



Common weights for the evaluation of decision-making units with nonlinear virtual inputs and outputs

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Abstract

In this paper, by investigating the common weights concept and DEA models with nonlinear virtual inputs/outputs, we introduce a model for evaluating the decision making units with nonlinear virtual inputs and outputs based on the common weights.

Keywords: Data envelopment analysis, Common weights, Nonlinear virtual inputs/outputs.

1. Introduction

Data envelopment analysis (DEA) obtains the efficiency measurement of decision making units (DMUs) with multiple inputs/outputs based on the ratio of weighted outputs to weighted inputs, where each DMU can take the desirable weights for inputs/outputs to provide the maximum performance. But the evaluation of DMUs based on the selection of different set of weights is unacceptable.

Therefore, some researchers by linking DEA with some other techniques such as multi-objective programming, have developed models to generate a set of common weights for evaluating and ranking the DMUs (see Roll and Golany [9], Doyle [6], Bylton and Vickres [1], Lee and Reeves [8], Kao and hung [7], Chen et al [2] and Zohrehbandian et al [10]).

In addition, Cook et al [3] and Cooper et al [4] have assumed linear input/output models in DEA as unreal properties and by considering Piecewise linear form for them, they tried to provide a model for evaluating DMUs in more realistic situations. Moreover, Despotis et al [5] have proposed a general formula by considering nonlinear virtual inputs/outputs in DEA models.

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In this paper, a common weight model with nonlinear virtual inputs/outputs will be presented, which is the idea of the combination of common weights models and nonlinear virtual inputs/outputs concept. The rest of the paper is as follows. In section 2, nonlinear virtual inputs/outputs and common weight concepts will be reviewed. Section 3 presents a common weight model with nonlinear virtual inputs/outputs. Section 4 includes numerical example while section 5 is devoted to concluding remarks

2. Preliminaries and notations

2.1 - Model with nonlinear virtual inputs/outputs

Suppose we have n DMUs with m inputs and s outputs. CCR model to assess the performance unit o is as the following form:

$$\begin{aligned} \text{Max} \quad & \sum_{r=1}^s u_r y_{ro} \\ \text{s.t.} \quad & \sum_{i=1}^m v_i x_{io} = 1, \\ & \sum_{r=1}^s u_r y_{rj} - \sum_{i=1}^m v_i x_{ij} \leq 0 \quad (j = 1, \dots, n) \\ & v \geq 0, u \geq 0 \end{aligned}$$

When $Y_j = (y_{1j}, y_{2j}, \dots, y_{sj})$ and $X_j = (x_{1j}, x_{2j}, \dots, x_{mj})$ are vectors of inputs/outputs for unit j . In this notation, the weighted sum of inputs and outputs can be seen in the following form:

$$\sum_{r=1}^s u_r y_{rj} = U(Y_j) \quad , \quad \sum_{i=1}^m v_i x_{ij} = V(X_j)$$

But by removing the assumption of linearity of these functions, approximating the output vector $Y_j = (y_{1j}, y_{2j}, \dots, y_{sj})$ can be done based on the following aggregate function, where U_1, U_2, \dots, U_s are nonlinear functions:

$$U(Y_j) = U_1(Y_{1j}) + U_2(Y_{2j}) + \dots + U_s(Y_{sj})$$

Moreover, piecewise linear approximation of function U_r ($r = 1, \dots, s$) can also be achieved as follows. Suppose that $[l_r, h_r]$ is range of output r for all decision units ($l_r = \min_j \{y_{rj}\}$, $h_r = \max_j \{y_{rj}\}$).

