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Research Article

Computer Extended Series and HAM For the Solution of Non-Linear Squeezing Flow of Casson Fluid Between Parallel Plates

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Abstract

The paper presents analysis of two-dimensional non-Newtonian incompressible viscous flow between parallel plates. The governing problem of momentum equations are reduced to nonlinear ordinary differential equation (NODE) using similarity transformations. The resulting equation is solved using computer extended series solution (CESS) and homotopy analysis method (HAM). These methods have advantages over pure numerical methods for obtaining the derived quantities accurately for various values of parameters and results are valid in much larger parameter domain compared with numerical schemes.

Keywords : Squeezing flow; CESS; HAM; Casson fluid; Pade approximants.

1 Introduction

The two-dimensional incompressible viscous flow between two parallel plates moving normal to the surface symmetrically about line of symmetry, gives rise to squeezing flow is very important because of its many widespread industrial, biological and practical applications in many engineering fields. Squeezing flow between parallel infinite plates or discs is an interesting area of study in fluid mechanics; it has many applications including hydrodynamical machines, compression / injection modeling, flow inside syringes and nasogastric tubes etc. Stephan [33] was the first researcher who initiated the pioneering work on the squeezing flow under lubrication approach. Later Reynolds [31] and Architabald [2] investigated the squeezing flow between elliptic plate and rectangular plate respectively.

The fluids involved are not simply Newtonian in most of realistic models, and a single model cannot capture the complex rheological properties of non-Newtonian fluids. The different types of non-Newtonian fluids have been studied in different mathematical approaches; one of such model is Casson fluid model. McDonald [24] and Mrill et al. [26] showed that the most compatible formulation to simulate blood type fluid flows. Many researchers have contributed their investigation efforts towards the better understanding of squeezing flow in different geometries. Mohammad et al. [25] and Ganesh et al. [15] discussed the problem for unsteady flow of MHD viscous fluid between two parallel plates through porous medium. Kirubhashankar et al. [18, 19] studied MHD flow of an electrically conducting viscous fluid between moving parallel permeable plates. Siddiqui et al. [32] examined the homotopy perturbation solution for the MHD squeezing flow of

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a viscous fluid between two plates. Mustafa et al. [27] analyzed heat and mass transfer characteristics in a fluid for the squeezing flow. Rashidi et al. [30] considered the problem of unsteady two dimensional squeezing flows between circular plates. The MHD squeezing flow of viscous fluid has been studied by Umar Khan et al. [17] and the same problem was discussed by Naveed et al. [1] by invoking a magnetic field parameter to the problem. Ellahi et al. [12] discussed the non-Newtonian nanofluids with Reynolds model and Vogels model which are the illustrative models of variable viscosity. In pure numerical methods, a set of discrete points on a curve has been considered to obtain the solution of differential equations. Therefore it is often difficult and time consuming to get a complete path (curve) of results. On the other hand, at each point within the domain of interest approximate analytical solutions are available. The analytical solutions provide an effective initial guess to get approximate solution of nonlinear problem within few iterations. Many researchers are interested in obtaining semi-analytical and semi-numerical solutions of various nonlinear problems related to science and engineering fields. We reinvestigate the problem of non-linear squeezing flow of a Casson fluid between parallel plates using semi-analytical / semi-numerical methods viz. computer extended series solution and found some useful and interesting results based on new type of series analysis. Van Dyke [34] and his associates have shown the probable applications of these methods in computational fluid dynamics. For simple models the semi-analytical and semi-numerical methods proposed here is to provide accurate results and have advantages over pure numerical schemes. The few manually calculated approximations in low Reynolds number perturbation solution of the boundary value problem, which enables us to propose systematic series expansion to generate the universal polynomial coefficients by using recurrence relation and Mathematica. The resulting series have limited radius of convergence by non-physical singularities, is extended to moderately high Reynolds numbers using an analytic continuation of series solution. The location and nature of singularity restricts the convergence of series is predicted by using Domb-Sykes plot [10]. The sign pattern of the coefficients decides the nature of singularity, then recast the series into continued fraction representation [7] and use Pade' approximants of various orders for summing it. Bujurke et al. [8, 9] studied the flow in a narrow channel of varying gap using computer extended series solution and his associates have used this technique successfully.

