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Research Article

A Fuzzy Power Series Method for Solving Fuzzy Differential Equations With Fractional Order

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Abstract

In this paper a new method for solving fuzzy differential equation with fractional order is considered. The fuzzy solution is construct by power series in the fuzzy Caputo derivatives sense. To illustrate the reliability of method, some examples are provided.

Keywords : Fuzzy Numbers; Generalized Hukuhara differentiability; Caputo differentiability; Fuzzy fractional differential equations.

1 Introduction

Fuzzy fractional calculus and fuzzy fractional differential equations have excited in recent years as a considerable interest both in mathematics and applications. In its turn, mathematical aspects of fuzzy fractional differential equations with fractional order and method of their solution were discussed by many authors: the concept of Riemann-Liouville fractional differential equations with uncertainly based on the Hukuhara differentiability was introduced by Agarwal et al [1]. Existence and uniqueness of solution of fuzzy fractional differential equation was investigated in [5, 2]. Allahviranloo et al in [3] give the explicit solutions of uncertain fractional differential equations under Riemann-Liouville H-differentiability using Mittag- Leffler functions and in [16] give the solutions of fuzzy fractional differential equations under Riemann-Liouville H-differentiability by fuzzy Laplace transforms.

In this paper, some theorems of the fractional power series are generalized for the fuzzy fractional power series by using fuzzy Caputo derivatives. We use the fuzzy fractional power series to solve the fractional differential equations subject to given fuzzy initial conditions.
In section 2, some basic definitions are brought. The fuzzy fractional power series and the proposed method for solving fuzzy fractional differential equation are introduced in section 3 and 4, finally conclusion is drawn.

2 Preliminaries

First notations which shall be used in this paper are introduced.
We denote by $\mathbb{R}_{\mathcal{F}}$, the set of fuzzy numbers, that is, normal, fuzzy convex, upper semi-continuous and compactly supported fuzzy sets which are defined over the real line.
For $0 < r \leq 1$, set $[u]^r = \Big\{t \in \mathbb{R}\Big|u(t) \geq r\Big\}$, and $[u]^0 = cl\Big\{t \in \mathbb{R}\Big|u(t) > 0\Big\}$. We represent $[u]^r = [\underline{u}(r), \overline{u}(r)]$, so if $u \in \mathbb{R}_{\mathcal{F}}$, the r-level set $[u]^r$ is a closed interval for all $r \in [0, 1]$.

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For arbitrary $u,\ v \in \mathbb{R}_{\mathcal{F}}$ and $k \in \mathbb{R}$, the addition and scalar multiplication are defined by $[u+v]^r = [u]^r + [v]^r$, $[ku]^r = k[u]^r$ respectively.

A triangular fuzzy number is defined as a fuzzy set in $\mathbb{R}_{\mathcal{F}}$, that is specified by an ordered triple $u = (a, b, c) \in \mathbb{R}^3$ with $a \le b \le c$ such that $\underline{u}(r) = a + (b-a)r$ and $\overline{u}(r) = c - (c-b)r$ are the endpoints of r-level sets for all $r \in [0, 1]$.

Definition 2.1 *[13]The Hausdorff distance between fuzzy numbers is given by $D : \mathbb{R}_{\mathcal{F}} \times \mathbb{R}_{\mathcal{F}} \longrightarrow \mathbb{R}^+ \cup \{0\}$ as*

$$D(u,\ v) = \sup_{r\in[0,\ 1]} \max\Big\{|\underline{u}(r) - \underline{v}(r)| \quad , \ |\overline{u}(r) - \overline{v}(r)|\Big\}. \tag{2.1}$$

Consider $u, v, w, z \in \mathbb{R}_{\mathcal{F}}$ and $\lambda \in \mathbb{R}$, then the following properties are well-known for metric D,

1. $D(u \oplus w,\ v \oplus w) = D(u,\ v)$;
2. $D(\lambda u,\ \lambda v) = |\lambda| D(u,\ v)$;
3. $D(u \oplus v,\ w \oplus z) \le D(u,\ w) + D(v,\ z)$;
4. $D(u \ominus v,\ w \ominus z) \le D(u,\ w) + D(v,\ z)$, *as long as $u \ominus v$ and $w \ominus z$ exist, where $u,\ v,\ w,\ z \in \mathbb{R}_{\mathcal{F}}$.*

where, $\ominus$ is the Hukuhara difference(H-difference), it means that $w \ominus v = u$ if and only if $u \oplus v = w$.

