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Interval Malmquist Productivity Index in DEA

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Abstract

Data envelopment analysis is a method for evaluating the relative efficiency of a collection of decision making units. The DEA classic models calculate each unit's efficiency in the best condition, meaning that finds a weight that the DMU is at its maximum efficiency. In this paper, utilizing the directional distance function model in the presence of undesirable outputs, the efficiency of each unit has been calculated in the best and worst condition and an efficiency interval for each DMU is designated and then with aid from these efficiency interval, we present an interval for each unit with a proportionate Malmquist productivity index, that these intervals indicate the progression or regression of each DMU.

Keywords: Interval data, Directional distance function, Undesirable output, Malmquist productivity index.

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1. Introduction

The data envelopment analysis is a non-parametric method for evaluation of efficiency that is utilized for evaluation of relative efficiency and performance of a single collection of comparable beings. These comparable beings are called decision making units which produces outputs with consumption of inputs. This method considers a frontier function around the input and output factors. This frontier includes linear sectors that provides not only the most efficient action units but also an analysis for inactive units. For the first time Farrell (1957), conducted the evaluation of a non-parametric method for evaluating efficiency. The data envelopment analysis was the subject of Rhodes study. The results of the preliminary studies which were conducted with cooperation of Cooper and Charnes, was published in the year of 1978 (Charnes et al., 1978). Their paper under the subject of CCR was generalization of Farrell's work into multi- inputs and multi-outputs for determination of the efficiency of decision making units efficiency with planning optimization method. The DEA technique was developed in the year of 1984 by Charnes et al., (1978) (Banker et al., 1984). The data envelopment analysis models are divided into two groups with the input and output oriented. In models with input oriented, by stabilization of outputs, the inputs would decrease and in models with output oriented, with stabilization of inputs, the outputs would increase. Returns to scale is also a concept which defines variation ratio of inputs and outputs. Variation ratio could be constant or variable, i.e. it could be increasing or decreasing. Advantages of this method include that it could enter multiple input and output in the analysis without no judgment, identify the improvement place for inefficient units, consider the input and output with different measuring units and also

separate the economic inefficiency to the technical and professional inefficiency. The disadvantages of this method include the possibility of reducing the power of model with increase in the number of inputs and outputs in relative to the number of units, Lack of structural determination to achieve the goals and determine the relative efficiency with respect to sample units under review. In the DEA models, an inefficient DMU could upgrade its efficiency with increasing the output levels (the results collected) or by reducing the input levels (resources used). But in the real world it is possible that we have two outputs in the forms of desirable and undesirable. For instance, air pollution is an undesirable output in the companies' production matters. One of the first studies about utilizing undesirable output in evaluation of efficiency is done by Pittman (1983). This study developed the model by Caves et al., (1982), to perform the evaluation of desirable and undesirable outputs. For instance, Maghbouli et al., (2014), evaluated the efficiency of 39 airports in Spain with consideration of undesirable outputs. Dong (2013), provided a complete ranking from the DMUs in the presence of undesirable outputs. Khoshandam et al., (2015), determined a method for determining the final substitute rate in the data envelopment analysis with presence of undesirable outputs. Eliat et al., (2008), proposed a multi-criteria approach for evaluating R&D evaluation projects. They combined the DEA and BSC models. They used the AHP method for evaluating value of balanced point cards. Asosheh et al., (2010) conducted the rating of IT projects with combination of data envelopment analysis and balanced point card. For defining the evaluation criteria for the project, they used the balanced point card as a main structure and also a model for defining the project's character with taking into account the main and

