and $b \leq 0$ (b running on real set). The ordering is lexicographic: $(a,b) \leq (c,d)$ if and only if $a<c$ or ($a=c$ and $b \leq d$). The ordered set $NS[0,1]$ endowed with the operations: $(a,b) \odot (c,d) =$

$$max\left((0,0),(\frac{1}{2}(a+c-1+ac),\frac{b(c+1)}{2})\right)$$

If $(a,b) \leq (c,d)$, then $(a,b) \to (c,d) = 1$, otherwise $(a,b) \to (c,d) = \left(\frac{2c-a+1}{1+a},\frac{2d-2b}{1+a}\right)$.

Also, if $(a,b) \leq (c,d)$, then $(a,b) \rightsquigarrow (c,d)=1$, otherwise $(a,b) \rightsquigarrow (c,d) = \left(\frac{2c-a+1}{1+a},\frac{-b(c+1)}{1+a}+d\right)$.
Then $(NS[0,1],\odot,\to,\rightsquigarrow,0,1)$ is a bounded pseudo-hoop. But it is not an n-fold obstinate pseudo-hoop, because $(1,b)\odot(1,b) = max((0,0),(1,b)) = (1,b)$, for $b \leq 0$.
(*iii*) [1] Let $A=[0,\frac{1}{2}]\cup\{1\}$ and operations $\odot,\to$ and $,\rightsquigarrow$ are defind by, $x\odot y = \max(0,x+y-1)$, and if $x \leq y$, then $x \to y = 1$, otherwise $x \to y = \min(1-x+y,1)$.
Then $(A,\odot,\to,1,0)$ is an 2-fold obstinate pseudo-hoop.

Proposition 3.1 *If A is an n-fold obstinate pseudo-hoop, then A is an $(n+1)$-fold obstinate pseudo-hoop.*

Proof. Let A be an n-fold obstinate pseudo-hoop. Then $x^n=0$, for any $x \in A\backslash\{1\}$. By Proposition 2.1(v), $x^{n+1} \leq x^n$. Hence, $x^{n+1}=0$, for any $x \in A\backslash\{1\}$ and so A is an $(n+1)$-fold obstinate pseudo-hoop.

Corollary 3.1 *Any n-fold obstinate pseudo-hoop is an $(n+k)$-fold obstinate pseudo-hoop, for all $k \geq 1$.*

Proposition 3.2 *If A is an n-fold obstinate pseudo-hoop, then A is not a cancellative pseudo-hoop.*

Proof. Let A be a cancellative pseudo-hoop, by the contrary. Then $x^{n+1}=x^n=0$. Hence $x^n \odot x = x^n \odot 1 = 0$. Thus $x=1=0$, which is a contradiction. Therefore, A is not a cancellative pseudo-hoop.

Proposition 3.3 *If A does not have idempotent element except $\{0,1\}$ and $\mathcal{A}(M)$ is the set of all atoms of A, then $\mathcal{A}(M)\cup\{1\}$ is an n-fold obstinate pseudo-hoop.*

Proof. If $x \in \mathcal{A}(M)$, then x is an atom and is not idempotent element of A. Thus $x^2 \neq x$. By Proposition 2.1(v), $x^n = x^2 = 0$.

Proposition 3.4 *If A is an n-fold obstinate pseudo-hoop, then A does not have idempotent element except 0,1.*

Proof. Let $0 \neq x$ be an idempotent element of A. Then $x^2=x$. Since A is an n-fold obstinate pseudo-hoop, $0=x^n=x$, which is a contradiction.

Notation: For any $x \in A$, we consider $m_x = ord(x)-1$, so $x^{m_x} \neq 0$.

Proposition 3.5 *Let A be an n-fold obstinate pseudo-hoop. Then x^{m_x} is an atom for any $0,1 \neq x \in A$ and $m_x \in \mathbb{N}$.*

Proof. Let $x \in A$. Then $x^n=0$ and $0=x^n \leq x^{n-1} \leq x^{n-2} \leq ... \leq x$. If $t=ord(x)$, then $x^{t-1} \neq 0$. So for $m_x=t-1$, x^{m_x} is an atom.