Definition 2.2 *[9] Let $u, v \in R_F$. If there exists $w \in R_F$ such that*

$$u \ominus_{gH} v = w \Leftrightarrow \begin{cases} \quad (i) \qquad u = v + w, \\ or \quad (ii) \quad\ v = u + (-1)w, \end{cases}$$

Then w is called the generalized Hukuhara difference of u and v.

A function $f : [a, b] \to R_F$ so called fuzzy-valued function. The r-level representation of fuzzy valued function f is expressed by $[f]^r(t) = [\underline{f}(t, r), \overline{f}(t, r)]$, $t \in [a, b]$, $r \in [0, 1]$.

Definition 2.3 *[9] The generalized Hukuhara derivative of a fuzzy-valued function $f : (a, b) \to R_F$ at t_0 is defined as*

$$f'_{gH}(t_0) = \lim_{h\to 0} \frac{f(t_0 + h) \ominus_{gH} f(t_0)}{h}, \tag{2.2}$$

if $f'_{gH}(t_0) \in R_F$, we say that f is generalized Hukuhara differentiable (gH-differentiable) at t_0. Also we say that f is $[i - gH]$-differentiable at t_0 if

$$[f'_{gH}]^r(t_0) = [\underline{f}'(t_0, r), \overline{f}'(t_0, r)], \quad 0 \le r \le 1, \tag{2.3}$$

and say f is $[ii - gH]$-differentiable at t_0 if

$$[f'_{gH}]^r(t_0) = [\overline{f}'(t_0, r), \underline{f}'(t_0, r)], \quad 0 \le r \le 1. \tag{2.4}$$

Definition 2.4 *[9] We say that a point $t_0 \in (a, b)$, is a switching point for the differentiability of f, if in any neighborhood V of t_0 there exist points $t_1 < t_0 < t_2$ such that*

type (I): at t_1 (2.3) holds while (2.4) does not hold and at t_2 (2.4) holds and (2.3) does not hold, or

type (II):at t_1 (2.4) holds while (2.3) does not hold and at t_2 (2.3) holds and (2.4) does not hold.

Definition 2.5 *[9] Let $f : (a,\ b) \to \mathbb{R}_{\mathcal{F}}$. Consider $f(t)$ is gH-differentiable of order i, $i = 1, ..., n-1$ at t_0 with no switching point on $[a, b]$. We say that $f(t)$ is gH-differentiable of the n^{th}-order at t_0, if $(f)^{(n)}_{gH}(t_0) \in \mathbb{R}_{\mathcal{F}}$ exists such that*

$$(f)^{(n)}_{gH}(t_0) = \lim_{h\to 0} \frac{f^{(n-1)}(t_0 + h) \ominus_{gH} f^{(n-1)}(t_0)}{h}.$$

In this paper $C_F[a, b]$ is the space of all continuous fuzzy-valued function on $[a, b]$. Also we denote the space of all Lebesgue integrable fuzzy-valued functions on the bounded interval $[a, b] \subset R$ by $L_F[a, b]$.

Definition 2.6 *[4] Let $f \in C_F[a, b] \cap L_F[a, b]$. The fuzzy Riemann-Liouville integral of fuzzy-valued function f is defined as following:*

$$(I^\alpha_a f)(t) = \frac{1}{\Gamma(\alpha)} \int_a^t \frac{f(s)ds}{(t-s)^{1-\alpha}}, \tag{2.5}$$

where $a < s < t$, $\quad 0 < \alpha \le 1$.

Definition 2.7 *[4] Let $f' \in C_F[a, b] \cap L_f[a, b]$. The fractional generalized Hukuhara Caputo derivative of fuzzy-valued function f is defined as follows:*

$$\begin{aligned} ({}_{gH}D^\alpha_* f)(t) &= I^{1-\alpha}_a (f'_{gH})(t) \qquad (2.6) \\ &= \frac{1}{\Gamma(1-\alpha)} \int_a^t \frac{(f'_{gH})(s)ds}{(t-s)^\alpha}, \end{aligned}$$

for $a < s < t, \quad 0 < \alpha \le 1.$
Also we say that f *is* $[i - gH]$*-differentiable at* t_0 *if*

$$(_{gH}D_*^{\alpha} f)^r(t)(t_0) = [D_*^{\alpha}\underline{f}(t_0, r), D_*^{\alpha}\overline{f}(t_0, r)], \tag{2.7}$$

and say f *is* $[ii - gH]$*-differentiable at* t_0 *if*

$$(_{gH}D_*^{q} f)^r(t)(t_0) = [D_*^{q}\overline{f}(t_0, r), D_*^{q}\underline{f}(t_0, r)], \tag{2.8}$$