For each r , the output interval $[l_r, h_r]$ with breakpoints $b_r^1, b_r^2, \dots, b_r^k, b_r^{k+1}, \dots, b_r^{a_r}$ can be segmentation where for any $y_{rj} > l_r$ there is exactly one k_j such that $y_{rj} \in [b_r^{k_j}, b_r^{k_j+1}]$. In other words, y_{rj} can be decomposed as follows:

$$y_{rj} = b_r^1 + (b_r^2 - b_r^1) + (b_r^3 - b_r^2) + \dots + (b_r^{k_j} - b_r^{k_j-1}) + (y_{rj} - b_r^{k_j})$$

Now, instead of considering a variable weight for output r for wide range of $[l_r, h_r]$, it will be allocated by different weighting factors for each sub-interval $[b_r^k, b_r^{k+1}]$ $k = 2, \dots, a_r - 1$ and the replacement:

$$\begin{aligned} \gamma_{r1}^1 &= b_r^1, \gamma_{r2}^1 = (b_r^2 - b_r^1), \gamma_{r3}^1 = (b_r^3 - b_r^2), \dots, \gamma_{rk_j}^1 = (b_r^{k_j} - b_r^{k_j-1}), \gamma_{rk_j+1}^1 = (y_{rj} - b_r^{k_j}), \gamma_{rk_j+2}^1 \\ &= 0, \dots, \gamma_{ra_r}^1 = 0 \end{aligned}$$

Then, amounts of function $U_r(Y_{rj})$ for each $y_{rj} \in [b_r^k, b_r^{k+1}]$ is obtained as follows:

$$U_r(Y_{rj}) = (\gamma_{r1}^1 + \gamma_{r2}^1)u_{r1} + \gamma_{r3}^1 u_{r2} + \dots + \gamma_{rk}^1 u_{rkj-1} + \gamma_{rkj+1}^1 u_{rkj} + \gamma_{rkj+2}^1 u_{rkj+1} + \dots + \gamma_{ra_r}^1 u_{ra_r-1}$$

By writing the above equation for each $r = 1, \dots, s$, and gathering on the r , virtual output for $U(Y_j)$ of unit j is a linear function of $u = (u_{11}, \dots, u_{1,a_1-1}, \dots, u_{r1}, \dots, u_{ra_r-1}, \dots, u_{s1}, \dots, u_{sa_s-1})$ as follows:

$$U(Y_j) = \sum_{r=1}^s [(\gamma_{r1}^j + \gamma_{r2}^j)u_{r1} + \sum_{k=3}^{a_r} \gamma_{rk}^j u_{rk-1}]$$

But, probably the assumption of non-linearity is acceptable only for a particular category of the outputs. In this case, call these outputs as non-linear and call others as linear outputs. Without loss of generality, we assume that the sorted output and $(d < s)$ are linear and the rest of them are non-linear. Therefore, the above formula will be changed as follows:

$$U(Y_j) = \sum_{r=1}^d y_{rj} u_r + \sum_{r=d+1}^s \left[(\gamma_{r1}^j + \gamma_{r2}^j)u_{r1} + \sum_{k=3}^{a_r} \gamma_{rk}^j u_{rk-1} \right]$$

Similarly, the virtual inputs can be assumed to have non-linear assumption and similarly by segmentation the final amount $U(X_j)$ for unit j has the following form:

$$U(X_j) = \sum_{i=1}^t x_{ij} v_i + \sum_{i=t+1}^m \left[(\mu_{i1}^j + \mu_{i2}^j)v_{i1} + \sum_{k=3}^{a_i} \mu_{ik}^j v_{ik-1} \right]$$

2.2 Common weight models

Choosing different weights in a DEA model for evaluation of DMUs is unacceptable. Therefore common weights concept was introduced in DEA literature. Most of the proposed methods are based on the solution of a multi-objective model which simultaneously maximizes the efficiency of all DMUs. In this section, we introduce the methods proposed by Kao and Hung [7].

