Available online at http://ijim.srbiau.ac.ir/
Int. J. Industrial Mathematics (ISSN 2008-5621)
Vol. 11, No. 2, 2019 Article ID IJIM-0884, 12 pages
Research Article

Application of Fuzzy Bicubic Splines Interpolation for Solving Two-Dimensional Linear Fuzzy Fredholm Integral Equations

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Received Date: 2018-01-30 Revised Date: 2018-10-02 Accepted Date: 2018-12-07

Abstract

In this paper, firstly, we review approximation of fuzzy functions by fuzzy bicubic splines interpolation and present a new approach based on the two-dimensional fuzzy splines interpolation and iterative method to approximate the solution of two-dimensional linear fuzzy Fredholm integral equation (2DLFFIE). Also, we prove convergence analysis and numerical stability analysis for the proposed numerical algorithm. Finally, by an example, we show the efficiency of the proposed method.

Keywords : Two-dimensional linear fuzzy Fredholm integral equations; Two dimensional fuzzy splines interpolation; Iterative method; Fuzzy numbers.

1 Introduction

Recently many authors proposed various numerical methods for solving one dimensional fuzzy integral equations [2, 3, 4, 5, 6, 13, 14, 16, 17, 18, 21, 22, 24, 25, 28, 39, 40, 42]. Also, two-dimensional fuzzy integral equations have been noticed by a lot of researchers because of their broad applications in engineering science. Some of the most important papers in this area are: trapezoidal quadrature rule and iterative method [7, 41, 42], triangular functions [23], quadrature iterative [26], Bernstein polynomials [12, 15], collocation fuzzy wavelet [20], homotopy analysis method (HAM) [35], open fuzzy cubature rule [8], kernel iterative method [29], modified homotopy pertubation [30], block-pulse functions [31], optimal fuzzy quadrature formula [34], hybrid of Block-Pulse functions and Bernstein polynomials [41] and finally, iterative method and fuzzy bivariate block-pulse functions [37]. Furthermore, some researchers have solved one-dimensional fuzzy Fredholm integral equations by using fuzzy interpolation via iterative method such as: iterative interpolation method [28], Lagrange interpolation base on the extension principle [18], spline interpolation [22]. Also, in 2018, Nouriani and Ezzati [27] solved two-dimensional linear fuzzy integral equation by using the fuzzy Lagrange interpolation.

As we know interpolation is one of the most substantial and the most applicable methods in numerical analysis. So, in this study, we want to solve 2DLFFIEs by applying two-dimensional fuzzy splines interpolation and iterative method. First of all an approximate solution of integral by applying splines interpolation and iterative method is provided. Then, convergence analysis and numerical stability analysis of the proposed method in theorems 4.1 and 5.1 is proved.

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The paper is organized as follows: Some notations and theorems about the structure of fuzzy sets are reviewed in Section 2. In Section 3, at first, we review two-dimensional fuzzy splines interpolation. Also, we present two-dimensional fuzzy splines interpolation and iterative method for solving 2DLFFIEs. Also, in Section 4, we verify convergence analysis for proposed method. In Section 5, we prove numerical stability analysis for the method. One numerical example is presented in Section 6.

2 Preliminaries

At first, we review some basic definitions and necessary results about fuzzy set theory.

Definition 2.1 ([1]) *A fuzzy number is a function $f:\mathbb{R}\to[0,1]$ with the following properties:*

1. $\exists\, x_0\in\mathbb{R}$ *such that* $f(x_0)=1$.
2. $f(\eta x+(1-\eta)y)\geq\min\{f(x),f(y)\},\ \forall\, x,y\in\mathbb{R},\forall\,\eta\in[0,1]$.
3. $\forall\, x_0\in\mathbb{R}$ *and* $\forall\,\epsilon>0$, $\exists$ *neighborhood* $U(x_0)$: $f(x)\leq f(x_0)+\epsilon,\ \forall\, x\in U(x_0)$.
4. *In* $\mathbb{R}$, *the set* $\overline{supp(f)}$ *is compact.*

The set of all fuzzy numbers is denoted by $\mathbb{R}_F$.

Definition 2.2 ([1]) *For* $f\in\mathbb{R}_F$ *and* $0<\alpha\leq 1$, *define* $[f]^0:=\overline{\{x\in\mathbb{R}:f(x)>0\}}$ *and*

$$[f]^\alpha:=\{x\in\mathbb{R}:f(x)\geq\alpha\}.$$

Then it is well known that for each $\alpha\in[0,1]$, $[f]^\alpha$ *is a bounded and closed interval of* $\mathbb{R}$. *We define uniquely the sum* $f\oplus g$ *and the product* $\mu\odot f$ *for* $f,\ g\in\mathbb{R}_F$ *and* $\mu\in\mathbb{R}$ *by*

$$[f\oplus g]^\alpha=[f]^\alpha+[g]^\alpha,$$
$$[\mu\odot f]^\alpha=\mu[f]^\alpha,\ \forall\alpha\in[0,1],$$

where $[f]^\alpha+[g]^\alpha$ means the usual addition of two intervals (as subsets of $\mathbb{R}$) and $\mu[f]^\alpha$ means the usual product between a scalar and a subset of $\mathbb{R}$. Notice $1\odot f=f$ and it holds $f\oplus g=g\oplus f$, $\mu\odot f=f\odot\mu$. If $0\leq\alpha_1\leq\alpha_2\leq 1$ then $[f]^{\alpha_2}\subseteq[f]^{\alpha_1}$. Actually $[f]^\alpha=[f_-^{(\alpha)},f_+^{(\alpha)}]$, where $f_-^{(\alpha)}\leq f_+^{(\alpha)}$, $f_-^{(\alpha)},\ f_+^{(\alpha)}\in\mathbb{R}$, $\forall\alpha\in[0,1]$. For $\mu>0$ one has $\mu f_\pm^{(\alpha)}=(\mu\odot f)_\pm^{(\alpha)}$, respectively.