3 Fuzzy Fractional Power Series

In this section, we will generalize some important definition and theorem related with the fuzzy power series into fractional case in the sense of the fuzzy Caputo definition.

Definition 3.1 *A fuzzy power series representation of the form*

$$\sum_{n=0}^{\infty} c_n \odot (t - t_0)^{n\alpha} = c_0 \oplus c_1 \odot (t - t_0)^{\alpha} \oplus c_2 \odot (t - t_0)^{2\alpha} \oplus \cdots, \tag{3.9}$$

where $0 \le m - 1 < \alpha \le m$ *and* $t \ge t_0$ *is called fuzzy power series (FPS) about* t_0*, where* t *is a variable and* c_n *are fuzzy constants called the coefficient of the series.*

Theorem 3.1 *If* $f(t)$ *is a fuzzy-valued function defined by* $f(t) = \sum_{n=0}^{\infty} c_n \odot t^{n\alpha}$*, then for* $0 \le m - 1 < \alpha \le m$*, If* f *is* $[(i) - gH]$*-differentiable we have:*

$$_{gH}D_*^{\alpha} f(t) = \sum_{n=1}^{\infty} c_n \odot \frac{\Gamma(n\alpha + 1)}{\Gamma((n-1)\alpha + 1)} t^{(n-1)\alpha}, \tag{3.10}$$

If f *is* $[(ii) - gH]$*-differentiable*

$$_{gH}D_*^{\alpha} f(t) = \ominus \sum_{n=1}^{\infty} c_n \odot \frac{\Gamma(n\alpha + 1)}{\Gamma((n-1)\alpha + 1)} t^{(n-1)\alpha}, \tag{3.11}$$

Proof: Define $g(x) = \sum_{n=0}^{\infty} c_n \odot x^n$. If f is $[(i) - gH]$-differentiable therefore $g(x)$ is $[(i) - gH]$-differentiable, therefore

$$[_{gH}D_*^{\alpha} g]^r(x) = \frac{1}{\Gamma(m - \alpha)} \int_0^x (x - \tau)^{m-\alpha-1} [g_{gH}^{(m)}]^r(\tau) d\tau,$$

thus

$$_{gH}D_*^{\alpha} \underline{g}(x, r) = \frac{1}{\Gamma(m - \alpha)} \int_0^x (x - \tau)^{m-\alpha-1} \underline{g}_{gH}^{(m)}(\tau, r) d\tau,$$

and

$$_{gH}D_*^{\alpha} \overline{g}(x, r) = \frac{1}{\Gamma(m - \alpha)} \int_0^x (x - \tau)^{m-\alpha-1} \overline{g}_{gH}^{(m)}(\tau, r) d\tau,$$

$$_{gH}D_*^{\alpha} \underline{g}(x, r) = \frac{1}{\Gamma(m - \alpha)} \int_0^x (x - \tau)^{m-\alpha-1} \Big(\sum_{n=0}^{\infty} \underline{c}_n(r) \frac{d^m}{d\tau^m} \tau^n\Big) d\tau,$$

$$_{gH}D_*^{\alpha} \overline{g}(x, r) = \frac{1}{\Gamma(m - \alpha)} \int_0^x (x - \tau)^{m-\alpha-1} \Big(\sum_{n=0}^{\infty} \overline{c}_n(r) \frac{d^m}{d\tau^m} \tau^n\Big) d\tau,$$

therefore

$$_{gH}D_*^{\alpha} g(x) = \sum_{n=0}^{\infty} c_n \odot \frac{1}{\Gamma(m-\alpha)} \int_0^x (x - \tau)^{m-\alpha-1} \Big(\frac{d^m}{d\tau^m} \tau^n\Big) d\tau = \sum_{n=0}^{\infty} c_n \odot_{gH} D_*^{\alpha}(x^n),$$

