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Effect of uncontrollable factors on constant returns-to-scale production technologies

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Abstract
This article discusses the effect of uncontrollable factors on constant returns-to-scale production technologies in data envelopment analysis(DEA). First we show that the Banker and Morey CRS technology is not convex and it is nevertheless a suitable reference technology for the assessment of scale efficiency and achieves model by the incorporation of variant inputs and outputs in data envelopment analysis. Second, we propose that if changes axiom returns to scale of production possibility set, model will secure variant another model with uncontrollable factors. Therefore returns to scale properties very depending on the manner in which uncontrollable factors are treated and approaches can be compared both theoretically and empirically.

Keywords: Data envelopment analysis; Uncontrollable factors; Controllable factors; Returns to scale.

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1. Introduction

Data envelopment analysis is nonparametric technique for evaluation relative efficiency of decision making units. An important properties that lead to increasing extend this science, ability for assessment relative efficiency of decision making units with multiple inputs and outputs by classic models of DEA. Constant (CRS) and variable (VRS) returns to scale models of data envelopment analysis presented by Charnes et al [1], Banker [2], do not take into account uncontrollable inputs and outputs. Thus this factors exclude model or treats as normal controllable factors. Banker and Mory [3], (below referred to as the BM model) extended this approaches and suggested CRS and VRS model when uncontrollable factors. This model can be considered standard DEA model contain of uncontrollable inputs and presented one stage models that factors is fully controllable or fully uncontrollable. Golany and Roll [5], extended the approach BM [3], and to account for both uncontrollable inputs and uncontrollable outputs simultaneously. In contrast with the BM model, the approaches of Lovell and Ruggiero [6], assumed that uncontrollable factors on convex space can not affect. Approaches introduce by Serjanen [4], Muniz [7], Yang and Pollitt [8], and Lober and Staat [9]. In these case, BM CRS model is based should be examined. Then number of different models and approaches introduced that those are based on different assumption and axioms. This differences have necessary in practical application. Even though uncontrollable, it is important to take account of such factors in a manner that is reflected in the measures of efficiency used. In this article examine CRS technology of Banker and Mory that constraints contain controllable and uncontrollable factors. We show that the BM CRS technology is not convex and it is nevertheless a suitable reference technology for the assessment of scale efficiency. To study that if change Ray unboundedness, achieves another model with uncontrollable factors. Frequently the returns to scale properties depending when treats uncontrollable factors and model analyse both theoretically based on formulate axioms, and empirically based on examples.

2. Background of DEA

Data envelopment analysis branch of operations research that evaluation decision making units with multiple factors by apply mathematic models. In classic models of DEA assume that all inputs and outputs can be varied at the discretion of management or other users. But exist in realistic situation uncontrollable factors not subject to management control, may also need to be consider for the sake exact evaluation relative efficiency of units. Suppose their exist n; DMU_s that nonnegative input vectors $x_j=(x_{1j},x_{2j},\dots,x_{mj})$ products nonnegative output vectors $y_j=(y_{1j},y_{2j},\dots,y_{sj})$, where $j=1,2,\dots,n$. Number of input factors is uncontrollable, objective evaluation technical efficiency DMU_o that $o\in\{1,2,\dots,n\}$ by the following model:

$$
\begin{array}{ll}
min\theta-\varepsilon(\sum_{r=1}^{s}s_r^{+}+\sum_{i\in D}s_i^{-}) & \\
s.t\sum_{j=1}^{n}\lambda_j x_{ij}+s_i^{-}=\theta x_{io}, & i\in D\\
\sum_{j=1}^{n}\lambda_j x_{ij}+s_i^{-}=x_{io}, & i\in ND\\
\sum_{j=1}^{n}\lambda_j y_{rj}-s_r^{+}=y_{ro}, & r=1,2,\dots,s\\
\lambda_j\geq 0, & j=1,2,\dots,n\\
s_i^{-}\geq 0, & i=1,2,\dots,m\\
s_r^{+}\geq 0, & r=1,2,\dots,s\\
\theta is free. &
\end{array}
\qquad (1)
$$