Definition 2.3 ([1]) *Define* $D:\mathbb{R}_F\times\mathbb{R}_F\to\mathbb{R}_+$ *by*

$$\begin{aligned}D(f,g)&\\ &=\sup_{\alpha\in[0,1]}\max\left\{\left|f_-^{(\alpha)}-g_-^{(\alpha)}\right|,\left|f_+^{(\alpha)}-g_+^{(\alpha)}\right|\right\}\\ &=\sup_{\alpha\in[0,1]}\ \textit{Hausdorff distance}\ ([f]^\alpha,[g]^\alpha),\end{aligned}$$

where $[g]^\alpha=[g_-^{(\alpha)},g_+^{(\alpha)}]$; $f,\ g\in\mathbb{R}_F$. Clearly D is a metric on $\mathbb{R}_F$. Also $(\mathbb{R}_F,D)$ is a complete metric space, with the following properties [1]:

$$\begin{aligned}D(f\oplus h,g\oplus h)&=D(f,g),\ \forall f,g,h\in\mathbb{R}_F,\\ D(k'\odot f,k'\odot g)&=|k'|\,D(f,g),\\ &\quad\forall f,g\in\mathbb{R}_F,\forall k'\in\mathbb{R},\\ D(f\oplus g,h\oplus e)&\leq D(f,h)+D(g,e),\\ &\quad\forall f,g,h,e\in\mathbb{R}_F.\end{aligned}$$

Definition 2.4 ([1]) *Suppose* $f,g:\mathbb{R}\to\mathbb{R}_F$ *be fuzzy number valued functions, then the distance between* $f,\ g$ *is defined by*

$$D^*(f,g):=\sup_{x\in\mathbb{R}}D(f(x),g(x)).$$

Lemma 2.1 ([1])

1. *If we denote* $\tilde{0}:=\chi_{\{0\}}$, *then* $\forall f\in\mathbb{R}_F$, $f\oplus\tilde{0}=\tilde{0}\oplus f=f$.
2. *With respect to* $\tilde{0}$, *none of* $f\in\mathbb{R}_F$, $f\neq\tilde{0}$ *has opposite in* $\mathbb{R}_F$.
3. *Let* $\alpha,\ \beta\in\mathbb{R}:\alpha.\beta\geq 0$, *and any* $f\in\mathbb{R}_F$, *we have* $(\alpha+\beta)\odot f=\alpha\odot f\oplus\beta\odot f$. *Notice that for general* $\alpha,\ \beta\in\mathbb{R}$, *the above property is false.*
4. *For any* $\gamma\in\mathbb{R}$ *and any* $f,\ g\in\mathbb{R}_F$, *we have* $\gamma\odot(f\oplus g)=\gamma\odot f\oplus\gamma\odot g$.
5. *For any* $\gamma,\ \eta\in\mathbb{R}$ *and any* $f\in\mathbb{R}_F$, *we have* $\gamma\odot(\eta\odot f)=(\gamma\odot\eta)\odot f$.

If we denote $\|f\|_F:=D(f,\tilde{0}),\forall f\in\mathbb{R}_F$, then $\|.\|_F$ has the properties of a usual norm on $\mathbb{R}_F$, i.e.,

$$\begin{aligned}&\|f\|_F=0\ \text{iff}\ f=\tilde{0},\\ &\|\mu\odot f\|_F=|\mu|\,.\,\|f\|_F\,,\\ &\|f\oplus g\|_F\leq\|f\|_F+\|g\|_F\,,\\ &\|f\|_F-\|g\|_F\leq D(f,g).\end{aligned}$$

Notice that $(\mathbb{R}_F, \oplus, \odot)$ is not a linear space over $\mathbb{R}$, and consequently $(\mathbb{R}_F, \|.\|_F)$ is not a normed space. Here $\sum^*$ denotes the fuzzy summation.

Definition 2.5 ([1]) *A fuzzy valued function $f : [a, b] \to \mathbb{R}_F$ is said to be continuous at $x_0 \in [a, b]$, if for each $\epsilon > 0$ there exists $\delta > 0$ such that $D(f(x), f(x_0)) < \epsilon$, whenever $x \in [a, b]$ and $|x - x_0| < \delta$. We say that f is fuzzy continuous on $[a, b]$ if f is continuous at each $x_0 \in [a, b]$, and denotes the space of all such functions by $C_F[a, b]$.*

Definition 2.6 ([41]) *Suppose that $f : [a, b] \times [c, d] \to \mathbb{R}_f$ is a bounded mapping. The function $\omega_{[a,b]\times[c,d]}(f, .) : \mathbb{R}_+ \cup \{0\} \to \mathbb{R}_+$ defined by*

$$\omega_{[a,b]\times[c,d]}(f, \delta) = \sup \Big\{ D(f(x, y), f(s, t));$$
$$x, s \in [a, b]; y, t \in [c, d];$$
$$\sqrt{(x - s)^2 + (y - t)^2} \le \delta \Big\},$$

is called modules of oscillation of f on $[a, b]\times[c, d]$. Also, if $f \in C_F([a, b] \times [c, d])$, then $\omega_{[a,b]\times[c,d]}(f, \delta)$ is called uniform modules of continuity of f.