The alternative approach is to obtain analytic solution of the proposed problem using homotopy analysis method (HAM). The HAM was developed by Liao [20] and further modified it in [21] to introduce a non-zero auxiliary parameter which is known as convergence-control parameter, which allows us to adjust the convergence region and rate of approximations of required solution. In general perturbation and asymptotic techniques are strongly dependent upon small / large parameters which are often valid for solving weakly nonlinear boundary value problems. This method is free from small / large physical parameter, flexibility on choice of base function and initial guess. Also, HAM is used to solve highly nonlinear differential equations arising in engineering applications, which has great flexibility and generality over all other analytical techniques. Recently, Ellahi [13, 14] discussed various aspects related to flow of non-Newtonian nanofluid and investigated the effects of MHD and temperature dependent viscosity by using HAM. Most recently, Awati et al. [4, 5, 6] used CESS and HAM for the solution of flow problems arising in fluid mechanics and shown potential applications of these methods.

The present paper is structured as follows. Section 1 describes the introduction of the problem. Section 2 Mathematical formulation of the proposed problem and relevant boundary conditions is given. Section 3 is devoted to the solution of the problem using CESS. Section 4 devoted to the solution by HAM. Section 5 presents the results and discussion, and Section 6 is about the conclusion.

2 Mathematical Formulation

Consider a two-dimensional viscous incompressible Casson fluid over an exponentially stretching / shrinking porous sheet with the flow being confined. For an isotropic and incompressible Casson fluid the rheological equation of state given

[11, 28] as

$$\tau_{ij} = \begin{cases} (\mu_B + \frac{p_y}{\sqrt{2\pi}})2e_{ij}, & \pi > \pi_c, \\ (\mu_B + \frac{p_y}{\sqrt{2\pi}})2e_{ij}, & \pi < \pi_c. \end{cases}$$

where τ_{ij} is the (i,j)-th component of stress tensor, $\pi = e_{ij}e_{ij}$ and e_{ij} are the (i,j)-th component of deformation rate, π is the product component of deformation rate with itself, π_c is critical value of product based on non-Newtonian model, μ_B is dynamic viscosity of non-Newtonian fluid and p_y is the yield stress of fluid. The governing equations of present flow problem become [24]

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0, \tag{2.1}$$

$$u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} = \left(1 + \frac{1}{\beta}\right)\left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2}\right), \tag{2.2}$$

$$u\frac{\partial v}{\partial x} + v\frac{\partial v}{\partial y} = -\frac{1}{\rho}\frac{\partial p}{\partial y} + \nu\left(1 + \frac{1}{\beta}\right)\left(\frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 v}{\partial y^2}\right) \tag{2.3}$$

where u and v are the velocity components along x and y directions respectively, ν is the kinematic fluid viscosity, p be the pressure, ρ is the density of fluid, $\beta = \mu_B\sqrt{\frac{2\pi_c}{p_y}}$ is the Casson fluid parameter. The appropriate boundary conditions for the flow equations are

$$\begin{array}{lll} u(x,y) = 0, v(x,y) = v_0 & \text{at} & y = h, \\ u_y(x,y) = 0, v(x,y) = 0 & \text{at} & y = 0. \end{array} \tag{2.4}$$

There is no slip condition at upper plate for the first two conditions and remaining two conditions follow from symmetry of the flow at $y = 0$. Eliminating the pressure terms from Eqn. (2.2) and (2.3), after cross differentiation and invoking vorticity ω then simplify the above system of equations, we get

$$u\frac{\partial \omega}{\partial x} + v\frac{\partial \omega}{\partial y} = \nu\left(1 + \frac{1}{\beta}\right)\left(\frac{\partial^2 \omega}{\partial x^2} + \frac{\partial^2 \omega}{\partial y^2}\right) \tag{2.5}$$

where $\omega = \frac{\partial v}{\partial x} - \frac{\partial u}{\partial y}$. Introducing the dimensionless similarity variable $\eta = \frac{y}{h}$, where h is the distance between plates, then Eqns. (2.1) and (2.5) becomes

$$\frac{\partial u}{\partial x} + \frac{1}{h}\frac{\partial v}{\partial \eta}, \tag{2.6}$$