If we make the change of variable $x = t^{\alpha}$, $t \ge 0$, we have

$$_{gH}D_*^{\alpha} f(t) = _{gH}D_*^{\alpha} g(t^{\alpha}) = \sum_{n=0}^{\infty} c_n \odot D_*^{\alpha}(t^{n\alpha}) = \sum_{n=1}^{\infty} c_n \odot \frac{\Gamma(n\alpha + 1)}{\Gamma((n-1)\alpha + 1)} t^{(n-1)\alpha},$$

By similarly way, If f is $[ii - gH]$-differentiable we have

$$_{gH}D_*^{\alpha} f(t) = \ominus \sum_{n=1}^{\infty} c_n \odot \frac{\Gamma(n\alpha + 1)}{\Gamma((n-1)\alpha + 1)} t^{(n-1)\alpha},$$

Theorem 3.2 *Suppose that fuzzy-valued function* $f(t)$ *has fuzzy power series representation at* t_0 *of the form:*

$$f(t) = \sum_{n=0}^{\infty} c_n \odot (t - t_0)^{n\alpha}, \tag{3.12}$$

where $0 \leq m-1 < \alpha \leq m$, *then*

$$c_n = \frac{{}_{gH}D_*^{n\alpha} f(t_0)}{\Gamma(n\alpha+1)} \tag{3.13}$$

Proof: First we put $t = t_0$ into equation (3.12), we get $f(t_0) = c_0$.
On the other aspect as well, by using (3.10) we have:

$$\begin{aligned} {}_{gH}D_*^{\alpha} f(t) &= c_1 \odot \Gamma(\alpha+1) \\ &+ c_2 \odot \frac{\Gamma(2\alpha+1)}{\Gamma(\alpha+1)}(t-t_0)^{\alpha} \\ &+ c_3 \odot \frac{\Gamma(3\alpha+1)}{\Gamma(2\alpha+1)}(t-t_0)^{2\alpha} + \cdots \end{aligned} \tag{3.14}$$

The substitution of $t = t_0$ into equation (3.14)then we have

$$\begin{aligned} {}_{gH}D_*^{\alpha} f(t_0) &= c_1 \odot \Gamma(\alpha+1) \\ \Rightarrow c_1 &= \frac{{}_{gH}D_*^{\alpha} f(t_0)}{\Gamma(\alpha+1)} \end{aligned} \tag{3.15}$$

Applying Equation (3.10) on the series representation in Equation (3.14), on can obtain that

$$\begin{aligned} {}_{gH}D_*^{2\alpha} f(t) &= c_2 \odot \Gamma(2\alpha+1) \\ &+ c_3 \odot \frac{\Gamma(3\alpha+1)}{\Gamma(\alpha+1)}(t-t_0)^{\alpha} \\ &+ c_4 \odot \frac{\Gamma(4\alpha+1)}{\Gamma(2\alpha+1)}(t-t_0)^{2\alpha} + \cdots \end{aligned} \tag{3.16}$$

at $t = t_0$ we have

$${}_{gH}D_*^{2\alpha} f(t) = c_2 \odot \Gamma(2\alpha+1) \Rightarrow c_2 = \frac{{}_{gH}D_*^{2\alpha} f(t_0)}{\Gamma(2\alpha+1)}$$

Now we can see pattern and discover the general formula for c_n,

$$c_n = \frac{{}_{gH}D_*^{n\alpha} f(t_0)}{\Gamma(n\alpha+1)},$$

By substituting of $c_n = \frac{{}_{gH}D_*^{n\alpha} f(t_0)}{\Gamma(n\alpha+1)}$, $n = 0, 1, 2, \cdots$ back into the series of Equation (3.9)will lead to the following expansion for fuzzy-valued function $f(t)$ about t_0:

$$f(t) = \sum_{n=0}^{\infty} \frac{{}_{gH}D_*^{n\alpha} f(t_0)}{\Gamma(n\alpha+1)} \odot (t-t_0)^{n\alpha}$$

which is the Generalized Taylor's series.

