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Estimating right and left returns to scales in data envelopment analysis: A new approach

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Abstract

In this research a new returns to scale (RTS) method is proposed to estimate the right and left returns to scales (RTSs) of the frontier decision making units (DMUs) in data envelopment analysis (DEA). This study modifies Golany and Yu's RTS method in such a matter that it always can fit within estimating the right and left returns to scales of efficient DMUs.

It is necessary to say that, since an inefficient decision making unit (DMU) has more than one projection on the empirical frontier function hence, the different right and left returns to scales can be determined for the inefficient DMU by using our proposed RTS method. Then, an illustrative example highlights the method and also the obtained results of the proposed RTS method are compared with Golany and Yu's RTS method. A concluding comment, future extensions and suggest possible future direction of research are all summarized in the last section.

Keywords: Data Envelopment Analysis (DEA), Right and Left Returns to Scale (RTSS), Efficiency

1. Introduction

Data envelopment analysis (DEA) is a non-parametric technique. The relative efficiency of decision making units (DMUs) can be evaluated by DEA [2,8]. The identification of returns to scale (RTS) is one of the most important issues in DEA. We can recognize the optimal size of the DMU under assessment with using the RTS behavior of the unit. So far, some methods have been presented to determine the returns to scale of DMUs in DEA models [6, 11, 12, 15, 17, 18, 22, 23, 24, and 25]. For instance, the concept of most productivity scale size (MPSS) was introduced by Banker [1] that CCR model is used to determine returns to scale by Banker's proposed DEA approach [1].

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Färe and Grosskopf [14] provided an alternative approach for estimating returns to scale based on optimal solutions of BCC, CCR, and CCR-BCC models. In this vein, Banker and Thrall [3] introduced a method for estimating the right and left returns to scales in data envelopment analysis.
Furthermore, a general definition of the right and left returns to scales has been presented by Hadjicostas and Soteriou [16]. Again, Khodabakhshi introduced a method for estimating most productive scale size with stochastic data in DEA [20]. Furthermore, an additive model approach for estimating returns to scale in imprecise DEA was introduced by Khodabakhshi et al. [21].
Golany and Yu [15] have proposed a method to estimate the right and left returns to scales (RTSs) of the DMU under assessment. Their method is not always feasible for all DMUs under assessment. In this current study, a new RTS method is proposed to determine the right and left returns to scales in DEA that it is always feasible for all DMUs under the assessment.
The structure of this paper is organized as follows: Section 2 provides preliminary information that will be used in the succeeding sections. Previous research efforts about the basic DEA models on determining RTS are documented in Section 3. An illustrative example is presented to make comparisons between our proposed approach and Golany and Yu's approach in Section 4. At last, in Section 5 the conclusion and some remarks are put forward.

2. Mathematical preliminaries

2.1. Basic DEA models

Suppose there are n DMUs $\{DMU_j \,|\, j = 1,2,\dots,n\}$ which product s outputs $\mathbf{Y}_j = (y_{1j},\dots,y_{rj},\dots,y_{sj}) \geq \mathbf{0}$ utilizing m inputs $\mathbf{X}_j = (x_{1j},\dots,x_{rj},\dots,x_{mj}) \geq \mathbf{0}$ that $\mathbf{Y}_j \neq \mathbf{0}$ and $\mathbf{X}_j \neq \mathbf{0}$. There are many alternative ways to characteristic the production technology. The most general representation is production possibility set (PPS) which is defined as a set of semipositive $(\mathbf{X},\mathbf{Y})$ as, $PPS = \{(\mathbf{X},\mathbf{Y}) \,|\, \mathbf{X}\, can\ produce\ \mathbf{Y}\}$. There are many models in DEA that the original and important models are CCR, BCC, BCC-CCR, and CCR-BCC models. Note that, they are different in RTS assumptions. In these models, RTS is constant, variable, non-decreasing and non-increasing, respectively. Their PPSs are defined as follows
[28, p. 42]:

$$PPS_{CCR} = \left\{ (\mathbf{X},\mathbf{Y}) \left| \sum_{j=1}^{n} \lambda_j \mathbf{X}_j \leq \mathbf{X} \,,\, \sum_{j=1}^{n} \lambda_j \mathbf{Y}_j \geq \mathbf{Y} \,,\, \lambda_j \geq 0 \,;\, j = 1,2,\dots,n \right. \right\}, \tag{1}$$