$$u\frac{\partial \omega}{\partial x} + \frac{v}{h}\frac{\partial \omega}{\partial \eta} = \nu\left(1 + \frac{1}{\beta}\right)\left(\frac{\partial^2 \omega}{\partial x^2} + \frac{1}{h^2}\frac{\partial^2 \omega}{\partial \eta^2}\right) \tag{2.7}$$

and the boundary conditions are

$$\begin{array}{lll} u(x,\eta) = 0, v(x,\eta) = v_0 & \text{at} & \eta = 1, \\ u_y(x,\eta) = 0, v(x,\eta) = 0 & \text{at} & \eta = 0. \end{array} \tag{2.8}$$

Let ψ be a stream function is of the form

$$\psi(x,\eta) = [hU(0) - v_0 x]\, f(\eta) \tag{2.9}$$

The velocity components u and v related to the physical stream function defined by

$$u(x,\eta) = \frac{\partial \psi}{\partial y} = \frac{1}{h}\frac{\partial \psi}{\partial \eta} \quad \text{and} \quad v(x,\eta) = -\frac{\partial \psi}{\partial x}, \tag{2.10}$$

where $U(0)$ is the average entrance velocity at $x = 0$ and velocity components becomes

$$u(x,\eta) = \left[U(0) - \frac{v_0}{h}x\right] f'(\eta) \quad \text{and} \quad v(x,\eta) = v_0 f(\eta) \tag{2.11}$$

Using Eqns. (2.11) the continuity equation satisfied automatically and the Eqns. (2.7) and (2.8) reduces to the nonlinear ordinary differential equation

$$f'''' + R\left(\frac{\beta}{1+\beta}\right)\left(f'f'' - ff'''\right), \tag{2.12}$$

where $R = \frac{v_0 h}{\nu}$ is Reynolds number. The relevant boundary conditions becomes

$$f(0) = 0, f'(1) = 0, f''(0) = 0, f(1) = 0. \tag{2.13}$$

3 Series Solution

We seek a solution of Eqn. (2.12) in powers of R in the form

$$f(\eta) = \sum_{n=0}^{\infty} R^n f_n(\eta) \tag{3.14}$$

Substituting Eqn. (3.14) into Eqn. (2.12) and equating like powers of R on both sides, we get

$$f_0'''' = 0, \tag{3.15}$$

$$f_{n+1}'''' = -(\frac{\beta}{1+\beta})[f_0' f_n'' + f_n' f_0'' - f_0 f_n''' - f_n f_0'''] - \sum_{L=1}^{n-1}(\frac{\beta}{1+\beta})[f_L' f_m'' - f_L f_m'''], \quad n = 1, 2, ... \tag{3.16}$$

where $m = n - L$. The boundary conditions of the flow equations are

$$f_0(0) = 0, f_0'(1) = 0, f_0''(0) = 0, f_0(1) = 0,$$
$$f_n(0) = 0, f_n'(1) = 0, f_n''(0) = 0, f_n(1) = 0, \quad n = 1, 2, ... \quad (3.17)$$

The required solutions of the above equations up to $O(R^2)$ are

$$\begin{aligned} f_0 &= \frac{1}{2}(3\eta - \eta^3) \\ f_1 &= -\frac{1}{280}\left(\frac{\beta}{1+\beta}\right)(2\eta - 3\eta^2 + \eta^7) \\ f_2 &= \left(\frac{\beta}{1+\beta}\right)(-\frac{703}{1293600}\eta + \frac{73}{107800}\eta^3 \\ &+ \frac{3}{19600}\eta^7 - \frac{1}{3360}\eta^9 + \frac{73}{92400}\eta^{11}) \end{aligned} \quad (3.18)$$

The calculation of higher order terms manually of the series (3.16) is very difficult because the algebra becomes cumbersome. It is essential to get the higher order terms to approximate the function properly and we cannot analyze the problem accurately by considering only these three terms of the series. The behavior of the solution (3.18) enables us to propose a systematic series expansion with universal polynomial coefficients, which is quite useful and more efficient in calculation of higher order approximations. The nature of solutions (3.18) suggest the general form $f_n(\eta)$ to be of the form

$$f_n(\eta) = \sum_{k=1}^{2n} A_{(n,2k-1)}(1-\eta^2)^2\eta^{2k-1}, n \geq 1. \quad (3.19)$$