descriptive numbers. Vinter et al., (2006) considered a collection of inputs and outputs for each project using the DEA method and also a three step algorithm for reducing the number of inputs and outputs, they have tried to compare the relative efficiency of the projects. They focused on the output combination of projects. Chen and Lin (2006) conducted a case study for evaluating the effectiveness of R&D related with 52 integrated semiconductor companies located in the industry and science park of SinCho in Taiwan using the DEA approach. They evaluated efficiency of these 52 companies using the CCR model and calculated the technical and comparable efficiency using the BCC model. The results showed that the R&D performance was different amongst the evaluated companies and most of the inefficient companies had to increase their economic scales. Barry and Mathieu (2008) tested and developed a model for evaluating the efficiency of the software projects with open resources in their studies. They performed the evaluation of projects by using the data envelopment analysis model and definition of a collection of inputs and outputs. Lu et al., (2008) conducted the comparison of relative efficiency of projects using the CCR model with input nature and also definition of a collection of inputs and outputs. They used three inputs and three outputs in their paper for each DMU. Chen et al., (2010), in their studies with definition of effective indicators and using the data envelopment analysis method tried to evaluate the systems for delivering the present projects in China. Zhong et al., (2011) used the data envelopment analysis method with input nature of CRS and VRS with consideration of two inputs and three outputs for evaluation of the efficiency of R&D investment of the industrial

companies from 30 provinces in China based on the first economic official census of China. Ghapanchi et al., (2012) applied the data envelopment analysis method for choosing the best sample of IT projects. They used models with uncertainty due to the uncertainty of available data in variables and interactions between projects.

First time the rate of progression and regression of each decision making unit presented with the idea of dividing efficiency in present time to the efficiency in the past .due to numerous problems, the Malmquist productivity index which shows the progression and regression of decision making units. Malmquist (1998) proposed an index which is the basic of Caves et al., (1982) works, for creating the productivity index. Currently, the measuring and analysis of efficiency variation, are highly considered especially amongst the researchers who conduct analysis of units' efficiency. Maniadakis et al., (2004) suggested a more generalized index which was a more generalized Malmquist index. The productivity index is utilizable when the cost of inputs is determined and the goal of producers is to reduce the costs. In this manner, they developed the productivity index that considers not only the technical efficiency and technological alterations but also the discount efficiency and impact of alterations of input costs should be considered. Fuentes and Lillo-Banuls (2015), by utilizing the DEA and Malmquist, conducted the research of productivity of tax offices in Spain between 2004 to 2006. Alimohammadlu and Mohammadi (2016) computed productivity of 20 cement firms by the Malmquist productivity index. Thanassoulis et al., (2015) created a criterion for comparing the group efficiency of operational units in costs at times that the input costs are reachable.

Falavigna et al., (2017) for evaluating the impact of politics with the goal of judicial efficiency, used the Malmquist productivity index.

The optimization process has been started in uncertain circumstances in the late 1950 's and progressed with high velocity in theoretical and also in algorithm aspects. Many approaches exist for optimization in uncertain circumstances. Now uncertainty in data could be as an interval. For instance, in meteorology matters we are forced to prediction data in interval manner. So because the data are interval and have upper and lower bounds, the relative efficiency is also related to one of the interval that upper bound of efficiency is the optimistic efficiency and lower bound of efficiency is the pessimistic efficiency. Many researches have been studied in different matters. Wanke et al., (2016), to cope with the uncertainty of the data, used a fuzzy approach and evaluated the efficiency of the banks. Aghayi et al., (2016) have presented a model with common set of weights based on robust optimization. Mashayekhi and Omrani (2016) presented the fuzzy approach in sorting the genetic algorithm with uncertain data. Hosseinzadeh Lotfi et al., (2007) proposed a method to evaluate cost efficiency and also Malmquist productivity index with interval data. Aghayi (2017), by using the DEA conducted the measuring of cost efficiency with fuzzy data. Toloo et al., (2008) conducted the measurement of overall profit efficiency using interval data. Hatami-Marbini et al., (2014) computed the banks' efficiency by interval data. Salehpour and Aghayi (2015) calculated the most revenue efficiency using the uncertain prices. Aghayi and Maleki (2016) evaluated the efficiency of 50 branches of banks of Ardebil in Iran using the directional distance function with uncertainty of data by two interval and robust approaches.

Aghayi (2016) measured revenue efficiency of DMUs with fuzzy and undesirable data. Hosseinzadeh Lotfi et al., (2007) ranked bank branches using DEA and Interval data.