Definition 3.2 *A proper filter F of A is called an* n-fold obstinate filter *if for all $x,y \notin F$, then $x^n \to y, y^n \to x \in F$ and $x^n \rightsquigarrow y, y^n \rightsquigarrow x \in F$, for $n \in \mathbb{N}$.*

Example 3.2 *In Example 3.1(i), $F=\{1\}$ is an 3-fold obstinate filter but since $b^2 \to 0 = a \to 0 = b \notin F$, F is not an 2-fold obstinate filter of A.*

Proposition 3.6 *Let F be a proper filter of A. Then the following statements are equivalent:*
(i) F is an n-fold obstinate filter of A,
(ii) $x \in F$ or $(x^n)^-,(x^n)^\sim \in F$, for all $x \in A$.

Proof. (*i*) $\Rightarrow$ (*ii*) Suppose F is an n-fold obstinate filter and $x \notin F$. Since F is a proper filter and A is bounded, $0 \notin F$. Then $(x^n)^- = x^n \to 0 \in F$ and $(x^n)^\sim = x^n \rightsquigarrow 0 \in F$.
(*ii*) $\Rightarrow$ (*i*) Let $x,y \notin F$. Then by assumption, $(x^n)^-$, $(x^n)^\sim$, $(y^n)^-$, $(y^n)^\sim \in F$. Thus,

by Proposition 2.2(vi), $(x^n)^- \leq x^n \to y$ and $(y^n)^- \leq y^n \to x$. Since F is a filter, by (F1), $x^n \to y \in F$ and $y^n \to x \in F$. The proof of other case is similar. Therefore, F is an n-fold obstinate filter of A.

Corollary 3.2 *F is an n-fold obstinate filter of A if and only if $x \notin F$ implies $((x^n)^-)^m, ((x^n)^\sim)^m \in F$, for all $m \in \mathbb{N}$ and $x \in A$.*

Proposition 3.7 *Let F be an n-fold obstinate filter of A. Then the following conditions hold:*

(i) for all $0, 1 \neq x \in A$, $x^n \to (x^n)^-$, $x^n \to (x^n)^\sim \in F$ or $(x^n)^- \to x^n$, $(x^n)^\sim \to x^n \in F$,
(ii) for all $x \in A$, $((x^n)^-)^\sim \to x^n$, $((x^n)^\sim)^- \rightsquigarrow x^n \in F$,
(iii) for all $x \notin F$, and for any $y \leq x^n$, then y^-, $y^\sim \in F$,
(iv) for all $x \in A$, $x^n \to x^{2n}$, $x^n \rightsquigarrow x^{2n} \in F$.

Proof. (i) Let $x \in F$. Then by Proposition 2.1(iii), $x^n \leq (x^n)^- \to x^n$. Since F is a filter, by $(F1)$, $(x^n)^- \to x^n \in F$. If $x \notin F$, then by Proposition 3.6(ii), $(x^n)^- \in F$. By Proposition 2.1(iii), $(x^n)^- \leq x^n \to (x^n)^-$, and so $x^n \to (x^n)^- \in F$. The proof of other cases is similar.
(ii) We consider the following cases:
Case 1: If $x \in F$, then by Proposition 2.1(iii), $x^n \leq ((x^n)^-)^\sim \to x^n$. Since F is a filter, by $(F1)$, $((x^n)^-)^\sim \to x^n \in F$.
Case 2: If $x \notin F$, then by Proposition 3.6(ii), $(x^n)^- \in F$. By Proposition 2.2(iv) and (vi), $(x^n)^- \leq (((x^n)^-)^\sim)^- \leq ((x^n)^-)^\sim \to x$, and so by $(F1)$, $((x^n)^-)^\sim \to x \in F$. The proof of other cases is similar, too.
(iii) Let $x \notin F$ and $y \leq x^n$. Then by Proposition 2.2(i), $(x^n)^- \leq y^-$. Since F is an n-fold obstinate filter, by Proposition 3.6(ii), $(x^n)^- \in F$ and by (F1), $y^- \in F$.
(iv) We consider the following cases:
Case 1: If $x \in F$, then by Proposition 2.1(iii), $x^{2n} \leq x^n \to x^{2n}$. Since F is a filter, by $(F1)$, $x^n \to x^{2n} \in F$.
Case 2: If $x \notin F$, then by Proposition 3.6(ii), $(x^n)^- \in F$. By Proposition 2.2(iv) and (vi), $(x^n)^- \leq (((x^n)^-)^\sim)^- \leq ((x^n)^-)^\sim \to x^{2n}$. Also, by Proposition 2.2(iv) and Proposition 2.1(iv), $x^n \leq ((x^n)^-)^\sim$ and $((x^n)^-)^\sim \to x^{2n} \leq x^n \to x^{2n}$. Therefore, by $(F1)$, $x^n \to x^{2n} \in F$.