Theorem 3.3 *Suppose that fuzzy-valued function* $f(t)$ *has a Generalized Taylor's series representation at* t_0 *of the form*

$$f(t) = \sum_{n=0}^{\infty} \frac{{}_{gH}D_*^{n\alpha} f(t_0)}{\Gamma(n\alpha+1)} \odot (t-t_0)^{n\alpha} \tag{3.17}$$

then

$$\begin{aligned} {}_{gH}\underline{D}_*^{n\alpha} f(t_0) &= \frac{\Gamma(n\alpha+1)}{n!} \odot {}_{gH}\underline{g}^{(n)}(t_0), \\ {}_{gH}\overline{D}_*^{n\alpha} f(t_0) &= \frac{\Gamma(n\alpha+1)}{n!} \odot {}_{gH}\overline{g}^{(n)}(t_0), \end{aligned}$$

where

$$g(t) = f((x-t_0)^{1/\alpha} + t_0).$$

Proof:By change of variable $t = (x-t_0)^{1/\alpha} + t_0$ into Equation (3.17) then we obtain

$$\begin{aligned} g(x) &= f((x-t_0)^{1/\alpha} + t_0) \\ &= \sum_{n=0}^{\infty} \frac{{}_{gH}D_*^{n\alpha} f(t_0)}{\Gamma(n\alpha+1)} \odot (x-t_0)^{n} \end{aligned} \tag{3.18}$$

The other hand, the power series of fuzzy-valued function $g(x)$ about t_0 take the form

$$g(x) = \sum_{n=0}^{\infty} g_{gH}^{(n)}(t_0) \odot \frac{(x-t_0)^n}{n!} \tag{3.19}$$

then the two series expansion in Equation (3.18) and (3.19) converge to the same function $g(x)$. Therefore

$$D(\sum_{n=0}^{\infty} \frac{1}{\Gamma(n\alpha+1)} \odot {}_{gH}D_*^{n\alpha} f(t_0) \odot (x-t_0)^n, \sum_{n=0}^{\infty} g_{gH}^{(n)}(t_0) \odot \frac{(x-t_0)^n}{n!}) = 0$$

this means

$$\begin{aligned} {}_{gH}\underline{D}_*^{n\alpha} f(t_0) &= \frac{\Gamma(n\alpha+1)}{n!} \odot {}_{gH}\underline{g}^{(n)}(t_0), \\ {}_{gH}\overline{D}_*^{n\alpha} f(t_0) &= \frac{\Gamma(n\alpha+1)}{n!} \odot {}_{gH}\overline{g}^{(n)}(t_0), \end{aligned}$$

4 Proposed Method

The idea of this method is to look for the solution in the form of a fuzzy power series, where coefficients of the series must be determined. In this section, we consider a few examples that demonstrate the performance and efficiency of the our method for solving fuzzy differential equations with fractional order.

Example 4.1 *We consider the fuzzy fractional differential equation, as follows:*

$$({}_{gH}D_*^{\alpha}y)(t) \quad = \quad f(t), \quad t > 0, \qquad (4.20)$$

where we suppose that $0 < \alpha < 1$.

We assume that the fuzzy-valued function $f(t)$ can be expand in the fuzzy power series. Then

$$f(t) = \sum_{n=0}^{\infty} c_n \odot t^{n\alpha} \qquad (4.21)$$
$$= c_0 \oplus c_1 \odot t^{\alpha} \oplus c_2 \odot t^{2\alpha} \oplus \cdots,$$

we look for the solution of the equation (4.20) in the form of the following power series

$$y(t) = \sum_{n=0}^{\infty} \beta_n \odot t^{n\alpha} \qquad (4.22)$$

Taking into account the formula in Theorem (3.1) we note that

$${}_{gH}D_*^{\alpha}y(t) \qquad (4.23)$$
$$= \sum_{n=1}^{\infty} \beta_n \odot \frac{\Gamma(n\alpha+1)}{\Gamma((n-1)\alpha+1)} t^{(n-1)\alpha},$$

Substituting the expressions (4.22) and (4.23) in (4.20) and by using definition of metric D in (2.1) we have

$$D({}_{gH}D_*^{\alpha}y(t), f(t)) = 0$$

therefore

$${}_{gH}\underline{D_*^{\alpha}}y(t,r) = \underline{f}(t,r),$$
$${}_{gH}\overline{D_*^{\alpha}}y(t,r) = \overline{f}(t,r),$$

this means

$$\sum_{n=1}^{\infty} \underline{\beta}_n(r).\frac{\Gamma(n\alpha+1)}{\Gamma((n-1)\alpha+1)} t^{(n-1)\alpha} = \sum_{n=0}^{\infty} \underline{c}_n(r).t^{n\alpha}$$

$$\sum_{n=0}^{\infty} \underline{\beta}_{n+1}(r).\frac{\Gamma((n+1)\alpha+1)}{\Gamma(n\alpha+1)} t^{n\alpha} = \sum_{n=0}^{\infty} \underline{c}_n(r).t^{n\alpha}$$