The sections of the study are as follows: section 2 gives the Malmquist productivity index based on directional distance function model considering the undesirable outputs with certain data and in section 3, a model is introduced for calculating the Malmquist productivity index based on directional distance function in the presence of undesirable outputs with interval data. In section 4, a numerical example is explained for showing the application of proposed methods and finally the conclusion section is expressed.

2. Malmquist productivity index with undesirable outputs

Assume that there are n DMUs with fix input and s desirable output and l undesirable output. Also

$y_j = (y_{1j}, \dots, y_{sj})$ and $b_j = (b_{1j}, \dots, b_{lj})$ should be assumed as the desirable and undesirable output vector corresponding to DMU_j , the following model is presented by Zanella et al., (2015) for evaluating the efficiency of DMU_o that the o is the under evaluation unit.

$$\theta_o^* = \min - \sum_{r=1}^s y_{ro} u_r + \sum_{k=1}^l b_{ko} d_k + v$$

$$s.t. \quad \sum_{r=1}^s g_y u_r + \sum_{k=1}^l g_b d_k = 1, \quad (1a)$$

$$- \sum_{r=1}^s y_{rj} u_r + \sum_{k=1}^l b_{kj} d_k + v \geq 0,$$

$$j = 1, \dots, n, \quad (2a)$$

$$u_r \geq 0, \quad r = 1, \dots, s,$$

$$d_k \geq 0, \quad k = 1, \dots, l.$$

The above model is in output oriented and is constant returns to scale. Also model (1) is based on directional distance function g that are considered as

$g = (-g_b, g_y) = (-b_{ko}, y_{ro})$. In fact, the goal of model (1) is decreasing the undesirable outputs and increasing desirable outputs. In model (1), d_k, u_r are the weights corresponding to the desirable and undesirable outputs, respectively. In model (1), we have $v \in R$ because of the fix inputs.

Definition 1: In model (1), if the objective function value is equal to zero i.e., that $\theta_o^* = 0$ then DMU_o is efficient. If not so, the DMU_o is inefficient.

Definition 2: The efficiency measure of DMU_o , in model (1), is calculated by

$$\phi_o^* = \frac{1}{1 + \theta_o^*}. \text{ Hence if } \phi_o^* = 1 \text{ then}$$

DMU_o is efficient. If $\phi_o^* < 1$ then DMU_o is inefficient.

In this part, we introduce the model for calculating the Malmquist productivity index based on the directional distance function in the presence of undesirable outputs. The Malmquist productivity index is based on using the distance functions. We define distance functions according to two different time periods as of $\theta_o^t(x^{t+1}, y^{t+1})$ and $\theta_o^{t+1}(x^t, y^t)$ in which the θ_o^{t+1} is related to the distance function by the frontier at time of $t + 1$ and (x^{t+1}, y^{t+1}) are the input and output at time of $t + 1$. The function $\theta_o^{t+1}(x^t, y^t)$ evaluates the input-output observed at $t + 1$ in related to the technology period at time t . The distance functions for a single input-output of a determined year in related to frontier in that year is shown with $\theta_o^t(x^t, y^t)$ and $\theta_o^{t+1}(x^{t+1}, y^{t+1})$ for years of t and $t + 1$. So the directional distance function with

undesirable outputs with the Malmquist productivity is presented as follows:

$$\begin{aligned} &\theta_o^p(y_o^p, b_o^p | p = t, t + 1) = \\ &\min - \sum_{r=1}^s y_{ro}^p u_r + \sum_{k=1}^l b_{ko}^p d_k + v \\ &s.t. \quad \sum_{r=1}^s g_y u_r + \sum_{k=1}^l g_b d_k = 1, \\ &\quad - \sum_{r=1}^s y_{rj}^p u_r + \sum_{k=1}^l b_{kj}^p d_k + v \geq 0, \quad (2) \\ &\quad j = 1, \dots, n, \\ &\quad u_r \geq 0, \quad r = 1, \dots, s, \\ &\quad d_k \geq 0, \quad k = 1, \dots, l. \end{aligned}$$