Model (1) known CCR model of Banker and Morey that symbols D and ND refer to the set controllable and uncontrollable input respectively. To be noted in the uncontrollable constraints variable θ is not possible to vary them at the discretion of user. Not that the slack $s_i^{-}, i\in ND$ are omitted from the objective. Hence these input do not enter directly the efficiency measures and effect in calculate efficiency via constraints. For further clear this subject that how uncontrollable variable effect the efficiency score, writing the dual of problem (1) in the from of the following model:

$$
\begin{aligned}
& max \sum_{r=1}^{s} u_r y_{ro} - \sum_{i \in ND} v_i x_{io} \\
& s.t \sum_{r=1}^{s} u_r y_{rj} - \sum_{i \in ND} v_i x_{ij} - \sum_{i \in D} v_i x_{ij} \le 0, \\
& j = 1,2,\dots,n \qquad (2) \\
& \sum_{i \in D} v_i x_{io} = 1, \\
& u_r \ge \varepsilon, \quad r = 1,2,\dots,s \\
& v_i \ge 0, \;\; i \in ND \\
& v_i \ge \varepsilon, \quad i \in D.
\end{aligned}
$$

Observe, the uncontrollable inputs enter into the objective of model (2). If negative number of $s_i^{-*}, i \in ND$ into constraint model (1) then according slackness complementarity theorem $v_i^* = 0$ and this x_{io} does not effect the evaluation model (1). On the other hand if $v_i^* < 0$ for $i \in ND$ frequently the efficiency score is reduced for DMU_o. This subject concludes from following relation that illustrate equality value objective function principle and dual in optimal:

$$[\theta^* - \varepsilon(\sum_{r=1}^{s} s_r^{+*} + \sum_{i \in D} s_i^{-*}) = \sum_{r=1}^{s} u_r^* y_{ro} - \sum_{i \in ND} v_i^* x_{io}] \qquad (3)$$

Thus uncontrollable inputs only when effect into efficiency that $v_i^* > 0$ for $i \in ND$.

3. The non convexity of the Banker and Morey CRS technology

Consider a production technology contain n decision making unite that each unite to have m inputs and s outputs. Any unit in this technology represented in the form $(X, Y) = (X_D, X_F, Y_D, Y_F)$, where X_D, X_F, Y_D, Y_F are, respectively vector of controllable and uncontrollable inputs, and controllable and uncontrollable outputs. Places $\bar{X}_D, \bar{X}_F, \bar{Y}_D, \bar{Y}_F$ be columns matrices respectively, controllable and uncontrollable parts of the vectors X_j and Y_j that $j = 1,2,\dots,n$. The Banker and Morey CRS technology show in form T_{CRS}^{BM}, set of nonnegative unite (X_D, X_F, Y_D, Y_F) for which there exists a vector $\lambda \in R_n^+$ and in working order following relation:

$$
\begin{aligned}
& \bar{Y}_D \lambda \ge Y_D, \quad (a) \\
& \bar{X}_D \lambda \le X_D, \quad (b) \\
& \bar{Y}_F \lambda \ge Y_F, \quad (c) \qquad (4) \\
& \bar{X}_F \lambda \le X_F, \quad (d).
\end{aligned}
$$

By the following example shows that technology T_{CRS}^{BM} is not convex. First its idea to study by units A and B then to generate subject for another unite by simple analysis. Consider 4 units in Table 1 and let output Y and input X_1 be controllable, and let input X_2 be uncontrollable.

Example 1. Suppose the technology T_{CRS}^{BM} contain only two observed units A and B shown in Table 1. The units C in Table 1 obtain from A by multiply uncontrollable and controllable factor of 2. Observe that $C \in T_{CRS}^{BM}$ because by places factors of units C in condition (a), (b)and (d):

$$
\begin{aligned}
& 2\lambda_A + 1\lambda_B \ge 4, \\
& 1\lambda_A + 2\lambda_B \le 2, \qquad (5) \\
& 2\lambda_A + 1\lambda_B \le 4.
\end{aligned}
$$

Table 1: Unite in Example 1

Unite	Output Y	Input X_1	Fixed input X_2
A	2	1	2
B	1	2	1
C	4	2	4
D	2.5	2	1

From solve (5) to derive a result that $\lambda_A = 2$ and $\lambda_B = 0$ therefor $C \in T_{CRS}^{BM}$. Consider unit D, substitution its in the (a), (b), and (d) obtain:

$$2\lambda_A + 1\lambda_B \geq 2.5,$$
$$1\lambda_A + 2\lambda_B \leq 2, \quad (6)$$
$$2\lambda_A + 1\lambda_B \leq 1.$$

Conclude that $\lambda_A = 1.5$ and $\lambda_B = -.5$. Then $D \not\in T_{CRS}^{BM}$ and technology T_{CRS}^{BM} is not convex.