By comparison of the coefficient of both series gives:

$$\begin{aligned}
\underline{\beta}_0(r) &= \underline{y}_0(r), \\
\underline{\beta}_1(r) &= \frac{\Gamma(1)}{\Gamma(\alpha+1)}\underline{c}_0(r), \\
\underline{\beta}_2(r) &= \frac{\Gamma(\alpha+1)}{\Gamma(2\alpha+1)}\underline{c}_1(r), \\
&\vdots \\
\underline{\beta}_{n+1}(r) &= \frac{\Gamma(n\alpha+1)}{\Gamma((n+1)\alpha+1)}\underline{c}_n(r)
\end{aligned}$$

by similarly way we obtain:

$$\begin{aligned}
\overline{\beta}_0(r) &= \overline{y}_0(r), \\
\overline{\beta}_1(r) &= \frac{\Gamma(1)}{\Gamma(\alpha+1)}\overline{c}_0(r), \\
\overline{\beta}_2(r) &= \frac{\Gamma(\alpha+1)}{\Gamma(2\alpha+1)}\overline{c}_1(r), \\
&\vdots \\
\overline{\beta}_{n+1}(r) &= \frac{\Gamma(n\alpha+1)}{\Gamma((n+1)\alpha+1)}\overline{c}_n(r)
\end{aligned}$$

therefore, under the above assumptions the solution of the equation (4.20), $y(t) = (\underline{y}(t,r), \overline{y}(t,r))$ is

$$\underline{y}(t,r) = \underline{y}_0(r) + \sum_{n=1}^{\infty} \frac{\Gamma((n-1)\alpha+1)}{\Gamma(n\alpha+1)}\underline{c}_{n-1}(r)t^{n\alpha}$$
$$\overline{y}(t,r) = \overline{y}_0(r) + \sum_{n=1}^{\infty} \frac{\Gamma((n-1)\alpha+1)}{\Gamma(n\alpha+1)}\overline{c}_{n-1}(r)t^{n\alpha}$$

Example 4.2 *[4] Consider Fractional Relaxation equation*

$$({}_{gH}D_*^{0.5}y)(x) = y(x), \qquad (4.24)$$

with fuzzy initial value $y(0) = (r, 2-r)$.

We consider fuzzy solution $y(x)$ as follows:

$$y(x) = \sum_{n=0}^{\infty} \beta_n \odot x^{n\alpha} \qquad (4.25)$$

therefore

$${}_{gH}D_*^{0.5}y(x) \qquad (4.26)$$
$$= \sum_{n=1}^{\infty} \beta_n \odot \frac{\Gamma(1+n\alpha)}{\Gamma(1+(n-1)\alpha)} x^{(n-1)\alpha}$$

by substituting (4.25) and (4.26) in (4.24) we have

$$D(\sum_{n=0}^{\infty} \beta_{n+1} \odot \frac{\Gamma(1+(n+1)\alpha)}{\Gamma(1+n\alpha)} x^{n\alpha},$$
$$\sum_{n=0}^{\infty} \beta_n \odot x^{n\alpha}) = 0,$$

by comparison of the coefficient of both series, we have

$$\begin{aligned} at \quad k=0, \quad & \underline{\beta}_1(r) = \frac{\underline{\beta}_0(r)}{\Gamma(1+\alpha)}, \\ & \overline{\beta}_1(r) = \frac{\overline{\beta}_0(r)}{\Gamma(1+\alpha)}, \\ at \quad k=1, \quad & \underline{\beta}_2(r) = \frac{\underline{\beta}_0(r)}{\Gamma(1+2\alpha)}, \\ & \overline{\beta}_2(r) = \frac{\overline{\beta}_0(r)}{\Gamma(1+2\alpha)}, \\ \vdots \\ at \quad k=n, \quad & \underline{\beta}_{n+1}(r) = \frac{\Gamma(1+n\alpha)}{\Gamma(1+(n+1)\alpha)}\underline{\beta}_0(r), \\ & \overline{\beta}_{n+1}(r) = \frac{\Gamma(1+n\alpha)}{\Gamma(1+(n+1)\alpha)}\overline{\beta}_0(r), \end{aligned}$$

the general fuzzy solution is

$$\underline{y}(x,r) = \underline{y}_0(r)E_{0.5}(x^{0.5}),$$

$$\overline{y}(x,r) = \overline{y}_0(r)E_{0.5}(x^{0.5})$$

the fuzzy solution is $[i-gH]$-differentiable and this is shown in figure 1.