If we show the optimal value produced by the CCR model as E_j^* , which is the best performance value for DMU_j , to produce a common weights, we can assume $E^* = (E_1^*, E_2^*, \dots, E_n^*)$ as an ideal efficiency vector and provide an efficiency vector for DMUs which is the nearest to this ideal. In other words, we want to obtain the efficiency vector $E(u, v) = (E_1(u, v), E_1(u, v), \dots, E_n(u, v))$ by the common set of weights which has the minimum distance from the ideal solution E^* . Hence,

$$\text{Min } D_p = \left[\sum_{j=1}^n (E_j^* - E_j(u, v))^p \right]^{1/p}$$

s.t

$$E_j(u, v) = \frac{\sum_{r=1}^s u_r y_{rj}}{\sum_{i=1}^m v_i x_{ij}} \leq 1 \quad j = 1, \dots, n$$

$$u_r, v_i \geq \varepsilon > 0 \quad r = 1, \dots, s, \quad i = 1, \dots, m$$

3. Common weight model with nonlinear virtual inputs/outputs

As it was explained in the previous section, nonlinear virtual inputs/outputs can be considered as follows:

$$U(X_j) = \sum_{i=1}^t x_{ij} v_i + \sum_{i=t+1}^m \left[(\mu_{i1}^j + \mu_{i2}^j) v_{i1} + \sum_{k=3}^{a_i} \mu_{ik}^j v_{ik-1} \right]$$

$$U(Y_j) = \sum_{r=1}^d y_{rj} u_r + \sum_{r=d+1}^s \left[(\gamma_{r1}^j + \gamma_{r2}^j) u_{r1} + \sum_{k=3}^{a_r} \gamma_{rk}^j u_{rk-1} \right]$$

Then, common weight model of Kao and Hung [7] for the case $p = \infty$ using these nonlinear virtual inputs/outputs is as follows:

Min w

s.t

$$E_j^* - \sum_{j=1}^n \left(\frac{\sum_{r=1}^s \hat{u}_r \hat{y}_{rj}}{\sum_{i=1}^m \hat{v}_i \hat{x}_{ij}} \right) \leq w \quad j = 1, \dots, n$$

$$\sum_{r=1}^s \hat{u}_r \hat{y}_{rj} - \sum_{i=1}^m \hat{v}_i \hat{x}_{ij} \leq 0 \quad j = 1, \dots, n$$

$$\hat{u}_r, \hat{v}_i \geq \varepsilon > 0 \quad r = 1, \dots, s, \quad i = 1, \dots, m$$

Where \hat{v} and \hat{u} are inputs and outputs weight vectors, respectively. Hence:

$$\hat{v} = (v_1, \dots, v_t, v_{t+1,1}, \dots, v_{t+1,a_{t+1}-1}, \dots, v_{m1}, \dots, v_{m,a_m-1})$$

$$\hat{u} = (u_1, \dots, u_d, u_{d+1,1}, \dots, u_{d+1,a_{d+1}-1}, \dots, u_{s1}, \dots, u_{s,a_s-1})$$

$$\hat{X} = \begin{pmatrix} x_{11} & x_{12} & \dots & x_{1n} \\ \dots & \dots & \dots & \dots \\ x_{t1} & x_{t2} & \dots & x_{tn} \\ (\mu_{t+1,1}^1 + \mu_{t+1,2}^1) & (\mu_{t+1,1}^2 + \mu_{t+1,2}^2) & \dots & (\mu_{t+1,1}^n + \mu_{t+1,2}^n) \\ \mu_{t+1,3}^1 & \mu_{t+1,3}^2 & \dots & \mu_{t+1,3}^n \\ \dots & \dots & \dots & \dots \\ \mu_{t+1,a_{t+1}-1}^1 & \mu_{t+1,a_{t+1}-1}^2 & \dots & \mu_{t+1,a_{t+1}-1}^n \\ \dots & \dots & \dots & \dots \\ (\mu_{m1}^1 + \mu_{m2}^1) & (\mu_{m1}^2 + \mu_{m2}^2) & \dots & (\mu_{m1}^n + \mu_{m2}^n) \\ \mu_{m3}^1 & \mu_{m3}^2 & \dots & \mu_{m3}^n \\ \dots & \dots & \dots & \dots \\ \mu_{m,a_m-1}^1 & \mu_{m,a_m-1}^2 & \dots & \mu_{m,a_m-1}^n \end{pmatrix}$$