$$\begin{aligned} &\theta_o^p(y_o^q, b_o^q | q, p = t, t + 1, p \neq q) = \\ &\min - \sum_{r=1}^s y_{ro}^q u_r + \sum_{k=1}^l b_{ko}^q d_k + v \\ &s.t. \quad \sum_{r=1}^s g_y u_r + \sum_{k=1}^l g_b d_k = 1, \\ &\quad - \sum_{r=1}^s y_{rj}^p u_r + \sum_{k=1}^l b_{kj}^p d_k + v \geq 0, \quad (3) \\ &\quad j = 1, \dots, n, \\ &\quad u_r \geq 0, \quad r = 1, \dots, s, \\ &\quad d_k \geq 0, \quad k = 1, \dots, l. \end{aligned}$$

In model (2), DMU is at time $t + 1$ and the frontier is at t , b^p, y^p are the desirable and undesirable output matrixes of the observed data, respectively, in p period. Hence, model (2) is solved for $p = t$ and $p = t + 1$. Also in model (3), the DMU is at time t and frontier is at time $t + 1$, it is solved by $q, p = t, t + 1, p \neq q$.

Definition 3: In models (2) and (3), if the objective function value is equal to zero then DMU_o is efficient, otherwise it is inefficient.

In calculating the Malmquist index, if the returns to scale is constant then only two efficiency growth source are separable (the EFCH and TCH). Mostly in calculation of Malmquist efficiency index the geometrical average of these two sources are used. But when the variable returns to scale is used, in addition to the two sources, the PTECH and scale efficiency (SECH) would also be considered. The Malmquist productivity index defined as follows that is based on the variable returns to scale and undesirable outputs at times t or t+1 and also it is like the method proposed by Ray and Disesly (1997).

$$M_o = \sqrt{\frac{\phi_v^t(y_o^{t+1}, b_o^{t+1}) \times \phi_v^{t+1}(y_o^{t+1}, b_o^{t+1})}{\phi_v^t(y_o^t, b_o^t) \times \phi_v^{t+1}(y_o^t, b_o^t)}} \times \sqrt{\frac{SE^t(y_o^{t+1}, b_o^{t+1}) \times SE^{t+1}(y_o^{t+1}, b_o^{t+1})}{SE^t(y_o^t, b_o^t) \times SE^{t+1}(y_o^t, b_o^t)}} \quad (4)$$

Where ϕ_v^t evaluates the productivity growth amongst times t and t+1 using technology at time t and ϕ_v^{t+1} measures this value using the technology at time t+1 as the source technology in the VRS. SE^t and SE^{t+1} are the scale efficiency when the frontier is at times t and t+1,

respectively, and are calculated as follows:

$$SE^t(y_o^{t+1}, b_o^{t+1}) = \frac{\phi_c^t(y_o^{t+1}, b_o^{t+1})}{\phi_v^t(y_o^{t+1}, b_o^{t+1})},$$

$$SE^{t+1}(y_o^{t+1}, b_o^{t+1}) = \frac{\phi_c^{t+1}(y_o^{t+1}, b_o^{t+1})}{\phi_v^{t+1}(y_o^{t+1}, b_o^{t+1})} \quad (5)$$

$$SE^t(y_o^t, b_o^t) = \frac{\phi_c^t(y_o^t, b_o^t)}{\phi_v^t(y_o^t, b_o^t)},$$

$$SE^{t+1}(y_o^t, b_o^t) = \frac{\phi_c^{t+1}(y_o^t, b_o^t)}{\phi_v^{t+1}(y_o^t, b_o^t)}$$

ϕ_c^t and ϕ_c^{t+1} calculate the productivity growth at times t and t+1 with the CRS. The results are described as follows:

- 1: If $M_o > 1$, it shows the progression or increase of productivity.
- 2: If $M_o < 1$, it shows the reduction of productivity.
- 3: If $M_o = 1$ it reflects the in-changeability of productivity during two times.

The developed pattern of general view of desirable and undesirable outputs could be seen in figure (1).