The above general form yields exactly the previous calculated terms $f_i (i = 1, 2)$, besides this it enables us to find for $(i > 2)$ using the following recurrence relation and FORTRAN programming. Substituting Eqn.(3.19) into an Eqn.(3.14) and equating like powers of η on both sides and obtained a recurrence relation for unknowns $A_{(n,2k-1)}$ in the form

$$\begin{aligned} A_{(n+1,4n-(2J+1))} &= 2A_{(n+1,4n-(2J-1))} \\ &- A_{(n+1,4n-(2J-3))} + \\ &\frac{1}{(4n-(2J-3))(4n-(2J-2))(4n-(2J-1))(4n-(2J))} \\ &\times \{[\sum_{i=1}^{4} A_{(4n-2i-2J+3)}P_i(4n-i-J+2) \\ &+ \sum_{L=1}^{n-1}[\sum_{r=-2}^{2}(\sum_{k_1=2L-J+r}^{2L} A_{(L,2k_1-1)} \\ &\cdot A_{(m,4n-2k_1-(2J+(1-2r)))} \\ &\cdot S_{(7-r)}(k_1, N_2 - k_1 - (J-r))))]]\} \end{aligned} \quad (3.20)$$

where $m = n - L$ and J varies from $-2, -1, 0, 1, ...(2n - 1)$.
For obtaining other A_{ij}'s, we use above recurrence relation. The expression for radial velocity is given by

$$f'(\eta) = \frac{3}{2}(1-\eta^2) + \sum_{n=0}^{\infty} R^n \sum_{k=1}^{2n} A_{(n,2k-1)} [(2k+3)\eta^{2k+2} - 2(2k+1)\eta^{2k} + (2k-1)\eta^{2k-2}] \quad (3.21)$$

The expression for pressure gradient of the series is given by

$$f'''(0) = -3 + \sum_{n=0}^{\infty} R^n a_n \quad (3.22)$$

where $a_n = -6A_{(n,1)} + 6A_{(n,3)}$. The analytic continuation of region and validity of series can be achieved by taking various Pade' approximants. The coefficients of the series (3.21) and (3.22) representing the radial velocity $f'(\eta)$ and pressure gradient $f'''(0)$ are decreasing in magnitude but having random sign pattern and results are further extrapolated using rational approximation for determining the radius of convergence. Fig.1 shows Domb-Sykes plot which confirms the radius of convergence after extrapolation of the series $f'''(0)$ to be R_0= 26.32272 and 15.78283 for Casson fluid parameter $\beta = 1$ and 5 respectively. The direct sums of the series for radial and axial velocities are valid only up to the radius of convergence. We use pade approximants for summing the series which gives a converging sum for sufficiently large Reynolds number R.

Figs. 8-10 show the effects of Casson fluid

Table 1: Comparison of $f'''(0)$ results with CESS, HAM and numerical results for different Casson fluid parameter β and Reynolds number R.

		$f'''(0)$		
β	R	CESS	HAM	NDSolve
1	0	-3.00000	-3.00000	-3.00000
	5	-2.81135	-2.81135	-2.81135
	10	-2.55884	-2.55884	-2.55884
	20	-1.87965	-1.87966	-1.87966
	25	-1.50986	-1.50986	-1.50986
	30	-1.16547	-1.16542	- - - -
	35	-0.86244	-0.86221	- - - -
	40	-0.5908	-0.59074	- - - -
	45	-0.31877	-0.3162	- - - -
2	5	-2.7347	-2.73471	-2.73471
	10	-2.35364	-2.35365	-2.35365
	20	-1.39104	-1.3911	-1.3911
	30	-0.5819	-0.59074	- - - -
	35	-0.15008	-0.21509	- - - -
3	5	-2.69358	-2.69358	-2.69358
	15	-1.69336	-1.69337	-1.69337
	25	-0.72517	-0.72414	- - - -
	30	-0.30432	-0.3162	- - - -
5	5	-2.65056	-2.65057	-2.65057
	10	-2.12414	-2.12414	-2.12414
	15	-1.50984	-1.50986	-1.50986
	25	-0.49493	-0.50192	- - - -