Proposition 3.8 *If F is an n-fold obstinate filter of A, then F is an $(n+k)$-fold obstinate filter of A, for any $k \in \mathbb{N}$.*

Proof. Let $x \notin F$. Then by Proposition 3.6(ii), $(x^n)^- \in F$. By Proposition 2.2(ii), $(x^n)^- \leq (x^{n+1})^-$ and so by $(F1)$, $(x^{n+1})^- \in F$. The proof of other case is similar.

Let $F \in \mathcal{F}(A)$. Define $x \equiv_F y$ if and only if $x \to y \in F$, $y \to x \in F$, and $x \rightsquigarrow y \in F$, $y \rightsquigarrow x \in F$ for any $x, y \in A$. Then we can see that $\equiv_F$ is a congruence relation on A. The set of all congruence classes is denoted by A/F, it means $A/F = \{[x] \mid x \in A\}$, where $[x] = \{y \in A \mid x \equiv_F y\}$. Define the operations $\odot, \to$ and $\rightsquigarrow$ on A/F by $[x] \odot [y] = [x \odot y]$, $[x] \to [y] = [x \to y]$ and $[x] \rightsquigarrow [y] = [x \rightsquigarrow y]$. Therefore, $(A/F, \odot, \to, \rightsquigarrow, [1], [0])$ is a bounded pseudo-hoop with respect to F and $[x] \leq [y]$ if and only if $x \to y, x \rightsquigarrow y \in F$. (See [8])

Notation: It is easy to show that every obstinate filter of A is an n-fold obstinate filter of A and every 1-fold obstinate filter of A is an obstinate filter of A.

Theorem 3.1 *Let F be an 1-fold obstinate filter of A. Then A/F is a Boolean algebra.*

Proof. Let $x \in A$. Since F is an 1-fold obstinate filter, by Proposition 3.6(ii), $x \in F$ or x^-, $x^\sim \in F$. Then, $[x] = [1]$ or $[x^-] = [x^\sim] = [1]$. Hence, $[x] = [1]$ or $[(x^-)^\sim] = [0]$. If $[(x^-)^\sim] = [0]$, since $[x] \leq [(x^-)^\sim]$, then $[x] = [0]$. Therefore, A/F is a Boolean algebra.

Theorem 3.2 *F is an n-fold obstinate filter of A if and only if A/F is an n-fold obstinate pseudo-hoop.*

Proof. $(\Rightarrow)$ Let F be an n-fold obstinate filter and $x \notin F$. Then $x/F \neq 1/F$. By Proposition 3.6(ii), $(x^n)^- \in F$, thus $(x^n)^-/F = 1/F$. By Proposition 2.2(ii) and (iii), $x^n/F = 0/F$.
$(\Leftarrow)$ Let A/F be an n-fold obstinate pseudo-hoop and $x \notin F$. Then $x^n/F = 0/F$ and by Proposition 2.2(iii), $(x^n)^-/F = 1/F$. Hence $(x^n)^- \in F$.

By Proposition 3.6(ii), F is an n-fold obstinate filter of A.

Proposition 3.9 *Let F and G be two filters of A such that $F \subseteq G$. If F is an n-fold obstinate filter of A, then G is an n-fold obstinate filter, too.*

Proof. Let F and G be two filters of A such that $F \subseteq G$ and F be an n-fold obstinate filter of A. Suppose $x \notin G$. Then $x \notin F$. Since F is an n-fold obstinate filter, by Proposition 3.6(ii), $(x^n)^-$, $(x^n)^\sim \in F$. Hence $(x^n)^-$, $(x^n)^\sim \in G$ and G is an n-fold obstinate filter of A.

Proposition 3.10 *Let F be an n-fold obstinate filter of A. Then:*

(i) $(x \odot y)^- \in F$, implies $(x^n)^- \in F$ or $(y^n)^- \in F$.
(ii) $(x \odot y)^\sim \in F$, implies $(x^n)^\sim \in F$ or $(y^n)^\sim \in F$.