$$\hat{Y} = \begin{pmatrix} y_{11} & y_{12} & \dots & y_{1n} \\ \dots & \dots & \dots & \dots \\ y_{d1} & y_{d2} & \dots & y_{dn} \\ (\gamma_{d+1,1}^1 + \gamma_{d+1,2}^1) & (\gamma_{d+1,1}^2 + \gamma_{d+1,2}^2) & \dots & (\gamma_{d+1,1}^n + \gamma_{d+1,2}^n) \\ \gamma_{d+1,3}^1 & \gamma_{d+1,3}^2 & \dots & \gamma_{d+1,3}^n \\ \dots & \dots & \dots & \dots \\ \gamma_{d+1,a_{d+1}-1}^1 & \gamma_{d+1,a_{d+1}-1}^2 & \dots & \gamma_{d+1,a_{d+1}-1}^n \\ \dots & \dots & \dots & \dots \\ (\gamma_{s1}^1 + \gamma_{s2}^1) & (\gamma_{s1}^2 + \gamma_{s2}^2) & \dots & (\gamma_{s1}^n + \gamma_{s2}^n) \\ \gamma_{s3}^1 & \gamma_{s3}^2 & \dots & \gamma_{s3}^n \\ \dots & \dots & \dots & \dots \\ \gamma_{s,a_s-1}^1 & \gamma_{s,a_s-1}^2 & \dots & \gamma_{s,a_s-1}^n \end{pmatrix}$$

4. Numerical examples

Table 1 presents 5 DMUs, each of which contains 2 inputs and 2 outputs:

Table 1:

Data inputs and outputs

	X ₁	X ₂	Y ₁	Y ₂
DMU A	5	15	60	8
DMU B	10	10	90	4
DMU C	15	5	80	9
DMU D	20	10	90	10
DMU E	7	4	75	6

Here, we consider the first input and first outputs linear and second input and second output as nonlinear. If the input range be [4,15], and we consider breakpoints as: 4,7,10,13,15, then input augment matrix is as follows:

$$\hat{X} = \begin{vmatrix} 5 & 10 & 15 & 20 & 7 \\ 7 & 7 & 5 & 7 & 4 \\ 3 & 3 & 0 & 3 & 0 \\ 3 & 0 & 0 & 0 & 0 \\ 2 & 0 & 0 & 0 & 0 \end{vmatrix}$$

And the weight vector is:

$$\hat{v} = (v_1, v_{1,1}, v_{2,1}, v_{3,1}, v_{4,1})$$

Moreover, for the output range [4.10], by considering the breakpoints 4,6,8,10, the output augment matrix would be as follows:

$$\hat{y} = \begin{vmatrix} 60 & 90 & 80 & 90 & 75 \\ 6 & 4 & 6 & 6 & 6 \\ 2 & 0 & 2 & 2 & 0 \\ 0 & 0 & 1 & 2 & 0 \end{vmatrix}$$

And the weight vector is:

$$\hat{u} = (u_1, u_{1,1}, u_{2,1}, u_{3,1})$$

Finally, following table gives CCR efficiency and common weight efficiency values with nonlinear virtual inputs/outputs assumption for each DMU.

Table 2:

Efficiency score

DMU	A	B	C	D	E
CCR efficiency	1	0.83	1	1	1
Common weight efficiency	1	0.75	1	0.73	1

5 Conclusion

This paper presents a common weight model with nonlinear virtual inputs/outputs assumption. Firstly, the DEA models with nonlinear virtual inputs/outputs were introduced. Then common weight models have been introduced. Finally, we proposed the common weight model with nonlinear virtual inputs/outputs based on the combination of these two concepts.

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