4. Scale efficiency

Scale efficiency is ratio of the overall efficiency of the unite in the CRS technology to its technical efficiency in VRS technology. Measure the scale efficiency of efficient units is always smaller or equal to one and for inefficient units is equal to the scale efficiency of their radial projections on the boundary efficiency. In the standard manner and presence of uncontrollable factors measuring scale efficiency only with respect to controllable factors and application technology for evaluation to suppose the BM CRS model despite before that employed convex technology. Question arises as to whether the use of the latter model leads to the correct evaluation of scale efficiency? Under shows that the use of the BM CRS technology instance reference technology for assessment the scale efficiency is fully justified. As a proof of this, we careful that in the standard case contain only controllable input and output, instead the scale efficiency of an efficient unit (X_o, Y_o) in technology T, to define optimal value h^o in the model applied by Banker [2] for the assessment of the most productive scale size:

$$h^o = min\{\frac{\beta}{\alpha} | (\beta X_o, \alpha Y_o) \in T, \alpha, \beta > 0\} (7)$$

Suppose their exist n, DMU_s that each containing m, input $X_j = (x_{1j}, x_{2j}, \dots, x_{mj})$ and s, output $Y_j = (y_{1j}, y_{2j}, \dots, y_{sj})$ for $j = 1,2, \dots, n$. Number of component input and output is uncontrollable. We wish assess scale efficiency of DMU_o that $o \in \{1,2, \dots, n\}$ with use definition (7) and apply it to the VRS technology of Banker and Morey [3], consider only the units that operate in the same or more demanding environment compared to the unit (X_o, Y_o). Taking into assumption obtains the following model:

$$\begin{aligned} & minh^o = \frac{\beta}{\alpha} \\ & s.t \bar{Y}_D \lambda \geq \alpha (Y_o)_D, \\ & \bar{X}_D \lambda \leq \beta (X_o)_D, \\ & \bar{Y}_F \lambda \geq (Y_o)_F, \\ & \bar{X}_F \lambda \leq (X_o)_F, \\ & \lambda 1 = 1, \\ & \lambda \geq 0, \alpha, \beta \geq 0. \end{aligned} \quad (8)$$

By dividing both sides of the constraint in the above model by $(1^T\lambda)$ and using the transformation of variables $\hat{\lambda} = \frac{\lambda}{(1^T\lambda)}$, $1 = \frac{\alpha}{(1^T\lambda)}$ and $\theta = \frac{\beta}{(1^T\lambda)}$ we achieve the following formulation:

$$\begin{aligned} & minh^o = \theta, (a) \\ & s.t \bar{Y}_D \hat{\lambda} \geq (Y_o)_D, (b) \\ & \bar{X}_D \hat{\lambda} \leq \theta (X_o)_D, (c) \\ & \bar{Y}_F \hat{\lambda} \geq (Y_o)_F / (1^T\lambda), (d) \\ & \bar{X}_F \hat{\lambda} \leq (X_o)_F / (1^T\lambda), (e) \\ & 1 = 1/(1^T\lambda), (f) \\ & \hat{\lambda} \geq 0, \theta \geq 0, (g). \end{aligned} \quad (9)$$

By replacing constraint (f-9) into the right hand side of (d-9) and (e-9) we observe that model (a-9) - (e-9) and (g-9) becomes the input oriented BM CRS model:

$$\begin{aligned} & BM_{CRS} = min\theta \\ & s.t \bar{Y}_D \hat{\lambda} - (Y_o)_D \geq 0, \\ & -(\bar{X}_D \hat{\lambda}) + \theta (X_o)_D \geq 0, \\ & \bar{Y}_F \hat{\lambda} - (Y_o)_F \geq 0, \\ & -(\bar{X}_F \hat{\lambda}) + (X_o)_F \geq 0, \\ & \hat{\lambda} \geq 0, \theta \geq 0. \end{aligned} \quad (10)$$