Proof. (i) Let F be an n-fold obstinate filter of A and $(x \odot y)^- \in F$. Since F is a proper filter, $x \odot y \notin F$. Then by (F2), $x \notin F$ or $y \notin F$. By Proposition 3.6(ii), $(x^n)^- \in F$ and $(y^n)^- \in F$.
(ii) The proof is similar to (i).

Lemma 3.1 (i) *Let $\varphi \in Hom(A, B)$ and G be an n-fold obstinate filter of B. Then the inverse image of G is an n-fold obstinate filter of A.*
(ii) *Let $\varphi : A \to B$ be a pseudo-hoop isomorphism and $F \in \mathcal{F}(A)$ be an n-fold obstinate filter. Then $\varphi(F)$ is an n-fold obstinate filter of B.*
(iii) *Let $\varphi : A \to B$ be a pseudo-hoop surjective and A be an n-fold obstinate pseudo-hoop. Then B is an n-fold obstinate pseudo-hoop.*

Proof. (i) Let G be an n-fold obstinate filter of B and $x \in A$ but $x \notin \varphi^{-1}(G)$. Then $\varphi(x) \notin G$, and so by Proposition 3.6(ii), $((\varphi(x))^n)^-$, $((\varphi(x))^n)^\sim \in G$. By Definition 2.7, we have $\varphi((x^n)^-)$, $\varphi((x^n)^\sim) \in G$. Then $(x^n)^-$, $(x^n)^\sim \in \varphi^{-1}(G)$. Therefore, $\varphi^{-1}(G)$ is an n-fold obstinate filter of A.
(ii) It is easy to see that, if $F \in \mathcal{F}(A)$, since φ is a pseudo-hoop isomorphism, then $\varphi(F) \in \mathcal{F}(B)$. Now, let $y_1, y_2 \notin \varphi(F)$. Then $\varphi^{-1}(y_1)$, $\varphi^{-1}(y_2) \notin F$. Since F is an n-fold obstinate filter, then $\varphi^{-1}((y_1)^n \to y_2) = (\varphi^{-1}(y_1))^n \to \varphi^{-1}(y_2) \in F$ and so $(y_1)^n \to y_2 \in \varphi(F)$. By the similar way, we can get that $(y_2)^n \to y_1$, $(y_1)^n \rightsquigarrow y_2$, $(y_2)^n \rightsquigarrow y_1 \in \varphi(F)$. Therefore, $\varphi(F)$ is an n-fold obstinate filter of B.
(iii) Let $y \in B$. Then there exists $x \in A$ such that $y = \varphi(x)$ and so $y^n = \varphi(x^n) = \varphi(0) = 0$. Therefore, B is an n-fold obstinate pseudo-hoop.

Theorem 3.3 *The following conditions are equivalent:*

(i) any filter $F \in \mathcal{F}(A)$ is an n-fold obstinate filter of A,
(ii) $\{1\}$ is an n-fold obstinate filter of A,
(iii) A is an n-fold obstinate pseudo-hoop.

Proof. $(i) \Rightarrow (ii)$ The proof is clear.
$(ii) \Rightarrow (i)$ By Proposition 3.9, the proof is clear.
$(ii) \Rightarrow (iii)$ Since $A \cong A/\{1\}$ and $\{1\}$ is an n-fold obstinate filter, then by Theorem 3.2 and Lemma 3.1(iii), A is an n-fold obstinate pseudo-hoop.
$(iii) \Rightarrow (ii)$ Let A be an n-fold obstinate pseudo-hoop and $1 \neq x \in A$. Since, $x^n = 0$, by Proposition 2.2(iii), $(x^n)^- = (x^n)^\sim = 1 \in \{1\}$. Then by Proposition 3.6(ii), $\{1\}$ is an n-fold obstinate filter of A.

Proposition 3.11 *Let F be an n-fold obstinate filter of A. Then the following conditions are hold:*

(i) $[F \cup G)$ is an n-fold obstinate filter of A, for any $G \in \mathcal{F}(A)$.
(ii) $F(x)$ is an n-fold obstinate filter of A, for all $x \in A$.

Proof. (i) Let $x \notin [F \cup G)$. Then $x \notin F$ and $x \notin G$. By Proposition 3.6(ii), $(x^n)^- \in F$. Thus $(x^n)^- \in [F \cup G)$. By Proposition 3.6(ii), $[F \cup G)$ is an n-fold obstinate filter of A.
(ii) We consider the following cases:
Case 1: If $x \in F$, then $F(x) = F$.
Case 2: If $x \notin F$ and $y \notin F(x)$, $y \neq x$, then $y \notin F$ and by Proposition 3.6(ii), $(y^n)^- \in F$. Hence $(y^n)^- \in F(x)$. By Proposition 3.6(ii), $F(x)$ is an n-fold obstinate filter of A.