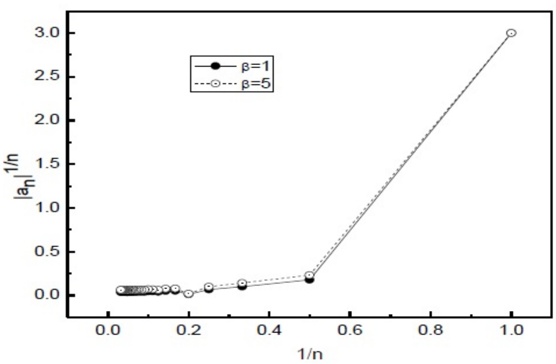

Figure 1: Domb-Sykes plot for the series $f'''(0)$ for $\beta = 1$ and 5.

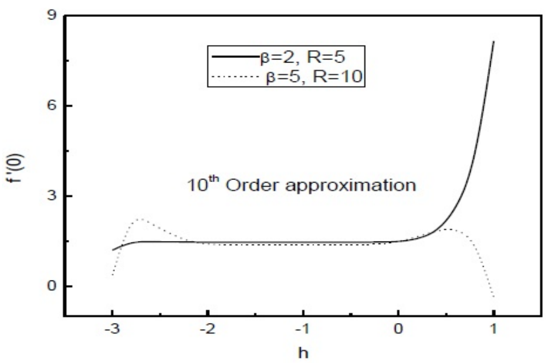

Figure 2: $\hbar$ curves for the series $f'(0)$ for different values of β and R.

parameter on radial velocity for moderately large Reynolds number. An identical behavior is observed for almost all velocity profiles for Casson fluid parameter and Reynolds number.

4 Homotopy Analysis Method

We employ HAM [22, 23] for the solution of the flow problem (2.12) with boundary conditions (2.13), it is straight forward to choose the initial approximation as

$$f_0(\eta) = \frac{3}{2}\eta - \frac{1}{2}\eta^3 \quad (4.23)$$

and auxiliary linear operator of the governing equation is defined as

$$L[f] = f'''' \quad (4.24)$$

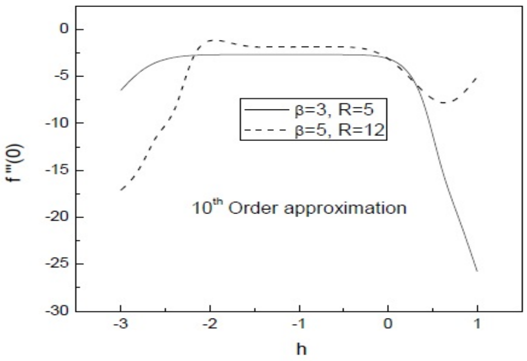

Figure 3: $\hbar$ curve for the series $f'''(0)$ for different values of β and R.

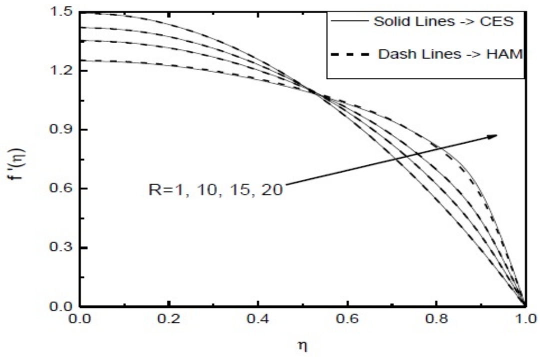

Figure 5: Radial velocity curves for different values of R when $\beta = 2$.

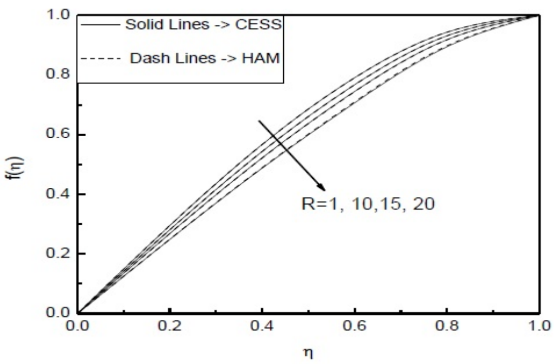

Figure 4: Axial velocity curves for different values of R when $\beta = 2$.