$$PPS_{BCC} = \left\{ (\mathbf{X},\mathbf{Y}) \left| (\mathbf{X},\mathbf{Y}) \in PPS_{CCR} \,,\, \sum_{j=1}^{n} \lambda_j = 1 \right. \right\}, \tag{2}$$

$$PPS_{BCC-CCR} = \left\{ (\mathbf{X},\mathbf{Y}) \left| (\mathbf{X},\mathbf{Y}) \in PPS_{CCR} \,,\, \sum_{j=1}^{n} \lambda_j \geq 1 \right. \right\}, \tag{3}$$

$$PPS_{CCR-BCC} = \left\{ (\mathbf{X},\mathbf{Y}) \left| (\mathbf{X},\mathbf{Y}) \in PPS_{CCR} \,,\, \sum_{j=1}^{n} \lambda_j \leq 1 \right. \right\}. \tag{4}$$

Assume that DMU_o $(o \in \{1,\dots,n\})$ is one of the observed DMUs. In order to determine the relative efficiency of DMU_o (θ_o^*) we need to solve an input-oriented model as,

$\theta_o^* = Min\left\{\theta_o \left|\left(\theta_o \mathbf{X}_o, \mathbf{Y}_o\right) \in PPS\right.\right\}$. Thus, CCR, BCC, BCC-CCR, and CCR-BCC models and their dual forms are represented by mathematical linear programming *M(I)* and *M(II)* as follows:

$$M\ (I): \ \theta_o^* = Max \quad \theta_o$$

$$s.t. \ \sum_{r=1}^{s} \lambda_j x_{ij} \le \theta_o x_{io}, \quad i = 1, \dots, m, \tag{5}$$

$$\sum_{i=1}^{m} \lambda_j y_{rj} \ge y_{ro}, \quad r = 1, \dots, s, \tag{6}$$

$$\lambda = \left(\lambda_1, \dots, \lambda_n\right) \in \Lambda, \tag{7}$$

$$M\ (II): \ \theta_o^* = Max \quad \sum_{r=1}^{s} u_r y_{ro} + u_o$$

$$s.t. \ \sum_{i=1}^{m} v_i x_{io} = 1, \tag{8}$$

$$\sum_{r=1}^{s} u_r y_{rj} - \sum_{i=1}^{m} v_i x_{ij} + u_o \le 0, \quad j = 1, \dots, n, \tag{9}$$

$$v_i \ge 0, \quad i = 1, \dots, m, \tag{10}$$

$$u_r \ge 0, \quad r = 1, \dots, s, \tag{11}$$

$$u_o \in \Lambda', \tag{12}$$

where the sets of Λ and Λ' are as follows:

$$\Lambda_{CCR} = \left\{\boldsymbol{\lambda} \left| \lambda_j \ge 0 \ ; \ j = 1, \dots, n \right.\right\}, \tag{13}$$

$$\Lambda'_{CCR} = \left\{u_o \left| u_o = 0 \right.\right\}, \tag{14}$$

$$\Lambda_{BCC} = \left\{\boldsymbol{\lambda} \left| \boldsymbol{\lambda} \in \Lambda_{CCR} \ , \sum_{j=1}^{n} \lambda_j = 1 \right.\right\}, \tag{15}$$

$$\Lambda'_{BCC} = \left\{u_o \left| u_o \ is \ free \right.\right\}, \tag{16}$$

$$\Lambda_{BCC-CCR} = \left\{\boldsymbol{\lambda} \left| \boldsymbol{\lambda} \in \Lambda_{CCR} \ , \sum_{j=1}^{n} \lambda_j \ge 1 \right.\right\}, \tag{17}$$

$$\Lambda'_{BCC-CCR} = \left\{u_o \left| u_o \ge 0 \right.\right\}, \tag{18}$$

$$\Lambda_{CCR-BCC} = \left\{\boldsymbol{\lambda} \left| \boldsymbol{\lambda} \in \Lambda_{CCR} \ , \sum_{j=1}^{n} \lambda_j \le 1 \right.\right\}, \tag{19}$$

$$\Lambda'_{CCR-BCC} = \left\{u_o \left| u_o \le 0 \right.\right\}. \tag{20}$$

Now suppose the feasible region of each of the above models in *M(I)* is denoted by S , then we will have the following relations among the feasible regions:

$$S_{BCC} \subset S_{CCR},\ S_{BCC-CCR} \subset S_{CCR},\ S_{CCR-BCC} \subset S_{CCR},\ S_{CCR} = S_{BCC-CCR} \cup S_{CCR-BCC},\ S_{BCC} = S_{BCC-CCR} \cap S_{CCR-BCC}. \quad (21)$$