4 Relation between n-fold filters in pseudo-hoops

In this section, we investigate the relationship between n-fold obstinate filters and other filters and

n-fold filters in pseudo-hoops.

Theorem 4.1 *Every n-fold obstinate filter of A is a maximal filter of A.*

Proof. Let F be an n-fold obstinate filter of A which is not a maximal filter of A. Then there exists a proper filter G of A such that $F \subsetneq G$. Let $x \in G \backslash F$. Since F is an n-fold obstinate filter, by Proposition 3.6(ii), $(x^n)^- \in F$. Since $(x^n)^- \in G$ and $x^n \in G$, by Proposition 2.2(v), $x^n \odot (x^n)^- = 0 \in G$, which is a contradiction. Therefore, F is a maximal filter.

The next example shows that the converse of Theorem 4.1, is not true, in general.

Example 4.1 *Let $(A = \{0, a, b, c, d, 1\}, \leq)$ be a poset. Define operations $\odot, \rightsquigarrow$ and $\rightarrow$ on A as follows,*

$\rightarrow$, $\rightsquigarrow$	0	a	b	c	d	1
0	1	1	1	1	1	1
a	c	1	b	c	b	1
b	d	a	1	b	a	1
c	a	a	1	1	a	1
d	b	1	1	b	1	1
1	0	a	b	c	d	1

$\odot$	0	a	b	c	d	1
0	0	0	0	0	0	0
a	0	a	d	0	d	a
b	0	d	c	c	0	b
c	0	0	c	c	0	c
d	0	d	0	0	0	d
1	0	a	b	c	d	1

By routine calculations, we can see that $(A, \odot, \rightarrow, \rightsquigarrow, 0, 1)$ is a bounded pseudo-hoop. It is clear that $F = \{1, a\}$ is a maximal filter but it is not an 1-fold obstinate filter. Because $b \notin F$ and $b^- = d \notin F$.

Corollary 4.1 *Every n-fold obstinate filter of pseudo $\vee$-hoop A is a prime filter of A.*

Proof. By Theorem 4.1 and Proposition 2.5, the proof is clear.

Proposition 4.1 *Any 1-fold obstinate filter F is a normal filter of A.*

Proof. Let F be an 1-fold obstinate filter and $x \rightarrow y \in F$. We consider the following cases:
Case 1: If $y \in F$, then by Proposition 2.1(iii), $y \leq x \rightsquigarrow y$. By (F1), $x \rightsquigarrow y \in F$.
Case 2: If $x, y \notin F$, then by assumption, $x \rightsquigarrow y \in F$.
Case 3: If $x \in F$, then by Proposition 2.1(v), $(x \rightarrow y) \odot x \leq y$. By (F1) and (F2), $y \in F$. Hence by Case 1, $x \rightsquigarrow y \in F$.
Therefore, F is a normal filter of A.

In the following example we show that the converse of Proposition 4.1, is not true, in general.

Example 4.2 *In Example 4.1, $F = \{1\}$, is a normal filter but it is not an n-fold obstinate filter. Because, $a^n \rightarrow b = b$ and $b^n \rightarrow a = a \notin F$.*

Theorem 4.2 *Let F be an n-fold obstinate filter of A. Then F is an n-fold implicative filter.*

Proof. Assume that F is not an n-fold implicative filter. Then there exist $x, y \in A$, such that $1 \rightarrow ((x^n \rightarrow y) \rightsquigarrow x) \in F$ but $x \notin F$. By Proposition 2.3(ii), $(x^n \rightarrow y) \rightsquigarrow x \in F$. We consider two cases:
Case 1: If $y \in F$, then since $y \leq x^n \rightarrow y$, so by (F1), $x^n \rightarrow y \in F$. By Proposition 2.3(iii), since $(x^n \rightarrow y) \rightsquigarrow x \in F$ and $x^n \rightarrow y \in F$, we get, $x \in F$, which is a contradiction.
Case 2: If $y \notin F$, then since F is an n-fold obstinate filter, $x^n \rightarrow y \in F$. By Proposition 2.3(iii), since $(x^n \rightarrow y) \rightsquigarrow x \in F$ and $x^n \rightarrow y \in F$, we get, $x \in F$, which is a contradiction.
Therefore, F is an n-fold implicative filter of A.