Hence make use of non convex BM CRS technology is a correct reference technology for the assessment of scale efficiency. If we divide in model [8] both side of the constraints by $(1^T\lambda)$ and using the transformation of variable $\hat{\lambda} = \frac{\lambda}{(1^T\lambda)}$,

$1 = \frac{\beta}{(1^T\lambda)}$ and $\varphi = \frac{\alpha}{(1^T\lambda)}$, by taking in to nonlinear programming we would obtain following model:

$$\begin{aligned} & max\varphi \\ & s.t\bar{Y}_D\hat{\lambda} - \varphi(Y_o)_D \geq 0, \\ & -(\bar{X}_D\hat{\lambda}) + (X_o)_D \geq 0, \\ & \bar{Y}_F\hat{\lambda} - (Y_o)_F \geq 0, \\ & -(\bar{X}_F\hat{\lambda}) + (X_o)_F \geq 0, \\ & \hat{\lambda} \geq 0, \varphi \geq 0. \end{aligned} \quad (11)$$

Model (11) is the output oriented BM CRS model.

5. Another constant returns to scale model with uncontrollable factors

We are able to get the following postulates describing the production possibility set of model (10) $T = \{(X,Y)|Y \geq 0 \text{ can be product from } X \geq 0\}$:

Axiom 1. Inclusion of observation: For any $j = 1,2,\dots,n; (X_j, Y_j) \in T$.

Axiom 2. In efficiency: Free disposability: If $(X,Y) \in T, Y \geq \tilde{Y} \geq 0$ and $X \leq \tilde{X}$ then $(\tilde{X}, \tilde{Y}) \in T$.

Axiom 3. Convexity: If $(X,Y) \in T$ and $(\tilde{X}, \tilde{Y}) \in T$, then for any scalar $\lambda \in [0,1], (\lambda(\tilde{X}, \tilde{Y}) + (1-\lambda)(X,Y)) \in T$.

Axiom 4. Ray unboundedness: Constant return to scale: If $(X,Y) \in T$ then $(X^\alpha, Y^\alpha) = (\alpha X_D, \alpha X_F, \alpha Y_D, \alpha Y_F) \in T$ for any $\alpha > 0$.

Axiom 5. Closedness: Technology T is a closed set.

To observed the constant return to scale stated base on controllable and uncontrollable factors, and scaler α to multiply all factors. Consequently production possibility set of model BM_{CRS} is similar to the CCR model, where moreover normal factors included uncontrollable factors. Also, the axioms behind the production possibility set are similar and BM_{CRS} model always lead to a lower or equal efficiency score than CCR model. BM_{CRS} model, to be accepted standard model and Cooper [10] examined this type of model in their textbook and Ruggiero compares this approach with others. If between 5 axiom above change Ray unboundedness to the following from, will obtain another constant returns to scale model.

Axiom 4^+. Ray unboundedness with controllable input and output:
If $(X,Y) \in T$ then $(X^\alpha, Y^\alpha) = (\alpha X_D, X_F, \alpha Y_D, Y_F) \in T$ for any $\alpha > 0$.

$$\begin{aligned} & BM_{+CRS} = min\theta \\ & s.t\bar{Y}_D\hat{\lambda} - (Y_o)_D \geq 0, \\ & -(\bar{X}_D\hat{\lambda}) + \theta(X_o)_D \geq 0, \\ & \bar{Y}_F\hat{\lambda} - (Y_o)_F(1^T\hat{\lambda}) \geq 0, \\ & -(\bar{X}_F\hat{\lambda}) + (X_o)_F(1^T\hat{\lambda}) \geq 0, \\ & \hat{\lambda} \geq 0, \theta \geq 0. \end{aligned} \quad (12)$$

This CRS model obtained hereafter called the original BM CRS model.