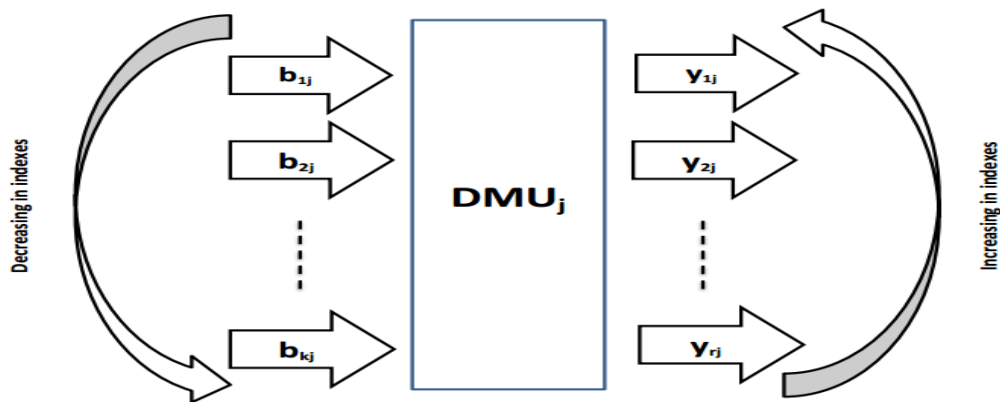


Figure1: Pattern of the desirable and undesirable outputs in the Malmquist productivity index

3. Malmquist productivity index with undesirable outputs and interval data

Assume there are n *DMUs* with with fix input and s desirable interval output and l undesirable interval output.

$\bar{y}_j = (\bar{y}_{1j}, \dots, \bar{y}_{sj})$ and $\bar{b}_j = (\bar{b}_{1j}, \dots, \bar{b}_{lj})$ are desirable and undesirable outputs corresponding to *DMU_j*, respectively.

So that $\bar{y}_j \in [y_j^L, y_j^U]$ and $\bar{b}_j \in [b_j^L, b_j^U]$. In fact, y_j^L and b_j^L are the lower bounds of desirable and undesirable outputs of *DMU_j*, respectively, y_j^U and b_j^U are upper bounds of desirable and undesirable outputs of *DMU_j*. Hence, we have:

$y_j^L \leq y_j^U$ and $b_j^L \leq b_j^U$. The following model is presented for calculating the Malmquist productivity index with uncertain data that o is the index of the under evaluation unit.

$$\begin{aligned} & \bar{\theta}_o^p(\bar{y}_o^p, \bar{b}_o^p | p = t, t + 1) = \\ & \min - \sum_{r=1}^s \bar{y}_{ro}^p u_r + \sum_{k=1}^l \bar{b}_{ko}^p d_k + v \\ & s.t. \quad \sum_{r=1}^s g_y u_r + \sum_{k=1}^l g_b d_k = 1, \\ & \quad - \sum_{r=1}^s \bar{y}_{rj}^p u_r + \sum_{k=1}^l \bar{b}_{kj}^p d_k + v \geq 0, \\ & \quad j = 1, \dots, n, \\ & \quad u_r \geq 0, \quad r = 1, \dots, s, \\ & \quad d_k \geq 0, \quad k = 1, \dots, l. \end{aligned} \tag{6}$$

$$\begin{aligned} & \bar{\theta}_o^p(y_o^q, b_o^q | q, p = t, t + 1, p \neq q) = \\ & \min - \sum_{r=1}^s y_{ro}^q u_r + \sum_{k=1}^l b_{ko}^q d_k + v \end{aligned} \tag{7}$$

$$\begin{aligned} & s.t. \quad \sum_{r=1}^s g_y u_r + \sum_{k=1}^l g_b d_k = 1, \\ & \quad - \sum_{r=1}^s y_{rj}^p u_r + \sum_{k=1}^l b_{kj}^p d_k + v \geq 0, \\ & \quad j = 1, \dots, n, \\ & \quad u_r \geq 0, \quad r = 1, \dots, s, \\ & \quad d_k \geq 0, \quad k = 1, \dots, l. \end{aligned}$$