Theorem 2.1 ([41]) *The following properties hold:*

1. $D(f(x, y), f(s, t)) \le \omega_{[a,b]\times[c,d]}(f, \sqrt{(x - s)^2 + (y - t)^2})$, $\forall x, s \in [a, b],\ y, t \in [c, d]$;
2. $\omega_{[a,b]\times[c,d]}(f, \delta)$ *is a nondecreasing mapping in* δ;
3. $\omega_{[a,b]\times[c,d]}(f, 0) = 0$;
4. $\omega_{[a,b]\times[c,d]}(f, \delta_1 + \delta_2) \le \omega_{[a,b]\times[c,d]}(f, \delta_1) + \omega_{[a,b]\times[c,d]}(f, \delta_2)$, $\forall \delta_1, \delta_2 \ge 0$;
5. $\omega_{[a,b]\times[c,d]}(f, n\delta) \le n\omega_{[a,b]\times[c,d]}(f, \delta)$, $\forall \delta \ge 0$, $n \in \mathbb{N}$;
6. $\omega_{[a,b]\times[c,d]}(f, \mu\delta) \le (\mu + 1)\omega_{[a,b]\times[c,d]}(f, \delta)$, $\forall \mu, \delta \ge 0$;

Theorem 2.2 ([41]) *If f and g are Henstock integrable mapping on $[a, b] \times [c, d]$ and if $D(f(s, t), g(s, t))$ is Lebesgue integrable, then*

$$D\left((FH)\int_c^d\int_a^b f(s, t)dsdt, (FH)\int_c^d\int_a^b g(s, t)dsdt\right)$$
$$\le (L)\int_c^d\int_a^b D(f(s, t), g(s, t))dsdt.$$

3 The main result

In this section, we review bicubic fuzzy splines interpolation. Then, we propose bicubic fuzzy splines interpolation and iterative method for solving Eq. (3.12).

In [19], authors presented approximation of fuzzy functions by fuzzy interpolating bicubic splines by using following definitions and theorems.

Definition 3.1 ([19]) *We denote by $< . >_n$ and $< ., . >_n$, respectively, the Euclidean norm and inner product in $\mathbb{R}^n$. For any real intervals (a, b) and (c, d), with $a < b$ and $c < d$, also, we consider the rectangle $R = (a, b) \times (c, d)$ and let $H^3(R)$ be the usual Sobolev space of functions u belonging to $L^2(R)$, together with all their partial derivatives $D^\beta(u)$ with $\beta = (\beta_1, \beta_2)$, in the distribution sense, of order $|\beta| = \beta_1 + \beta_2 \le 3$. This space is equipped with the norm*

$$\|u\| = \left(\sum_{|\beta|\le 3}\int_R (D^\beta u(p))^2 dp\right)^{\frac{1}{2}},$$

the seminorms

$$|u|_l = \left(\sum_{|\beta|=l}\int_R (D^\beta u(p))^2 dp\right)^{\frac{1}{2}},$$
$$0 \le l \le 3,$$

and the corresponding inner semiproducts

$$(u, v)_l = \sum_{|\beta|=l}\int_R D^\beta u(p) D^\beta v(p) dp,$$
$$0 \le l \le 3.$$

Moreover, for $m, n \in \mathbb{N}^*$, let $T_n = \{x_0, \cdots, x_n\}$, $T_m = \{y_0, \cdots, y_m\}$ be some subsets of distinct of $[a, b]$ and $[c, d]$, with $a = x_0 < x_1 < \cdots < x_n = b$ and $c = y_0 < y_1 < \cdots < y_m = d$. We denoted by

$S_3(T_n)$ and $S_3(T_m)$ the spaces of cubic splines of class C^2 given by

$$S_3(T_n) = \{s \in C^2[a,b] : s|_{[x_{i-1},x_i]} \in P_3[x_{i-1},x_i],\ i = 1,\cdots,n\},$$

and

$$S_3(T_m) = \{s \in C^2[c,d] : s|_{[y_{j-1},y_j]} \in P_3[y_{j-1},y_j],\ j = 1,\cdots,m\},$$

where $P_3[x_{i-1},x_i](P_3[y_{j-1},y_j])$ is the restriction on $[x_{i-1},x_i]([y_{j-1},y_j])$ of the linear space of real polynomials with total degree less than or equal to 3. It is known that $dim S_3(T_n) = n + 3$ $(dim S_3(T_m) = m+3)$. Let $\{\phi_1,\cdots,\phi_{n+3}\}$ and $\{\psi_1,\cdots,\psi_{m+3}\}$ be bases of functions with local support of $S_3(T_n)$ and $S_3(T_m)$ respectively, and consider the space $S_3(T_n \times T_m)$ of bicubic spline functions of class C^2 given by

$$S_3(T_n \times T_m) = span\{\phi_1,\cdots,\phi_{n+3}\} \otimes span\{\psi_1,\cdots,\psi_{m+3}\}.$$

This space is a Hilbert subspace of $H^3(R)$ equipped with the same norm, semi-norms and inner semi-products of such space, and verifies

$$S_3(T_n \times T_m) \subset H^3(R) \cap C^2(R). \tag{3.1}$$

Particulary, let

$$\{B_0^3(x),\cdots,B_{n+2}^3(x)\} \left(\{B_0^3(y),\cdots,B_{m+2}^3(y)\}\right),$$

be the C^2-cubic B-splines basis of $S_3(T_n)(S_3(T_m))$, then

$$\{B_r^3(x)B_s^3(y),\ r = 0,\cdots,n+2,\ s = 0,\cdots,m+2\},$$

is the C^2-bicubic B-splines basis of $S_3(T_n \times T_m)$, then $dim S_3(T_n \times T_m) = (n+3)(m+3)$ and we can define

$$B_k(x,y) = B_r^3(x)B_s^3(y),\ (x,y) \in R,$$

for $r = 0,\cdots,n+2,\ s = 0,\cdots,m+2,\ k = (m+3)r+s+1$. Then $1 \le k \le (n+3)(m+3)$ and if we denote $M = (n+3)(m+3)$, we have that

$$B_1(x,y),\cdots,B_M(x,y),$$

is the C^2-bicubic B-splines basis of $S_3(T_n \times T_m)$.