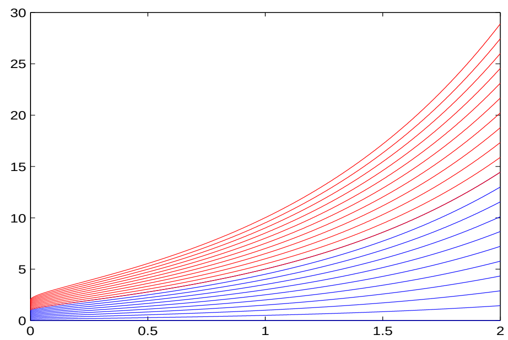

Figure 1: Fuzzy solution for example 4.2

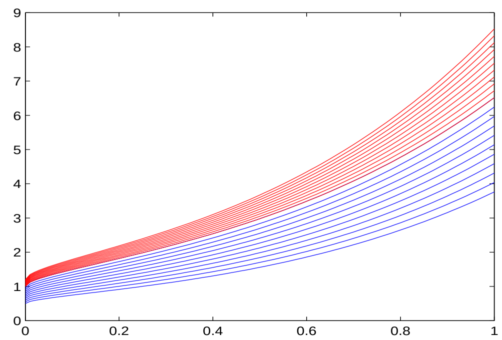

Figure 2: The fuzzy solution of Example 4.3

Example 4.3 *[4] Consider Fractional Relaxation equation*

$$({}_{gH}D_*^{0.5}y)(x) = \lambda y(x) \oplus c \odot x^2, \qquad (4.27)$$
$$y(0) = y_0 \in R_F$$

where

$$y(0) = (0.5+0.5r, 1.2-0.2r),$$
$$c = (1+0.2r, 2-0.8r),$$

Let $\lambda = 1$, we consider the fuzzy solution $y(x)$ as follows

$$y(x) = \sum_{n=0}^{\infty} \beta_n \odot x^{n\alpha} \qquad (4.28)$$

therefore

$$\begin{aligned} & {}_{gH}D_*^{0.5}y(x) \qquad (4.29) \\ & = \sum_{n=1}^{\infty} \beta_n \odot \frac{\Gamma(1+n\alpha)}{\Gamma(1+(n-1)\alpha)} x^{(n-1)\alpha} \end{aligned}$$

By putting (4.28) and (4.29) in (4.27) we have

$$\begin{aligned} D(\sum_{n=0}^{\infty} \beta_{n+1} \odot \frac{\Gamma(1+(n+1)\alpha)}{\Gamma(1+n\alpha)} x^{n\alpha}, \\ \sum_{n=0}^{\infty} \beta_n \odot x^{n\alpha} \oplus c \odot x^{4\alpha}) = 0, \end{aligned}$$

therefore

$$\begin{aligned} & \sum_{n=0}^{\infty} \underline{\beta}_{n+1}(r) \odot \frac{\Gamma(1+(n+1)\alpha)}{\Gamma(1+n\alpha)} x^{n\alpha} \\ & = \sum_{n=0}^{\infty} \underline{\beta}_n(r) \odot x^{n\alpha} \oplus \underline{c}(r) \odot x^{4\alpha} \end{aligned}$$

and

$$\begin{aligned} & \sum_{n=0}^{\infty} \overline{\beta}_{n+1}(r) \odot \frac{\Gamma(1+(n+1)\alpha)}{\Gamma(1+n\alpha)} x^{n\alpha} \\ & = \sum_{n=0}^{\infty} \overline{\beta}_n(r) \odot x^{n\alpha} \oplus \overline{c}(r) \odot x^{4\alpha} \end{aligned}$$