Whereas, in the (6) and (7) models, the output and input parameters alter in an interval, it is anticipated that the efficiency changes in an interval. For this purpose, we obtain a lower bound and upper bound for the efficiency score of *DMUs*. When the goal is calculation the lower bound of efficiency score, we set the under evaluation *DMU* at its worst condition. Therefore, the following model below is presented for measuring the efficiency of *DMU_o* in the pessimistic case:

$$\begin{aligned} & \theta_o^{Lp}(y_o^{Lp}, b_o^{Up} | p = t, t + 1) = \\ & \min - \sum_{r=1}^s y_{ro}^{Lp} u_r + \sum_{k=1}^l b_{ko}^{Up} d_k + v \\ & s.t. \quad \sum_{r=1}^s g_y u_r + \sum_{k=1}^l g_b d_k = 1, \\ & \quad - \sum_{r=1}^s y_{rj}^{Up} u_r + \sum_{k=1}^l b_{kj}^{Lp} d_k + v \geq 0, \\ & \quad j = 1, \dots, n, j \neq o \\ & \quad - \sum_{r=1}^s y_{ro}^{Lp} u_r + \sum_{k=1}^l b_{ko}^{Up} d_k + v \geq 0, \\ & \quad u_r \geq 0, \quad r = 1, \dots, s, \\ & \quad d_k \geq 0, \quad k = 1, \dots, l. \end{aligned} \tag{8}$$

$$\begin{aligned} & \theta_o^{Lp}(y_o^{Lq}, b_o^{Uq} | q, p = t, t + 1, p \neq q) = \\ & \min - \sum_{r=1}^s y_{ro}^{Lq} u_r + \sum_{k=1}^l b_{ko}^{Uq} d_k + v \end{aligned} \tag{9}$$

$$\begin{aligned}
 s.t. \quad & \sum_{r=1}^s g_y u_r + \sum_{k=1}^l g_b d_k = 1, \\
 & -\sum_{r=1}^s y_{rj}^{Up} u_r + \sum_{k=1}^l b_{kj}^{Lp} d_k + v \geq 0, \\
 & \quad j = 1, \dots, n, j \neq o, \\
 & -\sum_{r=1}^s y_{ro}^{Lq} u_r + \sum_{k=1}^l b_{ko}^{Uq} d_k + v \geq 0, \\
 & u_r \geq 0, \quad r = 1, \dots, s, \\
 & d_k \geq 0, \quad k = 1, \dots, l.
 \end{aligned}$$

Definition 4: If the $\theta_o^{Lp} = 0$, in models (8) and (9), then DMU_o is efficient in pessimistic condition. Otherwise it is inefficient.

Therefore, the lower bound of Malmquist productivity index is calculated through the below equation:

$$\begin{aligned}
 M^L = & \sqrt{\frac{\phi_v^{Lt}(y_o^{t+1}, b_o^{t+1})}{\phi_v^{Ut}(y_o^t, b_o^t)} \times \frac{\phi_v^{Lt+1}(y_o^{t+1}, b_o^{t+1})}{\phi_v^{Ut+1}(y_o^t, b_o^t)}} \times \\
 & \sqrt{\frac{SE^{Lt}(y_o^{t+1}, b_o^{t+1})}{SE^{Ut}(y_o^t, b_o^t)} \times \frac{SE^{Lt+1}(y_o^{t+1}, b_o^{t+1})}{SE^{Ut+1}(y_o^t, b_o^t)}} \quad (10)
 \end{aligned}$$