Hence, we have:

$$\theta^*_{CCR} \le \theta^*_{BCC},\ \theta^*_{CCR} \le \theta^*_{BCC-CCR},\ \theta^*_{CCR} \le \theta^*_{CCR-BCC},\ \theta^*_{BCC-CCR} \le \theta^*_{BCC},\ \theta^*_{CCR-BCC} \le \theta^*_{BCC}. \quad (22)$$

Note that in this study, the optimal value of variables is shown by superscript "$*$".
Furthermore, it can be shown that $\theta^*_o \le 1$ in all models and also, models *M(I)* and *M(II)* are called envelopment and multiplier forms of related models, respectively.

Definition 1. DMU_o is technically efficient in model *M(I)* if and only if $\theta^*_o = 1$.
Otherwise, DMU_o is called technically inefficient. Thus the projection point of DMU_o is defined as $\left(\theta^*_o \mathbf{X}_o - \mathbf{S}^{-^*}, \mathbf{Y}_o + \mathbf{S}^{+^*}\right)$ which is a technical efficient point in the related model. Note that $\mathbf{S}^{-^*}$ and $\mathbf{S}^{+^*}$ are respectively optimal vectors of slacks, corresponding with (5) and (6) in model *M(I)*.

2.2. Some customary methods for identifying RTS

So far, many methods have been presented to estimate RTS of DMUs in DEA. In this section, three common methods BCC, Färe et al., and Kerstens and Eechkaut are reviewed, respectively [3, 13, and 19]. It is note-worthy to say that, RTS is always determined for points on the BCC-efficient frontier. And also, RTS of inefficient DMUs is defined as that of their BCC-projections, [2].
In the following theorem, BCC model is used to estimate RTS of DMUs.

Theorem 1. *Suppose that* $\left(\mathbf{X}_o, \mathbf{Y}_o\right)$ *is a point on the BCC-efficient frontier. Then, the following conditions identify the situation for RTS at the point:*

(i) *Increasing RTS (IRS) prevail at* $\left(\mathbf{X}_o, \mathbf{Y}_o\right)$ *if and only if* $u^*_o > 0$ *for all optimal solutions of BCC model in multiplier form.*

(ii) *Decreasing RTS (DRS) prevail at* $\left(\mathbf{X}_o, \mathbf{Y}_o\right)$ *if and only if* $u^*_o < 0$ *for all optimal solutions of BCC model in multiplier form.*

(iii) *Constant RTS (CRS) prevail at* $\left(\mathbf{X}_o, \mathbf{Y}_o\right)$ *if and only if* $u^*_o = 0$ *for at least one optimal solution of BCC model in multiplier form.*

Proof. Refer to [3]. □

It is noticeable that, obtaining all optimal solutions of the multiplier form of BCC model can be onerous. In other words, the method is problematic when BCC model in multiplier form has alternative optimal solutions. Hence, this method has been developed by Banker et al. [4], which caused to solve a further model, whereas it is not pursued in this research. Note that, two LPs must be solved to estimate RTS of each DMU by Theorem 1 [5].
Färe and Grosskpof's method [14] is another method to estimate RTS of DMUs that is demonstrated in Theorem 2 as below:

Theorem 2. *Suppose that* $(\mathbf{X}_o, \mathbf{Y}_o)$ *is a point on the BCC-efficient frontier. Then, the following conditions identify the situation for RTS at this point:*

(i) *Increasing RTS (IRS) prevail at* $(\mathbf{X}_o, \mathbf{Y}_o)$ *if and only if* $\theta^*_{CCR} = \theta^*_{CCR-BCC} < \theta^*_{BCC}$ *.

(ii) *Decreasing RTS (DRS) prevail at* $(\mathbf{X}_o, \mathbf{Y}_o)$ *if and only if* $\theta^*_{CCR} < \theta^*_{CCR-BCC} = \theta^*_{BCC}$.

(iii) *Constant RTS (CRS) prevail at* $(\mathbf{X}_o, \mathbf{Y}_o)$ *if and only if* $\theta^*_{CCR} = \theta^*_{CCR-BCC} = \theta^*_{BCC}$.