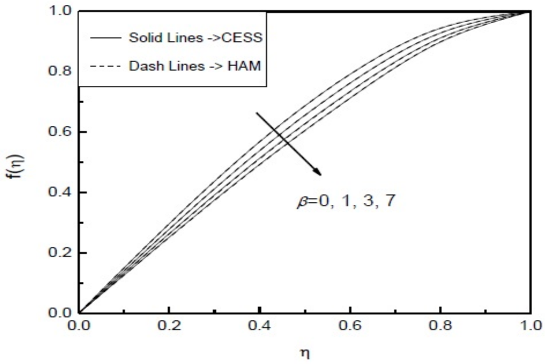

Figure 6: Axial velocity curves for different values of β when $R = 15$.

The above linear operator satisfies the following property

$$L[C_1\eta^3 + C_2\eta^2 + C_3\eta + C_4] = 0 \qquad (4.25)$$

where C_1, C_2, C_3 and C_4 are constants to be determined later. If $q \in [0,1]$ then the zeroth order deformation problem can be constructed as

$$(1-q)L[f(\eta,q) - f_0(\eta)] = qhH(\eta)N[f(\eta,q)], \qquad (4.26)$$

subjected to the boundary conditions

$$f(0,q) = 0, f'(1,q) = 0, \quad f''(0,q) = 0, f(1,q) = 1. \qquad (4.27)$$

where $q \in [0,1]$ is an embedding parameter, $\hbar$ and H are the non-zero auxiliary parameter and auxiliary function respectively. Further N is the nonlinear differential operator given by

$$N[f(\eta,q)] = \frac{\partial^4 f(\eta,q)}{\partial\eta^4} + R\left(\frac{\beta}{1+\beta}\right) \left[\frac{\partial f(\eta,q)}{\partial\eta}\frac{\partial^2 f(\eta,q)}{\partial\eta^2} + f\frac{\partial^3 f(\eta,q)}{\partial\eta^3}\right] \qquad (4.28)$$

For $q = 0$ and $q = 1$, Eqn. (4.26) have the solution

$$f(\eta,0) = f_0(\eta), f(\eta,1) = f(\eta) \qquad (4.29)$$

As q varies from 0 to 1, $f(\eta,q)$ also varies from the initial guess $f_0(\eta)$ to the exact (final) solution $f(\eta)$. By Taylor's theorem Eqn. (4.29) can be written as

$$f(\eta,q) = f_0(\eta) + \sum_{m=1}^{\infty} f_m(\eta)q^m \qquad (4.30)$$

where $f_m(\eta) = \frac{1}{m!}\frac{\partial^m f(\eta,q)}{\partial q^m}|_{q=0}$. The convergence of above series strictly depends upon the convergence control parameter $\hbar$, and also assume that $\hbar$ is selected in such a way that the series is convergent at $q = 1$, then we have

$$f(\eta) = f_0(\eta) + \sum_{m=1}^{\infty} f_m(\eta) \qquad (4.31)$$

Differentiating zeroth order deformation problem (4.26) $'m'$ times with respect to q and then dividing it by $m!$, finally setting $q = 0$. The resulting

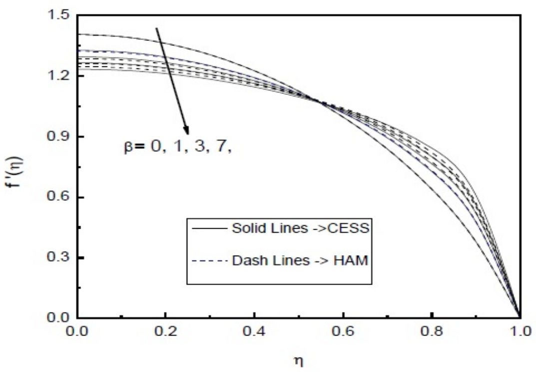

Figure 7: Axial velocity curves for different values of β when $R = 15$.