Lemma 4.1 *Any filter F of A is an n-fold positive implicative filter if and only if for all $x \in A$, $F_x = \{y \in A \mid x^n \rightarrow y$ and $x^n \rightsquigarrow y \in F\}$ is a filter of A.*

Proof. Let F be an n-fold positive implicative filter of A. Since $x^n \rightarrow 1 = 1 \in F$, we have $1 \in F_x$. Let $y, z \in A$ such that $y, y \rightarrow z \in F_x$. Then $x^n \rightsquigarrow y \in F$ and $x^n \rightarrow (y \rightarrow z) \in F$. Thus $x^n \rightarrow z \in F$, and so $z \in F_x$. Therefore, F_x is a filter of A.
Conversely, suppose F_x is a filter of A, for all $x \in A$. Let $x, y, z \in A$ such that $x^n \rightarrow (y \rightarrow z) \in F$ and $x^n \rightsquigarrow y \in F$. Then $y, y \rightarrow z \in F_x$. Thus $z \in F_x$, and so $x^n \rightarrow z \in F$. The proof of other cases is similar, too.

Theorem 4.3 *If F is a maximal and n-fold positive implicative filter of A, then F is an n-fold obstinate filter of A.*

Proof. Let F be a maximal and n-fold positive implicative filter of A and $x, y \in A\backslash F$. Then by Lemma 4.1, $F_x = \{b \in A \mid x^n \to b$ and $x^n \rightsquigarrow b \in F\}$ and $F_y = \{b \in A \mid y^n \to b,\ y^n \rightsquigarrow b \in F\}$ are filters of A .
Let $z \in F$. Then by Proposition 2.1(iii), $z \leq x^n \to z$ and by (F1), $x^n \to z \in F$. Thus, $z \in F_x$ and so $F \subseteq F_x$. On the other hand, $x^n \to x = 1 \in F$, so $x \in F_x$. By assumption, $x \notin F$. Hence $F \subsetneq F_x \subseteq A$. Since F is a maximal filter of A, $F_x = A$. Hence $y \in F_x$ or equivalently $x^n \to y \in F$. Similarly $x^n \rightsquigarrow y \in F$, $y^n \to x \in F$ and $y^n \rightsquigarrow x \in F$.

Proposition 4.2 [2] *Let F be a normal filter of A.*

(i) If for all $x \in A$, $x^n \to x^{2n} \in F$ or $x^n \rightsquigarrow x^{2n} \in F$, then F is an n-fold positive implicative filter of A.
(ii) If F is an n-fold implicative filter of A, then F is an n-fold fantastic filter of A.
(iii) $\{1\}$ is an n-fold fantastic filter, if and only if A is an n-fold fantastic pseudo-hoop.

Theorem 4.4 *Let A be a pseudo $\vee$-hoop. Then F is an n-fold obstinate filter if and only if F is a prime and n-fold implicative filter.*

Proof. If F is an n-fold obstinate filter, then by Corollary 4.1 and Theorem 4.2, the proof is clear.
Conversely, assume that F is a prime filter and n-fold implicative filter of A such that $x \in A\backslash F$. We show that $x \vee (x^n)^- \in F$ and $x \vee (x^n)^\sim \in F$, for all $x \in A$. Since F is an n-fold implicative filter, if $(x^n)^\sim \to x \in F$ then $x \in F$. Also $(x^n)^- \rightsquigarrow x \in F$ implies $x \in F$. Now, we must show that $t = x \vee (x^n)^- \in F$. Since $x \leq t$, we have $x^n \leq t^n$ and then by Proposition 2.2(i), $(t^n)^- \leq (x^n)^- \leq (x^n)^- \vee x = t$. So $(t^n)^- \rightsquigarrow t = 1 \in F$. Hence, we get that $t \in F$. The other case is similar. Thus $x \vee (x^n)^- \in F$. Since F is a prime filter and $x \notin F$, we have $(x^n)^- \in F$. Therefore, F is an n-fold obstinate filter of A.

Proposition 4.3 *Let F be a normal n-fold obstinate filter of A. Then:*
(i) F is an n-fold positive implicative filter,
(ii) F is an n-fold fantastic filter.