Definition 3.2 ([19]) *Let* $A^N = \{(x_i,y_j) \in T_n \times T_m,\ i = 0,\cdots,n,\ j = 0,\cdots,m\}$, *with* $N = (n+1)(m+1)$ *and suppose that*

$$\sup_{p \in R} \min_{a \in A^N} < p - a >_2 = O\left(\frac{1}{N}\right),\ N \to +\infty. \tag{3.2}$$

From (3.2) we deduce that $n \to +\infty$ and $m \to +\infty$. Let L_1^N be a Lagrangian operator defined from $H^3(R)$ into $\mathbb{R}^N$ given by

$$L_1^N v = (v(a))_{a \in A^N}, \tag{3.3}$$

and $L_2^N : H^3(R) \to \mathbb{R}^{2n+2m+8}$ given by

$$L_2^N v = (\mathcal{L}_l v)_{l=1,\cdots,2n+2m+8}, \tag{3.4}$$

where

$$\mathcal{L}_l v = \begin{cases} \frac{\partial^2 v}{\partial y^2}(x_{l-1},c), \\ l = 1,\cdots,n+1, \\ \frac{\partial^2 v}{\partial y^2}(x_{l-n-2},d), \\ l = n+2,\cdots,2n+2, \\ \frac{\partial^2 v}{\partial x^2}(a,y_{l-2n-3}), \\ l = 2n+3,\cdots,2n+m+3, \\ \frac{\partial^2 v}{\partial x^2}(b,y_{l-2n-m-4}), \\ l = 2n+m+3,\cdots,2n+2m+4, \\ \frac{\partial^4 v}{\partial x^2 \partial y^2}(x_{in},y_{jm}), \\ i = 0,1,\ j = 0,1, \\ l = 2n+2m+4+2i+j+1. \end{cases}$$

Let $B^N = \{u_l,\ l = 1,\cdots,N\} \subset \mathbb{R}$.

Theorem 3.1 ([19]) *There exists a unique* $S_N \in S_3(T_n \times T_m)$ *such that*

$$L_1^N S_N = (u_l)_{l=1,\cdots,N},$$
$$L_2^N S_N = 0 \in \mathbb{R}^{2n+2m+8},$$

called the interpolating natural C^2*-bicubic spline associated with* A^N *and* B^N.

Thus C^2-bicubic spline verifies that

$$S_N(x,y) = \sum_{k=1}^{M} \alpha_k B_k(x,y),\ (x,y) \in R, \tag{3.5}$$

where $\alpha = (\alpha_1, \cdots, \alpha_M)^T \in \mathbb{R}^M$ is the solution of the linear system

$$A\alpha = b, \tag{3.6}$$

with $A = \begin{bmatrix} A_1 \\ A_2 \end{bmatrix}$ and $b = \begin{bmatrix} b_1 \\ b_2 \end{bmatrix}$, where

$$A_i = (L_i^N B_k)_{k=1,\cdots,M}, \ (i = 1, 2) \tag{3.7}$$

$$b_1 = (u_l)_{l=1,\cdots,N}, \tag{3.8}$$

$$b_2 = (0)_{l=1,\cdots,M-N}. \tag{3.9}$$

Theorem 3.2 ([19]) *Let $f \in C^4(R)$ and let S_N be the interpolating natural C^2-bicubic spline associated with A^N and $L_1^N f$, then there exists a constant $C > 0$ such that*

$$|f - S_N|_l \leq Ch^{4-l}, \ l = 0, 1, 2, 3, \ N \to +\infty, \tag{3.10}$$

where $h = \max\{\frac{b-a}{n}, \frac{d-c}{m}\}$. Hence

$$\lim_{N \to +\infty} \|f - S_N\| = 0. \tag{3.11}$$

Consider 2DLFFIE as follows

$$G(s,t) = g(s,t) \oplus \mu \odot (FR) \int_c^d (FR) \int_a^b \mathcal{K}(x,y,s,t) \odot G(x,y)dxdy, \tag{3.12}$$

where $\mu > 0$, $\mathcal{K}(x,y,s,t)$ is an arbitrary positive function on $[a,b] \times [c,d] \times [a,b] \times [c,d]$ and $g : [a,b] \times [c,d] \to \mathbb{R}_F$. We assume that $\mathcal{K}$ is continuous, and therefore it is uniformly continuous with respect to (s,t). So, there exists $\mathcal{M} > 0$ such that $\mathcal{M} = \max_{s,x \in [a,b], t,y \in [c,d]} |\mathcal{K}(x,y,s,t)|$.
Therefore, the two-dimensional interpolation in the spline form is

$$p(x,y) = \sum_{j=0}^{n} \sum_{i=0}^{n} f(s_i,t_j) \odot sp_{ij}(x,y), \tag{3.13}$$

where the coefficients $f(s_i,t_j)$ are the fuzzy numbers.
Here, we consider the two-dimensional fuzzy spline interpolation in the given points $a = s_0 < s_1 < \cdots < s_n = b$ and $c = t_0 < t_1 < \cdots < t_n = d$ such that

$$\mathcal{K}(x,y,s,t) \odot G(x,y) \approx \sum_{j=0}^{n} \sum_{i=0}^{n} sp_{ij}(x,y) \odot \mathcal{K}(s_i,t_j,s,t) \odot G(s_i,t_j). \tag{3.14}$$