$\underline{\beta}_0(r) = \underline{y}_0(r),$

$\overline{\beta}_0(r) = \overline{y}_0(r),$

$\underline{\beta}_1(r) = \frac{\underline{\beta}_0(r)}{\Gamma(1+\alpha)},$

$\overline{\beta}_1(r) = \frac{\overline{\beta}_0(r)}{\Gamma(1+\alpha)},$

$\underline{\beta}_2(r) = \frac{\underline{\beta}_0(r)}{\Gamma(1+2\alpha)},$

$\overline{\beta}_2(r) = \frac{\overline{\beta}_0(r)}{\Gamma(1+2\alpha)},$

$\underline{\beta}_3(r) = \frac{\underline{\beta}_0(r)}{\Gamma(1+3\alpha)},$

$\overline{\beta}_3(r) = \frac{\overline{\beta}_0(r)}{\Gamma(1+3\alpha)},$

$\underline{\beta}_4(r) = \frac{\underline{\beta}_0(r)}{\Gamma(1+4\alpha)},$

$\overline{\beta}_4(r) = \frac{\overline{\beta}_0(r)}{\Gamma(1+4\alpha)},$

$\underline{\beta}_5(r) = \underline{c}(r).\frac{\Gamma(1+4\alpha)}{\Gamma(1+5\alpha)}.\frac{\underline{\beta}_0(r)}{\Gamma(1+5\alpha)},$

$\overline{\beta}_5(r) = \overline{c}(r).\frac{\Gamma(1+4\alpha)}{\Gamma(1+5\alpha)}.\frac{\overline{\beta}_0(r)}{\Gamma(1+5\alpha)},$

$\underline{\beta}_6(r) = \underline{c}(r).\frac{\Gamma(1+4\alpha)}{\Gamma(1+6\alpha)}.\frac{\underline{\beta}_0(r)}{\Gamma(1+6\alpha)},$

$\overline{\beta}_6(r) = \overline{c}(r).\frac{\Gamma(1+4\alpha)}{\Gamma(1+6\alpha)}.\frac{\overline{\beta}_0(r)}{\Gamma(1+6\alpha)},$

$\vdots$

$\underline{\beta}_n = \underline{c}(r).\frac{\Gamma(1+4\alpha)}{\Gamma(1+n\alpha)}.\frac{\underline{\beta}_0(r)}{\Gamma(1+n\alpha)},$

$\overline{\beta}_n = \overline{c}(r).\frac{\Gamma(1+4\alpha)}{\Gamma(1+n\alpha)}.\frac{\overline{\beta}_0(r)}{\Gamma(1+n\alpha)}, \quad n \geq 5,$

The general fuzzy solution is as follows

$$\begin{aligned}\underline{y}(x,r) &= \underline{y}_0(r).(1+\frac{x^{\alpha}}{\Gamma(1+\alpha)} \\ &+ \frac{x^{2\alpha}}{\Gamma(1+2\alpha)} + \cdots + \frac{x^{n\alpha}}{\Gamma(1+n\alpha)} + \cdots) \\ &+ \underline{c}(r).\Gamma(1+4\alpha)(\frac{x^{5\alpha}}{\Gamma(1+5\alpha)} \\ &+ \frac{x^{6\alpha}}{\Gamma(1+6\alpha)} + \cdots + \frac{x^{n\alpha}}{\Gamma(1+n\alpha)} + \cdots),\end{aligned} \tag{4.30}$$

and

$$\begin{aligned}\overline{y}(x,r) &= \underline{y}_0(r).(1+\frac{x^{\alpha}}{\Gamma(1+\alpha)} \\ &+ \frac{x^{2\alpha}}{\Gamma(1+2\alpha)} + \cdots + \frac{x^{n\alpha}}{\Gamma(1+n\alpha)} + \cdots) \\ &+ \underline{c}(r).\Gamma(1+4\alpha)(\frac{x^{5\alpha}}{\Gamma(1+5\alpha)} \\ &+ \frac{x^{6\alpha}}{\Gamma(1+6\alpha)} + \cdots + \frac{x^{n\alpha}}{\Gamma(1+n\alpha)} + \cdots),\end{aligned} \tag{4.31}$$

the fuzzy solution is $[i - gH]$-differentiable and this is shown in figure 2.

5 Conclusion

In this paper we generalized fuzzy power series to fractional fuzzy power series in sense of Caputo derivatives. Then we used fuzzy fractional power series for solving fractional fuzzy differential equation.

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