Proof. Refer to [14]. □

Banker et al. [2] proved the equivalence of this method with other related methods. There is no problem about alternative optimal solutions in Färe and Grosskpof's method. It is obvious that, its computation is expensive because three LP models must be solved to determine RTS of each DMU in the method. In this vein, there are other methods for determining RTS of DMUs such as the CCR method [2] and Kerstens and Eechkaut's method [19]. If there are alternative optimal solutions then the CCR method is problematic. As four LPs must be solved to determine RTS of each DMU, thus computation of Kerstens and Eechkaut's method is problematic, too. Next Kerstens and Eechkaut's method will be discussed in Theorem 3.

Theorem 3. *Suppose that* $(\mathbf{X}_o, \mathbf{Y}_o)$ *is a point on the BCC-efficient frontier. Then, the following conditions identify the situation for RTS at this point:*

(i) *Increasing RTS (IRS) prevail at* $(\mathbf{X}_o, \mathbf{Y}_o)$ *if and only if* $\theta^*_{BCC-CCR} = \textit{strict} \max\{\theta^*_{BCC-CCR}, \theta^*_{CCR-BCC}, \theta^*_{CCR}\}$.

(ii) *Decreasing RTS (DRS) prevail at* $(\mathbf{X}_o, \mathbf{Y}_o)$ *if and only if* $\theta^*_{CCR-BCC} = \textit{strict} \max\{\theta^*_{BCC-CCR}, \theta^*_{CCR-BCC}, \theta^*_{CCR}\}$.

(iii) *Constant RTS (CRS) prevail at* $(\mathbf{X}_o, \mathbf{Y}_o)$ *if and only if* $\theta^*_{CCR} = \max\{\theta^*_{BCC-CCR}, \theta^*_{CCR-BCC}, \theta^*_{CCR}\}$.

Proof. Refer to [19]. □

"Strict max" is defined as $p = \textit{strict} \max\{p, q, r\}$ if and only if $p > q$ and $p > r$, but it is not used in Kerstens and Eechkant's method. Also, the validity of this method is described in Kerstens and Eechkant method [19]. Note that here expression "Strict max" shows the further accuracy.

3. Proposed method

In this section, we propose a RTS method to determine the right and left returns to scales of the frontier DMUs that it is always feasible for all DMUs. Our proposed approach is capable of modifying infeasibility in Golany and Yu's method [15].

According to Section 2, let's consider the production possibility set (T) with variable RTS assumption as follows:

---

* $q^*_{BCC} = 1$.

$$T = PPS_{BCC} = \left\{ (\mathbf{X},\mathbf{Y}) \middle| \sum_{j=1}^{n} \lambda_j \mathbf{X}_j \le \mathbf{X} \, , \sum_{j=1}^{n} \lambda_j \mathbf{Y}_j \ge \mathbf{Y} \, , \sum_{j=1}^{n} \lambda_j = 1 \, , \lambda_j \ge 0 \, ; \, j = 1,2,\ldots,n \right\}. \quad (23)$$

Therefore [9, Theorem 2, p. 95], the empirical frontier function is a concave and piece-wise linear function that lies above T .

Consider that DMU_o $\left(o \in \{1,\ldots,n\}\right)$. If there is no convex combination of other DMUs with all outputs greater than or equal to the outputs of DMU_o and all inputs smaller than or equal to the inputs of DMU_o with at least one strict inequality then DMU_o is technically efficient. Now, we solve the following additive DEA model:

$$\begin{aligned}
Max \quad & \sum_{i=1}^{m} s_i^- + \sum_{r=1}^{s} s_r^+ \\
s.t. \quad & \sum_{j=1}^{n} \lambda_j x_{ij} + s_i^- = x_{io}, \quad i = 1,\ldots,m, \\
& \sum_{j=1}^{n} \lambda_j y_{rj} - s_r^+ = y_{ro}, \quad r = 1,\ldots,s, \\
& \sum_{j=1}^{n} \lambda_j = 1, \qquad\qquad\qquad\qquad\qquad\qquad (24) \\
& \lambda_j \ge 0, \quad j = 1,\ldots,n, \\
& s_i^- \ge 0, \quad i = 1,\ldots,m, \\
& s_r^+ \ge 0, \quad r = 1,\ldots,s,
\end{aligned}$$

Where input and output slacks are represented by s_i^- $(i = 1,\ldots,m)$ and s_r^+ $(r = 1,\ldots,s)$, respectively.