Now, we propose a numerical method to solve (3.12). To do this, we suppose the following iterative procedure to approximate the solution of (3.12) in point (s,t)

$$u_0(s,t) = g(s,t),$$
$$u_k(s,t) = g(s,t) \oplus \mu \odot \sum_{j=0}^{n} \sum_{i=0}^{n} C_{ij} \mathcal{K}(s_i,t_j,s,t) \odot u_{k-1}(s_i,t_j), \tag{3.15}$$

where

$$C_{ij} = \int_c^d \int_a^b sp_{ij}(x,y)dxdy. \tag{3.16}$$

In Theorem 3.3 of [41], authors proved the existence and uniqueness solution of Eq. (3.12) by using the Banach fixed point theorem.

Theorem 3.3 ([41]) *Let the function $\mathcal{K}(x,y,s,t)$ be continuous and positive for $x,s \in [a,b]$, and $y,t \in [c,d]$, and let $g : [a,b] \times [c,d] \to \mathbb{R}_F$ be continuous on $[a,b] \times [c,d]$. If $B = \mu\mathcal{M}(d-c)(b-a) < 1$ then the fuzzy integral equation (3.12) has a unique solution $G^* \in X$ where*

$$X = \{g : [a,b] \times [c,d] \to \mathbb{R}_F; g \text{ is continous}\},$$

be the space of two-dimensional fuzzy continuous functions with the metric D^* and it can be obtained by the following successive approximations method

$$G_0(s,t) = g(s,t),$$
$$G_k(s,t) = g(s,t) \oplus \mu \odot (FR) \int_c^d (FR) \int_a^b \mathcal{K}(x,y,s,t) \odot G_{k-1}(x,y)dxdy, \ \forall k \geq 1. \tag{3.17}$$

Moreover, the sequence of successive approximations, $(G_k)_{k \geq 1}$ converges to the solution G^*. Furthermore, the following error bound holds

$$D^*(G^*, G_k) \leq \frac{B^{k+1}}{1-B} \mathcal{M}_1, \quad \forall k \geq 1, \tag{3.18}$$

where $\mathcal{M}_1 = \sup_{s \in [a,b], t \in [c,d]} \|G(s,t)\|$. $\square$

4 Convergence analysis

In this section, we obtain an error estimate between the exact solution and the approximate solution of 2DLFFIE (3.12).

Theorem 4.1 *Under the hypotheses of Theorem 3.3 and $\mu > 0$, the iterative procedure (3.15) converges to the unique solution of (3.12), G^*, and its error estimate is as follows:*

$$D^*(G^*, u_k) \leq \frac{B^{k+1}}{(1-B)}\mathcal{M}_1 + \frac{B}{1-B}\Big(\omega(u_m, \upsilon(\Delta)) . \Big(1 + \frac{2\overline{C}}{(d-c)(b-a)}\Big)m_l\Big), \tag{4.19}$$

where

$$m_k = \sup_{(s,t)\in[a,b]\times[c,d]} \|u_k(s,t)\|,$$

$$m_l = \max\{m_0, \cdots, m_{k-1}\},$$

and

$$\omega(u_m, \upsilon(\Delta)) = \max\{\omega(u_0, \upsilon(\Delta)), \cdots, \omega(u_k, \upsilon(\Delta))\}.$$

Proof. Clearly, we have

$$D(G^*(s,t), u_k(s,t)) \leq D(G^*(s,t), G_k(s,t)) + D(G_k(s,t), u_k(s,t)).$$

From (3.17) and (3.15), we conclude that

$$D(G_k(s,t), u_k(s,t)) = D\Big(g(s,t) \oplus \mu \odot (FR)\int_c^d (FR)\int_a^b \mathcal{K}(x,y,s,t) \odot G_{k-1}(x,y)dxdy, g(s,t) \oplus \mu \odot \sum_{j=0}^{n}\sum_{i=0}^{n} C_{ij}\mathcal{K}(s_i,t_j,s,t) \odot u_{k-1}(s_i,t_j)\Big)$$

$$= \mu D\Big((FR)\int_c^d (FR)\int_a^b \mathcal{K}(x,y,s,t) \odot G_{k-1}(x,y)dxdy, \sum_{j=0}^{n}\sum_{i=0}^{n} C_{ij}\mathcal{K}(s_i,t_j,s,t) \odot u_{k-1}(s_i,t_j)\Big)$$

$$= \mu D\Big(\sum_{j=1}^{n}\sum_{i=1}^{n}(FR)\int_{t_{j-1}}^{t_j}(FR)\int_{s_{i-1}}^{s_i} \mathcal{K}(x,y,s,t) \odot G_{k-1}(x,y)dxdy, \sum_{j=0}^{n}\sum_{i=0}^{n} C_{ij}\mathcal{K}(s_i,t_j,s,t) \odot u_{k-1}(s_i,t_j)\Big)$$