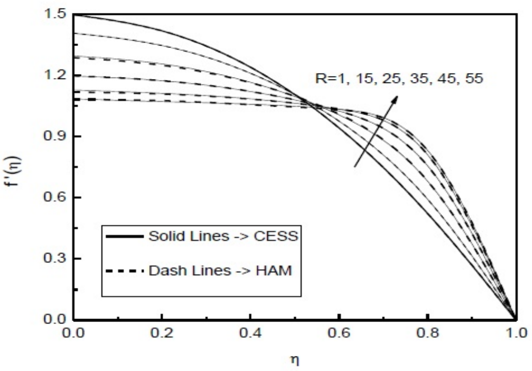

Figure 8: Velocity profiles for moderately larger values of R when $\beta = 1$.

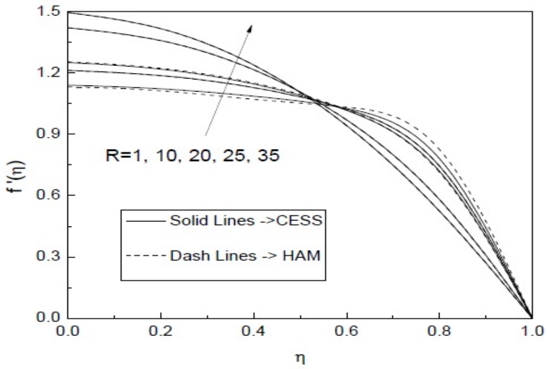

Figure 9: Velocity curves for moderately larger values of R when $\beta = 2$.

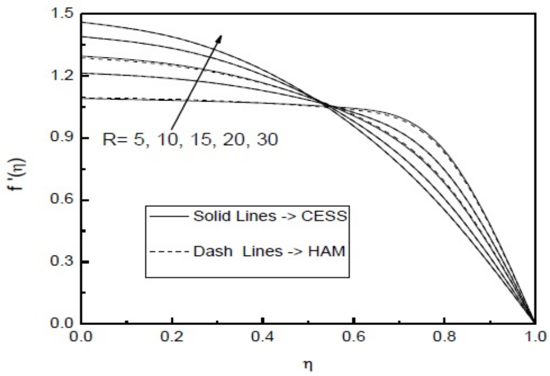

Figure 10: Velocity curves for moderately larger values of R when $\beta = 5$.

mth-order deformation problem becomes

$$L[f_m(\eta) - \chi_m f_{m-1}(\eta)] = qHR_m(\eta) \quad (4.32)$$

and the homogeneous boundary conditions are

$$f_m(0) = 0, f_m^{'}(1) = 0, f_m^{''}(0) = 0, \\ f_m^{'}(1) = 0, m \geq 1. \quad (4.33)$$

where

$$R_m(\eta) = f_m^{''''} + R\left(\frac{\beta}{1+\beta}\right) \sum_{n=0}^{m-1}[f_n^{'}f_{m-n-1}^{''} - f_n f_{m-n-1}^{'''}] \quad (4.34)$$

and

$$\chi_m = \begin{cases} 0, & \text{if } m \leq 1, \\ 1, & \text{if } m > 1. \end{cases} \quad (4.35)$$

To solve the system of linear equations (4.32) with homogeneous boundary conditions (4.33) by using Mathematica software, we obtain solutions as

$$\begin{aligned} f_1(\eta) =& \left(\frac{\beta\hbar R}{(1+\beta)}\right)\left(\frac{1}{140}\eta - \frac{3}{280}\eta^3 + \frac{1}{280}\eta^7\right) \\ f_2(\eta) =& \Big(\frac{\beta\hbar R}{140(1+\beta)} + \frac{\beta\hbar^2 R}{140(1+\beta)^2} \\ &+ \frac{\beta^2\hbar^2 R}{140(1+\beta)^2} + \frac{703\beta^2\hbar^2 R^2}{1293600(1+\beta)^2}\Big)\eta \\ &+ \Big(-\frac{3\beta\hbar R}{280(1+\beta)} - \frac{3\beta\hbar^2 R}{280(1+\beta)^2} \\ &- \frac{3\beta^2\hbar^2 R}{280(1+\beta)^2} + \frac{73\beta^2\hbar^2 R^2}{107800(1+\beta)^2}\Big)\eta^3 \\ &+ \Big(\frac{\beta\hbar R}{280(1+\beta)} + \frac{\beta\hbar^2 R}{280(1+\beta)^2} \\ &+ \frac{\beta^2\hbar^2 R}{280(1+\beta)^2} + \frac{3\beta^2\hbar^2 R^2}{19600(1+\beta)^2}\Big)\eta^7 \\ &- \frac{\beta^2\hbar^2 R^2}{3360(1+\beta)^2}\eta^9 + \frac{3\beta^2\hbar^2 R^2}{92400(1+\beta)^2}\eta^{11} \end{aligned} \quad (4.36)$$