Similarity, in the optimistic case, the under evaluation DMU should be put at its best case and other DMUs at the worst case. The below model is presented for measuring the efficiency of DMU_o in optimistic case based on directional distance function:

$$\begin{aligned}
 & \theta_o^{Up}(y_o^{Up}, b_o^{Lp} | p = t, t + 1) = \\
 \min \quad & -\sum_{r=1}^s y_{ro}^{Up} u_r + \sum_{k=1}^l b_{ko}^{Lp} d_k + v \\
 s.t. \quad & \sum_{r=1}^s g_y u_r + \sum_{k=1}^l g_b d_k = 1, \\
 & -\sum_{r=1}^s y_{rj}^{Lp} u_r + \sum_{k=1}^l b_{kj}^{Up} d_k + v \geq 0, \\
 & \quad j = 1, \dots, n, j \neq o,
 \end{aligned} \quad (11)$$

$$\begin{aligned}
 & -\sum_{r=1}^s y_{ro}^{Up} u_r + \sum_{k=1}^l b_{ko}^{Lp} d_k + v \geq 0, \\
 & u_r \geq 0, \quad r = 1, \dots, s, \\
 & d_k \geq 0, \quad k = 1, \dots, l. \\
 & \theta_o^{Up}(y_o^{Uq}, b_o^{Lq} | q, p = t, t + 1, p \neq q) = \\
 \min \quad & -\sum_{r=1}^s y_{ro}^{Uq} u_r + \sum_{k=1}^l b_{ko}^{Lq} d_k + v \\
 s.t. \quad & \sum_{r=1}^s g_y u_r + \sum_{k=1}^l g_b d_k = 1, \\
 & -\sum_{r=1}^s y_{rj}^{Lp} u_r + \sum_{k=1}^l b_{kj}^{Up} d_k + v \geq 0, \\
 & \quad j = 1, \dots, n, j \neq o, \\
 & -\sum_{r=1}^s y_{ro}^{Uq} u_r + \sum_{k=1}^l b_{ko}^{Lq} d_k + v \geq 0, \\
 & u_r \geq 0, \quad r = 1, \dots, s, \\
 & d_k \geq 0, \quad k = 1, \dots, l.
 \end{aligned} \quad (12)$$

Definition 5: If $\theta_o^{Up} = 0$, in models (11) and (12), then DMU_o is efficient in the optimistic case. Otherwise it is inefficient.

Now the high border for Malmquist productivity index criterion would be as follows:

$$\begin{aligned}
 M^U = & \sqrt{\frac{\phi_v^{Ut}(y_o^{t+1}, b_o^{t+1})}{\phi_v^{Lt}(y_o^t, b_o^t)} \times \frac{\phi_v^{Ut+1}(y_o^{t+1}, b_o^{t+1})}{\phi_v^{Lt+1}(y_o^t, b_o^t)}} \times \\
 & \sqrt{\frac{SE^{Ut}(y_o^{t+1}, b_o^{t+1})}{SE^{Lt}(y_o^t, b_o^t)} \times \frac{SE^{Ut+1}(y_o^{t+1}, b_o^{t+1})}{SE^{Lt+1}(y_o^t, b_o^t)}} \quad (13)
 \end{aligned}$$

Theorem 1: Prove $M^L \leq M^U$.

Proof: We show $\theta_o^{Up} \leq \theta_o^{Lp}$. So we have to prove that the objective function value of models (8) and (9) are less than the objective function value of models (11) and (12), i.e., $\theta_o^{Lp} - \theta_o^{Up} \geq 0$. The objective function value of the optimistic and pessimistic models are considered as follows:

$$\theta_o^{Lp} = -\sum_{r=1}^s u_r y_{ro}^{Lp} + \sum_{k=1}^l d_k b_{ko}^{Up} + v, \tag{14}$$

$$\theta_o^{Up} = -\sum_{r=1}^s u_r y_{ro}^{Up} + \sum_{k=1}^l d_k b_{ko}^{Lp} + v,$$

By calculation $\theta_o^{Lp} - \theta_o^{Up}$, we have:

$$\begin{aligned} \theta_o^{Lp} - \theta_o^{Up} &= \sum_{r=1}^s u_r (y_{ro}^{Up} - y_{ro}^{Lp}) + \\ &\sum_{k=1}^l d_k (b_{ko}^{Up} - b_{ko}^{Lp}) + v - v, \end{aligned} \tag{15}$$