$$= \mu D\Big(\sum_{j=1}^{n}\sum_{i=1}^{n}(FR)\int_{t_{j-1}}^{t_j}(FR)\int_{s_{i-1}}^{s_i} \mathcal{K}(x,y,s,t) \odot G_{k-1}(x,y)dxdy, \sum_{j=1}^{n}\sum_{i=1}^{n}(t_j - t_{j-1})(s_i - s_{i-1})\frac{C_{ij}}{(t_j - t_{j-1})(s_i - s_{i-1})} \mathcal{K}(s_i,t_j,s,t) \odot u_{k-1}(s_i,t_j) \oplus C_{00}\mathcal{K}(s_0,t_0,s,t) \odot u_{k-1}(s_0,t_0)\Big)$$

$$\leq \mu \sum_{j=1}^{n}\sum_{i=1}^{n} D\Big((FR)\int_{t_{j-1}}^{t_j}(FR)\int_{s_{i-1}}^{s_i} \mathcal{K}(x,y,s,t) \odot G_{k-1}(x,y)dxdy, (FR)\int_{t_{j-1}}^{t_j}(FR)\int_{s_{i-1}}^{s_i}\frac{C_{ij}}{(t_j - t_{j-1})(s_i - s_{i-1})} \mathcal{K}(s_i,t_j,s,t) \odot u_{k-1}(s_i,t_j)dxdy\Big) \oplus \mu D\Big(C_{00}\mathcal{K}(s_0,t_0,s,t) \odot u_{k-1}(s_0,t_0), \tilde{0}\Big)$$

$$\leq \mu \sum_{j=1}^{n}\sum_{i=1}^{n} \int_{t_{j-1}}^{t_j}\int_{s_{i-1}}^{s_i}$$

$$\Big[D\Big(\mathcal{K}(x,y,s,t)\odot G_{k-1}(x,y), \mathcal{K}(x,y,s,t)\odot u_{k-1}(s_i,t_j)\Big) + D\Big(\mathcal{K}(x,y,s,t)\odot u_{k-1}(s_i,t_j), \frac{C_{ij}}{(t_j-t_{j-1})(s_i-s_{i-1})}\mathcal{K}(s_i,t_j,s,t) \odot u_{k-1}(s,t)\Big)\Big]dxdy + \mu D\Big(C_{00}\mathcal{K}(s_0,t_0,s,t) \odot u_{k-1}(s_0,t_0),\tilde{0}\Big)$$
$$\leq \mu \sum_{j=1}^{n}\sum_{i=1}^{n}\int_{t_{j-1}}^{t_j}\int_{s_{i-1}}^{s_i}\Big(|\mathcal{K}(x,y,s,t)| D(G_{k-1}(x,y),u_{k-1}(s_i,t_j)) + |\mathcal{K}(x,y,s,t)|\,D(u_{k-1}(s_i,t_j),\tilde{0}) + \left|\frac{C_{ij}}{(t_j-t_{j-1})(s_i-s_{i-1})}\mathcal{K}(s_i,t_j,s,t)\right| D(u_{k-1}(s_i,t_j),\tilde{0})\Big)dxdy + \mu D(C_{00}\mathcal{K}(s_0,t_0,s,t) \odot u_{k-1}(s_0,t_0),\tilde{0}).$$

By supposing $m_k = \sup_{(s,t)\in[a,b]\times[c,d]} \|u_k(s,t)\|$, we get

$$D(G_k(s,t),u_k(s,t)) \leq \mu \sum_{j=1}^{n}\sum_{i=1}^{n}\int_{t_{j-1}}^{t_j}\int_{s_{i-1}}^{s_i} \Big(|\mathcal{K}(x,y,s,t)|\Big(D(G_{k-1}(x,y),u_{k-1}(s_i,t_j)) + \|u_{k-1}(s_i,t_j)\|\Big) + \left|\frac{C_{ij}}{(t_j-t_{j-1})(s_i-s_{i-1})}\mathcal{K}(s_i,t_j,s,t)\right| \|u_{k-1}(s_i,t_j)\|\Big)dxdy + |\mu C_{00}\mathcal{K}(s_0,t_0,s,t)|\,\|u_{k-1}(s_i,t_j)\|.$$

So, we have

$$D(G_k(s,t),u_k(s,t)) \leq \mu \sum_{j=1}^{n}\sum_{i=1}^{n}\int_{t_{j-1}}^{t_j}\int_{s_{i-1}}^{s_i} \Big(\mathcal{M}\Big(D(G_{k-1}(x,y),u_{k-1}(s_i,t_j)) + m_{k-1}\Big) + \left|\frac{C_{ij}}{(t_j-t_{j-1})(s_i-s_{i-1})}\mathcal{K}(s_i,t_j,s,t)\right| m_{k-1}\Big) dxdy + \mu\,|C_{00}\mathcal{K}(s_0,t_0,s,t)|\,m_{k-1}.$$

We know that $\upsilon(\Delta) = \sup\{\sqrt{(s_i-s_{i-1})^2+(t_j-t_{j-1})^2}\}, \quad \forall i,j = 1,\cdots,n$, so we conclude

$$D(G_k(s,t),u_k(s,t)) \leq \mu \sum_{j=1}^{n}\sum_{i=1}^{n}\int_{t_{j-1}}^{t_j}\int_{s_{i-1}}^{s_i} \Big(\mathcal{M}D^*(G_{k-1},u_{k-1}) + \mathcal{M}\omega(u_{k-1},\upsilon(\Delta))\Big)dxdy + \mu \sum_{j=1}^{n}\sum_{i=1}^{n}\int_{t_{j-1}}^{t_j}\int_{s_{i-1}}^{s_i} \mathcal{M}m_{k-1}dxdy + \sum_{j=1}^{n}\sum_{i=1}^{n}\int_{t_{j-1}}^{t_j}\int_{s_{i-1}}^{s_i} \left|\frac{C_{ij}}{(t_j-t_{j-1})(s_i-s_{i-1})}\mathcal{K}(s_i,t_j,s,t)\right| m_{k-1} dxdy + \mu\,|C_{00}\mathcal{K}(s_0,t_0,s,t)|\,m_{k-1}.$$