It is clear that, if $\left(\lambda_o = 1, \ \lambda_j = 0 \ (\forall j, \ j \ne o), \ s_i^- = 0 \ (\forall i), \ s_r^+ = 0 \ (\forall r)\right)$ is an optimal solution of the above additive model, then DMU_o is technically efficient. If DMU_o is technically inefficient, then it can be projected to an efficient position $\left(\mathbf{X}_o, \mathbf{Y}_o\right)$ as follows:

$$\begin{aligned}
x_{io} &= x_{io} - s_i^{-*}, \quad i = 1,\ldots,m, \\
y_{ro} &= y_{ro} + s_r^{+*}, \quad r = 1,\ldots,s.
\end{aligned} \qquad (25)$$

Now, let α_o be a proportional change in all the inputs of DMU_o and β_o be a proportional change in all the outputs. Then, the set of all feasible proportional changes associated with DMU_o is defined by $P\left(\mathbf{X}_o, \mathbf{Y}_o\right)$ as follows, [15]:

$$P\left(\mathbf{X}_o, \mathbf{Y}_o\right) = \left\{ (\alpha_o, \beta_o) \middle| \sum_{j=1}^{n} \lambda_j \mathbf{X}_j \le \alpha_o \mathbf{X}_o \ , \sum_{j=1}^{n} \lambda_j \mathbf{Y}_j \ge \beta_o \mathbf{Y}_o \ , \sum_{j=1}^{n} \lambda_j = 1 \ , \lambda_j \ge 0 \ ; \ j = 1,\dots,n \right\}. \quad (26)$$

Now, to identify the best possible improvement in the productivity of DMU_o , we would like to minimize α_o and maximize β_o . This improvement can be achieved by maximizing ratio $\frac{\beta_o}{\alpha_o}$, or minimizing its reciprocal $\frac{\alpha_o}{\beta_o}$, with respect to the constraints that the production possibilities set is defined by them as follows:

$$\begin{aligned} Min \ & \frac{\alpha_o}{\beta_o} \\ s.t. \ & \sum_{j=1}^{n} \lambda_j x_{ij} \le \alpha_o x_{io}, \quad i = 1,\dots,m, \\ & \sum_{j=1}^{n} \lambda_j y_{rj} \ge \beta_o y_{ro}, \quad r = 1,\dots,s, \\ & \sum_{j=1}^{n} \lambda_j = 1, \\ & \alpha_o \ge 0, \\ & \beta_o > 0. \end{aligned} \quad (27)$$

In the following theorem, RTS of DMU_o is indicated by the optimal values of α_o and β_o .

Theorem 4. *If* $\left(\boldsymbol{\lambda}^*, \alpha_o^*, \beta_o^*\right)$ *be an optimal solution of model (27), then we will have:*

- *(i)* *If* $1 < \alpha_o^* < \beta_o^*$ *then RTS of* DMU_o *is increasing.*
- *(ii)* *If* $\alpha_o^* < \beta_o^* < 1$ *then RTS of* DMU_o *is decreasing.*
- *(iii)* *If* $\alpha_o^* = \beta_o^*$ *then RTS of* DMU_o *is constant.*

Proof. Refer to [15, p. 30].□

It is worth stressing that, as $\left(\alpha_o = \beta_o = \lambda_o = 1, \ \lambda_j = 0 \ \ (j = 1,\dots,n; \ j \ne o)\right)$ is always a feasible solution of model (27) for any assessment of DMU_o , thus optimal solution of model (27) must always satisfy $\alpha_o^* \le \beta_o^*$.
Now, let's consider the following CCR model:

$$
\begin{aligned}
Min\ \ & \theta_o - \varepsilon\left(\sum_{i=1}^{m} s_i^- + \sum_{r=1}^{s} s_r^+\right) \\
s.t.\ \ & \sum_{j=1}^{n} \lambda_j x_{ij} + s_i^- = \theta_o x_{io}, \quad i = 1,\dots,m, \\
& \sum_{j=1}^{n} \lambda_j y_{rj} - s_r^+ = y_{ro}, \quad r = 1,\dots,s, \\
& \lambda_j \ge 0, \quad j = 1,\dots,n, \\
& s_i^- \ge 0, \quad i = 1,\dots,m, \\
& s_r^+ \ge 0, \quad r = 1,\dots,s,
\end{aligned} \tag{28}
$$

Where ε is a non-Archimedean small and positive number.