In the returns to scale properties of model (12) do not multiplying all the input and output by any positive scalar, then BM_{+CRS} is model that treats similar variable returns to scale. To observe from the models and the postulates,there are clear differences between the models and differences have important practical implications. These are one stage DEA model that factors to be either fully controllable or fully uncontrollable. So far approaches be compared theoretically based on formulated axioms, at this moment we wish explains effect circumstance difference in the assumption behind the models on efficiency analysis in practice. Namely we use a simple examples to illustrate the difference between formulations and interpretation of controllable factors.

Example 2. Consider data set consists of consists of 4 DMU_s, which are described by one output, one input and uncontrollable input. These DMU_s and their efficiencies based on the constant return to scale models(10) and(12) are introduced in Table 2.

Unite	Output Y	Input X_1	Fixed inputX_2	BM_{CRS}	BM_{+CRS}
A	6	7	6	1	1
B	4	5	4	.933	.968
C	2	3	2	.778	1
D	7	9	8	.907	.907

Table2: Illustrative data set 2 and efficiencies based on different model BM_{CRS} = model (10), BM_{+CRS} = model (12)

When we apply the BM_{CRS} model (10) for assessment existing data in Table 2, DMU A becomes the reference unit for all the other units, and slacks in the model are zero. In assessment units by BM_{+CRS}, DMU C becomes efficient and DMU A is reference units B and D. Differences model in the interpretation of uncontrollable factors when apply the same models to another data set, the values of the uncontrollable input are modified. The data and the efficiencies are presented in Table 3.

Unite	Output Y	Input X_1	Fixed inputX_2	BM_{CRS}	BM_{+CRS}
A	6	7	2	1	1
B	4	5	2	.933	.933
C	2	3	2	.778	.788
D	7	9	2	1	.907

Table3: Illustrative data set 3 and efficiencies based on different model BM_{CRS} = model (10), BM_{+CRS} = model (12)

In this case assessment units by BM_{CRS} model resulted that unit A is reference for DMU B and DMU C, but does not such for DMU D. The efficiency scores of these units are the same with data set 2, but the number of slack are positive. Thus despite the equal values of the uncontrollable input, the units operate different manner in environments. When we utilized BM_{+CRS} model, DMU A become references another units and efficiency score units are equal model (10) in the case of data set 2. Thus all the units operate in similar environment. We shall analyse the relationships of efficiency score from different models. As can be seen that BM_{CRS} and BM_{+CRS} model lead to an efficiency score higher than or equal to the DEA model without any uncontrollable input. Furthermore the results comparison example show that it is impossible find a general the relationship between BM_{CRS} and BM_{+CRS} models. This subject illustrated by the examples above, i.e DMU B and DMU C in data set 2 and DMU D in data set 3.

Then we conclude that no exist unique ranking for the efficiency scores when different models are applied to the same data and difference models are in the interpretation of uncontrollable inputs. In the manner constant returns to scale BM model (12) allow no scaling in the uncontrollable input. Thus we consider that the uncontrollable input are independent of scale on the other hand, units in similar environments and different scales have the same value for the indicator. Then uncontrollable inputs can be analysed separately from the other factor in the model. BM_{CRS} model treats is similar to the CCR model and all the inputs, outputs and uncontrollable inputs be scale dependent. This means that the value of the indicator for units with similar environment varies relative to the scale of operation. Thus value of the uncontrollable factor does not directly determine, it analyses relative to another inputs and outputs indicator measure on the ratio scale.

6. Conclusion

In this paper, we considered that CRS technology offered Banker and Morey [3] with uncontrollable factors is not convex and nevertheless a suitable reference technology for the assessment of scale efficiency. Analysed that if modified Ray unboundedness, obtain difference returns to scale model called the original BM model then returns to scale properties depending on the manner in which uncontrollable factors are treated. Models have been compared both theoretically and empirically. Comparison these shows in the manner empirically that there is no unique ranking of the efficiency scores when the models are applied to the same data. Difference between approaches in the interpretation uncontrollable factors. If we suppose returns to scale properties of the model are based on both the controllable and uncontrollable factors, approach that suggested by Banker and Morey model (10) be the best choice for assessment units contain factors and these values determines dependent on the scale of operation. In the case the definition of constant returns to scale only includes the controllable factors, original approach of Banker and Morey is suitable and the values of the uncontrollable factors are compared independently of the scale of operation.

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