4.1 Convergence of HAM

The series (4.31) contains the auxiliary parameter $\hbar$. The convergence of series strictly depends on the parameter $\hbar$ and is known as convergence control parameter, which influences the convergence rate and region of series. We draw $\hbar$ curves for velocity and pressure gradient at 10th order of approximations. To ensure that the permissible ranges of parameter $\hbar$, by drawing the line segment of $\hbar$ curves parallel to η-axis. From Fig. 2 and Fig. 3, it is observed that admissible ranges for $\hbar$ are $-2 \le \hbar \le 1.7$ and $-1.5 \le \hbar \le 1.7$ for the series $f'(0)$ and $f'''(0)$ for different values of Casson fluid parameter β and Reynolds number R. The computation shows that series converges in the whole region of $0 \le \eta \le 1$ when $\hbar = -0.9$.

5 Results and discussion

The nonlinear squeezing flow of an incompressible Casson fluid between two parallel plates is analyzed. The resulting nonlinear ordinary differential Eqn. (2.12) with boundary conditions (2.13) is solved by using CESS and HAM. By using both the methods the graphs of axial and radial velocity profiles have been drawn for different Reynolds number R and Casson fluid parameter β. The series expansion scheme with polynomial coefficients proposed here enables in obtaining recurrence relation. Using recurrence relation (3.20) and Mathematica, we generate large number ($n = 30$) of universal coefficients $A_{(n,2k-1)}, k = 1, 2, ...2n$ and $n = 1, 2, ...30$. The velocity profiles are further improved by using Pade approximants for much larger values of R for different values of Casson parameter β which are shown in Figs. 8-10 and the profiles agree closely with HAM curves.

Fig. 4 shows that an increasing value of R corresponds to a decrease in the velocity along the y-direction for fixed Casson parameter β. Fig. 5 presents radial velocity profiles for increasing values of R corresponds to a decrease in velocity in the region $\eta \le 0.5$ and increase in the region $0.5 \le \eta \le 1$. Fig. 6 depicts the behavior of Casson fluid parameter β, it is observed that the axial velocity decreases corresponds to increase in the values of β.

Fig. 7 shows that the radial velocity for increase in β corresponds to decrease in velocity in the region $\eta \le 0.5$ and increase in the region $0.5 \le \eta \le 1$.

The coefficient of the series (3.22) representing pressure gradient $f'''(0)$ are decreasing in magnitude and having random sign patterns. Fig.1 shows Domb-Sykes plot which confirms the radius of convergence after extrapolation of series $f'''(0)$ to be $R_0 = 26.32272$ and 15.78283 for Casson fluid parameter $\beta = 1$ and $\beta = 5$ respectively. The direct sum of the series (3.22) is valid only up to the radius of convergence for different values of Casson fluid parameter β. We use the Pade' approximant's for summing up series which give a converging sum for sufficiently large Reynolds number. The pressure gradient results are compared between computer extended series solution and homotopy analysis method with numerical results, which agree very well with pure numerical solutions for different Casson fluid parameter β and Reynolds number R. The results are given in Table 1.

6 Conclusion

In the present analysis, the non-linear squeezing flow of an incompressible Casson fluid between parallel plates is studied using CESS and HAM. The major observations are:

- The location and nature of singularity restricts the convergence of series is predicted quite accurately using Domb-Sykes plot.
- The region of validity of the series is extended for much larger value of Reynolds number R.
- From the $\hbar$ curves it is observed that 10th order approximations are enough to obtain the solutions of flow problem.
- Table 1 presents the validity of semi-numerical / semi-analytical methods results with numerical results.

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