Consider Banker's variable variation (h_o) as $h_o = \sum_{j=1}^{n} \lambda_j$. Note that, h_o , is always positive. Hence the constraints of model (28) can be divided by h_o as:

$$
\begin{aligned}
Min\ \ & \theta_o - \varepsilon\left(\sum_{i=1}^{m} s_i^- + \sum_{r=1}^{s} s_r^+\right) \\
s.t.\ \ & \sum_{j=1}^{n} \frac{\lambda_j}{h_o} x_{ij} + \frac{s_i^-}{h_o} = \frac{\theta_o}{h_o} x_{io}, \quad i = 1,\dots,m, \\
& \sum_{j=1}^{n} \frac{\lambda_j}{h_o} y_{rj} - \frac{s_r^+}{h_o} = \frac{1}{h_o} y_{ro}, \quad r = 1,\dots,s, \\
& \sum_{j=1}^{n} \frac{\lambda_j}{h_o} = 1, \\
& \lambda_j \ge 0, \quad j = 1,\dots,n, \\
& s_i^- \ge 0, \quad i = 1,\dots,m, \\
& s_r^+ \ge 0, \quad r = 1,\dots,s.
\end{aligned} \tag{29}
$$

Here by defining new variables α_o , β_o , μ_j $(j = 1,\dots,n)$, $s_i^{-'}$ $(i = 1,\dots,m)$, and $s_r^{+'}$ $(r = 1,\dots,s)$ as follows:

$$
\alpha_o = \frac{\theta_o}{h_o}, \quad \beta_o = \frac{1}{h_o}, \quad \mu_j = \frac{\lambda_j}{h_o}\ (j = 1,\dots,n), \quad s_i^{-'} = \frac{s_i^-}{h_o}\ (i = 1,\dots,m), \quad s_r^{+'} = \frac{s_r^+}{h_o}\ (r = 1,\dots,s). \tag{30}
$$

Thus, we will have the following model:

$$
\begin{aligned}
Min\ & \frac{\alpha_o}{\beta_o} - \varepsilon\left(\sum_{i=1}^{m} s_i^{-'} + \sum_{r=1}^{s} s_r^{+'}\right) \\
s.t.\ & \sum_{j=1}^{n} \mu_j x_{ij} + s_i^{-'} = \alpha_o x_{io}, \quad i = 1,\dots,m, \\
& \sum_{j=1}^{n} \mu_j y_{rj} - s_r^{+'} = \beta_o y_{ro}, \quad r = 1,\dots,s, \\
& \sum_{j=1}^{n} \mu_j = 1, \\
& \mu_j \geq 0, \quad j = 1,\dots,n, \\
& s_i^{-'} \geq 0, \quad i = 1,\dots,m, \\
& s_r^{+'} \geq 0, \quad r = 1,\dots,s.
\end{aligned} \tag{31}
$$

If h_o has a unique optimal value then the value of h_o is interpreted by Banker as an indicator of RTS [1]. Chang and Guh [7] showed that h_o can have alternate optima with values grater and less than one for the same model. Therefore, in the CCR formulation, observing an optimal value of h_o tells us nothing about RTS. Similarly, in the CCR model, observing the optimal θ_o value is not enough to obtain the direction of RTS. For instance, we can obtain the same optimal value, say $\theta_o^* = 0.8$, from $\alpha_o^* = 0.64$ and $\beta_o^* = 0.8$ (indicating decreasing RTS) or from $\alpha_o^* = 1.2$ and $\beta_o^* = 1.5$ (indicating increasing RTS).

Thus, in order to obtain correct estimates of RTS, we cannot rely on the CCR model (28). For this reason, we pay attention to non-linear model (31) and propose to solve its two linear variants instead of solving non-linear model (31). Moreover, since RTS are defined as a local property, hence we only need to examine them in as $\delta-$neighborhood of DMU_o.