Considering that $u_r, d_k \geq 0$ and $y_{ro}^{Up} \geq y_{ro}^{Lp}$ and $b_{ko}^{Up} \geq b_{ko}^{Lp}$, equation (15) would be a positive. So $\theta_o^{Up} \leq \theta_o^{Lp}$ and it also evident that $\phi_o^{Lp} \leq \phi_o^{Up}$. Now if we consider the upper and lower bounds as follows:

$$M^U = \sqrt{\frac{\phi_v^{Ut}(y_o^{t+1}, b_o^{t+1})}{\phi_v^{Lt}(y_o^t, b_o^t)} \times \frac{\phi_v^{Ut+1}(y_o^{t+1}, b_o^{t+1})}{\phi_v^{Lt+1}(y_o^t, b_o^t)}} \times$$

$$\begin{aligned} M^L &= \sqrt{\frac{\phi_v^{Lt}(y_o^{t+1}, b_o^{t+1})}{\phi_v^{Ut}(y_o^t, b_o^t)} \times \frac{\phi_v^{Lt+1}(y_o^{t+1}, b_o^{t+1})}{\phi_v^{Ut+1}(y_o^t, b_o^t)}} \times \\ &\sqrt{\frac{SE^{Lt}(y_o^{t+1}, b_o^{t+1})}{SE^{Ut}(y_o^t, b_o^t)} \times \frac{SE^{Lt+1}(y_o^{t+1}, b_o^{t+1})}{SE^{Ut+1}(y_o^t, b_o^t)}} \\ &\sqrt{\frac{SE^{Ut}(y_o^{t+1}, b_o^{t+1})}{SE^{Lt}(y_o^t, b_o^t)} \times \frac{SE^{Ut+1}(y_o^{t+1}, b_o^{t+1})}{SE^{Lt+1}(y_o^t, b_o^t)}} \end{aligned} \tag{16}$$

So, with considering $\phi_o^{Lp} \leq \phi_o^{Up}$, it is obvious that $M^L \leq M^U$.

Conclusion 1: If M_o^L and M_o^U be the lower and upper bounds of the Malmquist productivity index, respectively, then $\bar{M}_o \in [M_o^L, M_o^U]$.

4. Sample example

Assume there exists 5 DMUs with two desirable and undesirable interval outputs at times t and t+1 that the data are presented in tables 1 and 2. The results of models (2), (3), (8), and (9) are presented in table 3.

Table 1: The desirable outputs at times t and t+1

DMU	y_j^{lt}	y_j^{lt+1}	y_j^{ut}	y_j^{ut+1}
1	7	12	11	15
2	8	13	12	14
3	16	21	22	23
4	15	20	25	25
5	34	39	35	41

Table 2: The undesirable outputs at times t and t+1

DMU	b_j^{lt}	b_j^{lt+1}	b_j^{ut}	b_j^{ut+1}
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1	8	13	11	17
2	16	21	21	26
3	25	30	37	35
4	42	47	44	52
5	20	25	22	30

Table 3: The results

<i>DMU</i>	M_j	$[M_j^L, M_j^U]$
1	1.18	[1.02,1.21]
2	1.41	[1.08,1.19]
3	1.64	[1.01,1.04]
4	1.32	[1.04,1.07]
5	1	[1,1.01]

Considering table 3, we have increase of productivity for all DMUs in the optimistic condition and there are also progresses in pessimistic case. For DMU_5 , the productivity is increased in the optimistic condition but there is no alteration in the pessimistic condition.

in addition, the results from the proposed model are evaluated in a simple numerical example.

5. Conclusion

In this paper, a combinational approach of data envelopment analysis and Malmquist productivity index is presented for evaluating the efficiency of decision making units using the directional distance function with undesirable interval outputs. The calculation of Malmquist index is used for comparing the units' productivity in distant time periods. Advantages of this index include that it is applicable in areas of producing multi inputs and multi outputs and it is easy to computerize. In this paper, we consider undesirable outputs same as inputs. First we calculated the Malmquist productivity index of each unit in its best and worst condition and then an interval is determined for Malmquist productivity index of each unit. The intervals that obtained show the progression or regression of each unit. In

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