Therefore,

$$\begin{aligned} D(G_k(s,t),u_k(s,t)) &\leq \mu\mathcal{M}(d-c)(b-a)D^*(G_{k-1},u_{k-1}) \\ &+ \mu\mathcal{M}(d-c)(b-a)\omega(u_{k-1},\upsilon(\Delta)) \\ &+ \mu\mathcal{M}(d-c)(b-a)m_{k-1} \\ &+ \mu\mathcal{M}(d-c)(b-a)m_{k-1}\frac{\overline{C}}{(d-c)(b-a)} \\ &+ \mu\mathcal{M}\overline{C}m_{k-1} \\ &\leq BD^*(G_{k-1},u_{k-1}) \\ &+ B\omega(u_{k-1},\upsilon(\Delta)) \\ &+ Bm_{k-1} + 2Bm_{k-1}\frac{\overline{C}}{(d-c)(b-a)} \\ &= BD^*(G_{k-1},u_{k-1}) \\ &+ B\omega(u_{k-1},\upsilon(\Delta)) \\ &+ Bm_{k-1}\Big(1+\frac{2\overline{C}}{(d-c)(b-a)}\Big), \end{aligned}$$

where $\overline{C} = \max|C_{ij}|$. Hence, we conclude that

$$D^*(G_{k-1}, u_{k-1}) \leq BD^*(G_{k-2}, u_{k-2}) + B\omega(u_{k-2}, \upsilon(\Delta)) + Bm_{k-2}\Big(1 + \frac{2\overline{C}}{(d-c)(b-a)}\Big),$$

$$\vdots$$

$$D^*(G_1, u_1) \leq BD^*(G_0, u_0) + B\omega(u_0, \upsilon(\Delta)) + Bm_0\Big(1 + \frac{2\overline{C}}{(d-c)(b-a)}\Big).$$

So,

$$D^*(G_k, u_k) \leq \sum_{i=1}^{k} \omega(u_{k-i}, \upsilon(\Delta))B^i + \Big(1 + \frac{2\tilde{C}}{(d-c)(b-a)}\Big)\sum_{i=1}^{k} B^i m_{k-i}.$$

If $m_l = \max\{m_0, \cdots, m_{k-1}\}$ and $\omega(u_m, \upsilon(\Delta)) = \max\{\omega(u_0, \upsilon(\Delta)), \cdots, \omega(u_k, \upsilon(\Delta))\}$, then we obtain

$$D^*(G_k, u_k) < \frac{B}{1-B}\Big(\omega(u_m, \upsilon(\Delta)) + \Big(1 + \frac{2\overline{C}}{(d-c)(b-a)}\Big)m_l\Big),$$

therefore

$$D^*(G^*, u) \leq \frac{B^{k+1}}{1-B}\mathcal{M}_1 + \frac{B}{1-B}\Big(\omega(u_m, \upsilon(\Delta)) + \Big(1 + \frac{2\overline{C}}{(d-c)(b-a)}\Big)m_l\Big).$$

5 Numerical stability analysis

To show the numerical stability analysis of the proposed method in previous section, we consider another starting approximation $f(s,t) = Y_0(s,t)$ such that $\exists\, \epsilon > 0$ for which $D(G_0(s,t), Y_0(s,t)) < \epsilon$, $\forall s, t \in [a,b] \times [c,d]$. The obtained sequence of successive approximations is

$$Y_k(s,t) = f(s,t) \oplus \mu \odot (FR)\int_c^d (FR)\int_a^b \mathcal{K}(x,y,s,t) \odot Y_{k-1}(x,y)dxdy,$$

and using the same iterative method as (3.15), the terms of produced sequence are

$$\upsilon_0(s,t) = Y_0(s,t) = f(s,t),$$
$$\upsilon_k(s,t) = f(s,t) \oplus \mu \odot \sum_{j=0}^{n}\sum_{i=0}^{n} C_{ij}\mathcal{K}(s_i, t_j, s, t) \odot \upsilon_{k-1}(s_i, t_j).$$

Theorem 5.1 *The proposed method (3.15), under the assumptions of Theorem 4.1 is numerically stable with respect to the choice of the first iteration.*

Proof. At first, we obtain

$$D(u_k(s,t), \upsilon_k(s,t)) \leq D(u_k(s,t), G_k(s,t)) + D(G_k(s,t), Y_k(s,t)) + D(Y_k(s,t), \upsilon_k(s,t))$$

$$\leq \frac{B}{1-B}\Big(\omega(u_m, \upsilon(\Delta)) + \Big(1 + \frac{2\overline{C}}{(d-c)(b-a)}\Big)m_l\Big) + D(G_k(s,t), Y_k(s,t)) + \frac{B}{1-B}\Big(\omega(\upsilon_m, \upsilon(\Delta)) + \Big(1 + \frac{2\overline{C}}{(d-c)(b-a)}\Big)m_l\Big).$$