Now, we first set α_o to be equal to one plus a small arbitrary number (i.e., $\alpha_o = 1+\delta, \delta > 0$), then model (31) is transformed to the following model:

$$
\begin{aligned}
Max\ & \beta_o + \varepsilon\left(\sum_{i=1}^{m} s_i^{-'} + \sum_{r=1}^{s} s_r^{+'}\right) \\
s.t.\ & \sum_{j=1}^{n} \mu_j x_{ij} + s_i^{-'} = (1+\delta) x_{io}, \quad i = 1,\dots,m, \\
& \sum_{j=1}^{n} \mu_j y_{rj} - s_r^{+'} = \beta_o y_{ro}, \quad r = 1,\dots,s, \\
& \sum_{j=1}^{n} \mu_j = 1, \\
& \mu_j \geq 0, \quad j = 1,\dots,n, \\
& s_i^{-'} \geq 0, \quad i = 1,\dots,m, \\
& s_r^{+'} \geq 0, \quad r = 1,\dots,s.
\end{aligned} \tag{32}
$$

Since $\left(\beta_o = \lambda_o = 1,\ \lambda_j = 0\ \ (j = 1,\dots,n;\ j \neq o),\ s_i^{-'} = \delta x_{io}\ \ (i = 1,\dots,m),\ s_r^{+'} = 0\ \ (r = 1,\dots,s)\right)$ is a feasible solution of model (32) for any assessment of DMU_o, so the optimal solution of model (32) must always satisfy $\beta_o^* \geq 1$.

Second, we set β_o equal to one minus a small number (i.e., $\beta_o = 1-\delta,\ \delta > 0$), then this transforms model (31) into the following model:

$$
\begin{aligned}
& Min\ \ \alpha_o - \varepsilon\left(\sum_{i=1}^{m} s_i^{-'} + \sum_{r=1}^{s} s_r^{+'}\right) \\
& s.t.\ \ \sum_{j=1}^{n} \mu_j x_{ij} + s_i^{-'} = \alpha_o x_{io}, \quad i = 1,\dots,m, \\
& \qquad \sum_{j=1}^{n} \mu_j y_{rj} - s_r^{+'} = (1-\delta) y_{ro}, \quad r = 1,\dots,s, \\
& \qquad \sum_{j=1}^{n} \mu_j = 1, \qquad\qquad\qquad\qquad (33) \\
& \qquad \mu_j \geq 0, \quad j = 1,\dots,n, \\
& \qquad s_i^{-'} \geq 0, \quad i = 1,\dots,m, \\
& \qquad s_r^{+'} \geq 0, \quad r = 1,\dots,s.
\end{aligned}
$$

Note that, the optimal solution of model (33) must always satisfy $\alpha_o^* \leq 1$. Because $\left(\alpha_o = \lambda_o = 1,\ \lambda_j = 0\ \ (j = 1,\dots,n;\ j \neq o),\ s_i^{-'} = 0\ \ (i = 1,\dots,m),\ s_r^{+'} = \delta y_{ro}\ \ (r = 1,\dots,s)\right)$ is a feasible solution of model (33) for any assessment of DMU_o.

Now, we provide a new method to determine the right and left RTSs using the models (32) and (33):

A new method for determining the right and left RTSs

Suppose that DMU_o is a point on the BCC-efficient frontier.

1. Solve model (32) to determine the "right" RTS of DMU_o, then associated with Theorem 4:

- **1.1.** If $1+\delta < \beta_o^*$ then the right RTS of DMU_o is increasing.
- **1.2.** If $1+\delta = \beta_o^*$ then the right RTS of DMU_o is constant.
- **1.3.** If $1+\delta > \beta_o^*$ then the right RTS of DMU_o is decreasing.

2. Solve model (33) to determine the "left" RTS of DMU_o, then associated with Theorem 4:

- **2.1.** If $1-\delta < \alpha_o^*$ then the left RTS of DMU_o is increasing.
- **2.2.** If $1-\delta = \alpha_o^*$ then the left RTS of DMU_o is constant.
- **2.3.** If $1-\delta > \alpha_o^*$ then the left RTS of DMU_o is decreasing.

For convenience, in the case of a single input and output $(m = s = 1)$, the following figures depict to determine the right RTS:

Case 1.1.:

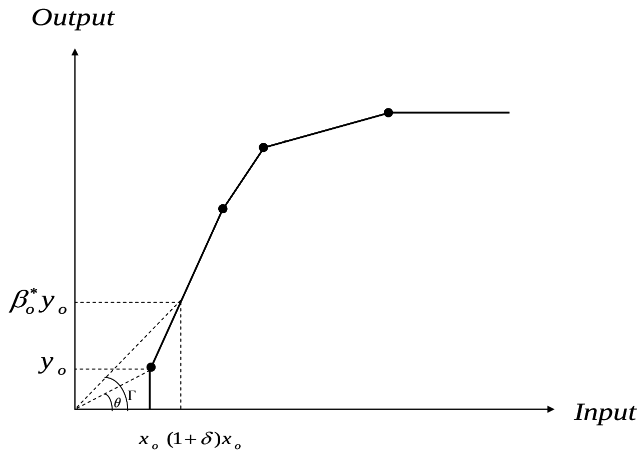

Fig.1. Increasing right RTS.