Proof. *(i)* We consider two cases:
Case 1: Let $x \in F$. Then by (F2), $x^{2n} \in F$ and by Proposition 2.1(iii), $x^{2n} \leq x^n \to x^{2n}$. By (F1), $x^n \to x^{2n} \in F$.
Case 2: Let $x \notin F$. Then by assume $(x^n)^- \in F$. By Proposition 2.2(vi), $(x^n)^- \leq x^n \to x^{2n}$ and by (F1), $x^n \to x^{2n} \in F$. Therefore, by Proposition 4.2(i), F is an n-fold positive implicative filter of A.
(ii) By Theorems 4.2 and 4.2(ii), F is an n-fold fantastic filter of A.

Theorem 4.5 *(i) If F is an n-fold fantastic filter of A, then $((x^n)^-)^\sim \to x \in F$ and $((x^n)^\sim)^- \to x \in F$.*
(ii) If $D_e(A) = \{x \in A \mid x^- = x^\sim = 0\} = A\backslash\{0\}$, then every n-fold fantastic filter is an n-fold obstinate filter of A.
(iii) Let F be an n-fold fantastic filter and for all $x, y \in A$, if $(x^n \odot y^n)^- \in F$, then $(x^n)^- \in F$ or $(y^n)^- \in F$. Also, $(x^n \odot y^n)^\sim \in F$ implies $(x^n)^\sim \in F$ or $(y^n)^\sim \in F$. Then F is an n-fold obstinate filter of A.

Proof. *(i)* Since $0 \to x = 0 \rightsquigarrow x = 1 \in F$ and F is an n-fold fantastic filter, then $((x^n)^-)^\sim \to x \in F$ and $((x^n)^\sim)^- \to x \in F$.
(ii) Let F be a proper n-fold fantastic filter of A. Then $0 \notin F$, and so $(x^n)^-$, $(x^n)^\sim \notin F$, for any $0 \neq x \in A$ and $n \geq 1$. By assumption and (i), $((x^n \to 0) \rightsquigarrow 0) \to x = (0 \rightsquigarrow 0) \to x = 1 \to x = x \in F$. Hence, by Proposition 3.6(ii), F is an n-fold obstinate filter.
(iii) Assume F is an n-fold fantastic filter of A such that $x \notin F$. It is enough to prove that $(x^n)^-, (x^n)^\sim \in F$. Let $(x^n)^- \notin F$, by the contrary. Then by Proposition 2.2(v), $(x^n \odot (x^n)^-)^\sim = 0^\sim = 1 \in F$. By assumption $((x^n)^-)^\sim \in F$. Since F is an n-fold fantastic filter, by (i), $((x^n)^-)^\sim \to x \in F$. By Proposition 2.3(ii), $x \in F$, which is a contradiction. Hence, $(x^n)^\sim \in F$. By the similar way, we get that $(x^n)^- \in F$. Therefore, F is an n-fold obstinate filter of A.

Proposition 4.4 *Let A be an n-fold fantastic pseudo-hoop and if for all $x, y \in A$, $x^n \odot y^n = 0$*

implies $x^n = 0$ or $y^n = 0$. Then A is an n-fold obstinate pseudo-hoop.

Proof. If A is an n-fold fantastic pseudo-hoop, then by Proposition 4.2(iii), $\{1\}$ is an n-fold fantastic filter of A. By hypothesis and Theorem 4.5(iii), $\{1\}$ is an n-fold obstinate filter of A and so by Theorem 3.3(iii), A is an n-fold obstinate pseudo-hoop.

Proposition 4.5 *Let A be a bounded simple pseudo-hoop. Then A is an n-fold obstinate pseudo-hoop, for some $n \in \mathbb{N}$.*

Proof. If $1 \neq x \in A$, then $[x) = A$ and so $0 \in [x)$. Hence for some $m \in \mathbb{N}$, $x^m = 0$. Let $n = max\{m \mid x \in A\}$. Then A is an n-fold obstinate pseudo-hoop.

Theorem 4.6 *Let F be an n-fold obstinate filter of A. Then A/F is a local and simple pseudo-hoop.*

Proof. Let F be an n-fold obstinate filter of A. Then by Theorem 4.1, F is a maximal filter of A and so A/F is a local and simple pseudo-hoop.

Notation: A partially ordered set $(P, \leq)$ is called to be of *the finite length* if the length of all chains in P are finite.