However,

$$D(G_k(s,t), Y_k(s,t)) = D\Big(g(s,t) \oplus \mu \odot (FR)\int_c^d (FR)\int_a^b \mathcal{K}(x,y,s,t) \odot G_{k-1}(x,y)dxdy,$$
$$f(s,t) \oplus \mu \odot (FR)\int_c^d (FR)\int_a^b \mathcal{K}(x,y,s,t)$$

$$\odot Y_{k-1}(x,y)dxdy\Big)$$
$$\le D(g(s,t), f(s,t))$$
$$+\mu D\Big((FR)\int_c^d (FR)\int_a^b \mathcal{K}(x,y,s,t)\odot$$
$$G_{k-1}(x,y)dxdy,$$
$$(FR)\int_c^d (FR)\int_a^b \mathcal{K}(x,y,s,t)\odot$$

$$Y_{k-1}(x,y)dxdy\Big)$$
$$\le \epsilon$$
$$+\mu \int_c^d \int_a^b |\mathcal{K}(x,y,s,t)|$$
$$D(G_{k-1}(x,y), Y_{k-1}(x,y))dxdy.$$

We conclude that

$$D^*(G_k, Y_k) \le \epsilon$$
$$+\mu \int_c^d \int_a^b \mathcal{M}D^*(G_{k-1}, Y_{k-1})dxdy$$
$$= \epsilon + BD^*(G_{k-1}, Y_{k-1}),$$

and thus

$$\begin{aligned} D^*(G_k, Y_k) &\le \epsilon + BD^*(G_{k-1}, Y_{k-1}) \\ D^*(G_{k-1}, Y_{k-1}) &\le \epsilon + BD^*(G_{k-2}, Y_{k-2}) \\ \vdots \quad &\quad \vdots \\ D^*(G_1, Y_1) &\le \epsilon + BD^*(G_0, Y_0). \end{aligned}$$

So,

$$D^*(G_k, Y_k) \le \epsilon + B\Big(\epsilon + BD^*(G_{k-2}, Y_{k-2})\Big)$$
$$\le \epsilon + B\epsilon + B^2\Big(\epsilon + BD^*(G_{k-3}, Y_{k-3})\Big)$$
$$\vdots$$
$$\le \epsilon + B\epsilon + B^2\epsilon + B^3\epsilon + \cdots + B^k D^*(G_0, Y_0)$$
$$\le \epsilon\Big(1 + B + B^2 + B^3 + \cdots + B^k\Big)$$
$$\le \frac{\epsilon}{1-B}.$$

Therefore,

$$D^*(u_k, v_k) \le \frac{B}{1-B}\Big(\omega(u_m, v(\Delta))$$
$$+\omega(v_m, v(\Delta))$$
$$+2m_l\Big(1 + \frac{2\overline{C}}{(d-c)(b-a)}\Big) + \epsilon\Big).$$

6 Numerical examples

In this section, we use the proposed method in Section 3 for solving a two-dimensional linear fuzzy Fredholm integral equations for solving example. By using the proposed method for $n = 3$, $k = 5$ and $r \in \{0.00, 0.25, 0.50, 0.75, 1.00\}$ in $(s,t) = (0.5, 0.5)$, we present the absolute errors in Table 1.

Example 6.1 **([23])** *Consider the linear integral equation*

$$G(s,t) = g(s,t)\oplus \qquad (6.20)$$
$$(FR)\int_0^1 (FR)\int_0^1 \mathcal{K}(s,t,x,y)\odot G(x,y)dxdy,$$

with

$$\underline{g}(s,t,r) = r\left(st + \frac{1}{676}(s^2+t^2-2)\right),$$
$$\overline{g}(s,t,r) = (2-r)\left(st + \frac{1}{676}(s^2+t^2-2)\right),$$

and kernel

$$\mathcal{K}(s,t,x,y) = \frac{1}{169}(s^2+t^2-2)(x^2+y^2-2),$$
$$0 \le s,t,x,y \le 1.$$

The exact solution is

$$\underline{G}^*(s,t,r) = rst,\ \overline{G}^*(s,t,r) = (2-r)st.$$

Table 1: The absolute errors on the level sets with $n = 3$, $k = 5$ for Example 6.1 by using the proposed method in $(s,t) = (0.5, 0.5)$.

$r-level$	$\underline{e}^r = \lvert\underline{\tilde{G}}^*(s,t,r) - \underline{\tilde{u}}_k(s,t,r)\rvert$
0.00	0.
0.25	2.07813e − 5
0.50	4.15625e − 5
0.75	6.23438e − 5
1.00	8.31251e − 5

Table 2: The absolute errors on the level sets with $n = 3$, $k = 5$ for Example 6.1 by using the proposed method in $(s,t) = (0.5, 0.5)$.

$r - level$	$\bar{e}^r = \left\lvert \bar{\tilde{G}}^*(s,t,r) - \bar{\tilde{u}}_k(s,t,r) \right\rvert$
0.00	$1.6625e-4$
0.25	$1.45469e-4$
0.50	$1.24688e-4$
0.75	$1.03906e-4$
1.00	$8.31251e-5$

7 Conclusion

The 2DLFFIE is solved by utilizing iterative method and fuzzy bicubic spline interpolation. As it was expected the method used to approximate the integral in this equation is a suitable one since convergence analysis and stability analysis have been proved and also absolute error in example is good. As a result, considering the fact that the proposed method does not lead to solve fuzzy linear system, it can be utilized as an efficient method to solve this type of equations. As future researches, we can use finite and divided differences methods for solving two-dimensional fuzzy Fredholm integral equations.

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