Theorem 4.7 *Let A be a pseudo-hoop of finite length. Then there exists $n \in \mathbb{N}$ such that every maximal filter of A is an n-fold obstinate filters of A.*

Proof. Let n be the length of the greatest chain in A. Then by Theorem 4.1, every n-fold obstinate filter of A is a maximal one. Now, let $F \in Max(A)$. Then, we show that F is an n-fold obstinate filter. Assume $x \notin F$. Since F is a maximal filter of A, by Proposition 2.4, then $(x^t)^- \in F$, for some $t \in \mathbb{N}$. If $t \leq n$, then by Proposition 2.1(v), $x^n \leq x^t$, so by Proposition 2.2(i), $(x^t)^- \leq (x^n)^-$. By (F1), $(x^n)^- \in F$. Let $n < t$. Since $0 \leq x^n \leq x^{n-1} \leq ... \leq x^2 \leq x \leq 1$ and A is finite length. Then by assumption, there is a $s \in \{1, 2, ..., n\}$ such that $x^s = x^{s+1}$, so $x^n = x^t$. It follows that $(x^n)^- \in F$. Therefore, F is an n-fold obstinate of A.

Theorem 4.8 *Let A be an n-fold obstinate pseudo-hoop. Then the following conditions are hold:*

(i) A is an n-fold fantastic pseudo-hoop,
(ii) A is an n-fold positive implicative pseudo-hoop,
(iii) A is an n-fold implicative pseudo-hoop,
(iv) A is a local pseudo-hoop,
(v) A is a simple pseudo-hoop.

Proof. (i) Let A be an n-fold obstinate pseudo-hoop. Then by Theorem 3.3(ii), $\{1\}$ is an n-fold obstinate filter of A. By Proposition 4.3(ii), $\{1\}$ is an n-fold fantastic filter of A. then by Proposition 4.2(iii), A is an n-fold fantastic pseudo-hoop.
(ii) Let A be an n-fold obstinate pseudo-hoop. Then $x^n = 0$, and so $x^{n+1} = x^n$. Hence, A is an n-fold positive implicative pseudo-hoop.
(iii) Let A be an n-fold obstinate pseudo-hoop. Then by Proposition 2.1(ii), $(x^n \to 0) \rightsquigarrow x = 1 \rightsquigarrow x = x$ and $(x^n \rightsquigarrow 0) \to x = 1 \to x = x$. Therefore, A is an n-fold implicative pseudo-hoop.
(iv) Since for any $1 \neq x \in A$, $x^n = 0$, then $ord(x) < \infty$. Hence, A is a local pseudo-hoop.
(v) Let A be an n-fold obstinate pseudo-hoop and $1 \neq x \in F$. Then by $(F2)$, $0 = x^n \in F$. Therefore, A is a simple pseudo-hoop.

In the following diagram, we show the relationship between n-fold obstinate filter and other filters of pseudo-hoop, where the condition $(*)$ is $x^n \odot y^n = 0 \Rightarrow x^n = 0$ or $y^n = 0$.

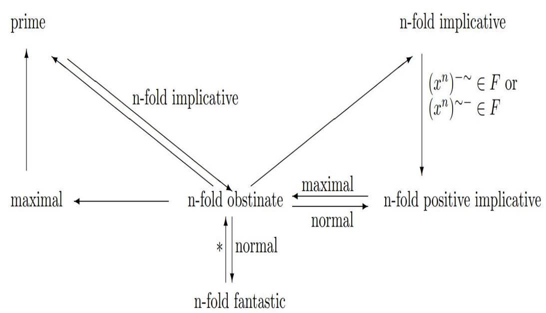

Figure 1: First-type nanostar dendrimer, $NS_1[2]$

5 Conclusion

In this paper, we have considered the folding theory of a filter which is a generalization of a filter in pseudo-hoop. We have provided conditions for a filter to be an n-fold obstinate filter of a pseudo-hoop. So we discuss on concept n-fold obstinate pseudo-hoops. Then we studied relationships between n-fold obstinate pseudo-hoops and some other special pseudo-hoops, such as simple pseudo-hoop and local pseudo-hoop. On the other hands, we introduced the notion of an n-fold obstinate filter in pseudo-hoop. Then we studied relationships between an n-fold obstinate filter and some other special n-fold filter, such as n-fold fantastic, n-fold positive implicative and n-fold implicative filter.

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