$$\theta < \Gamma \Rightarrow tg\,\theta < tg\,\Gamma \Rightarrow \frac{y_o}{x_o} < \frac{\beta_o^* y_o}{(1+\delta)x_o} \Rightarrow 1+\delta < \beta_o^*.$$

Case 1.2.:

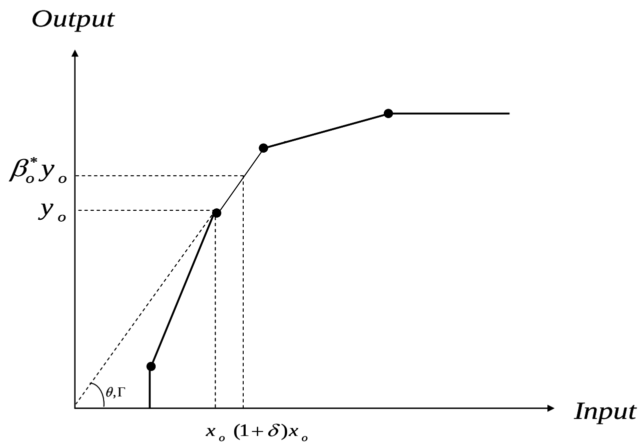

Fig.2. Constant right RTS.

$$\theta = \Gamma \Rightarrow tg\,\theta = tg\,\Gamma \Rightarrow \frac{y_o}{x_o} = \frac{\beta_o^* y_o}{(1+\delta)x_o} \Rightarrow 1+\delta = \beta_o^*.$$

Case 1.3.:

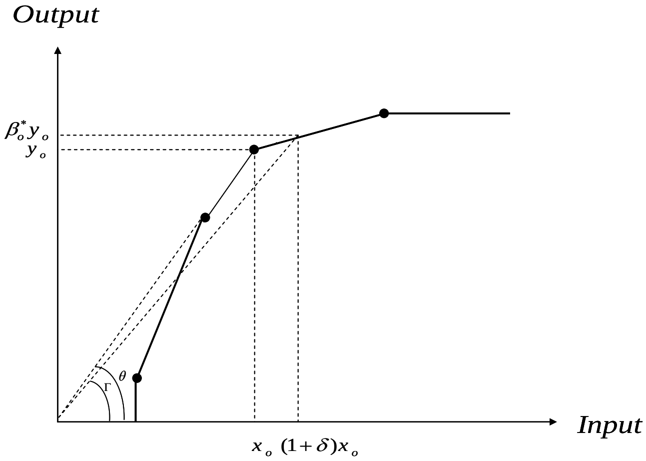

Fig 3.Decreasing right RTS.

$$\Gamma < \theta \Rightarrow tg\,\Gamma < tg\,\theta \Rightarrow \frac{\beta_o^* y_o}{(1+\delta)x_o} < \frac{y_o}{x_o} \Rightarrow 1+\delta > \beta_o^*.$$

As can be seen in figures 1, 2, and 3, it can be seen that $DMU_o = (\mathbf{X}_o, \mathbf{Y}_o)$ has increasing left RTS, increasing left RTS, and furthermore constant left RTS, respectively.

Since inefficient DMUs have more than one projection on the empirical frontier function, therefore the different right and left returns to scales of them can be determined by using our proposed method.
In next section, we will present an illustrative example to demonstrate our method then we compare the proposed method with Golany and Yu's method [15] to estimate right and left RTSs of DMUs.

4. An illustrative example

To illustrate how to use the proposed DEA approach, this study selects a small example related to Golany and Yu [15].
Table 1 summarizes the data set which was previously analyzed by Golany and Yu [15] on determining the right and left RTSs of 9 DMUs which consists of two inputs and one output by our method. Note that DMUs 1, 3, 5, 6, 7, and 8 are BCC-efficient DMUs. Furthermore, DMUs 2, 4, and 9 are inefficient DMUs. We then make comparisons between our proposed approach and Golany and Yu's method in Table 2.

Table 1
The obtained results from models (32) and (33) for BCC- efficient DMUs. ($\delta = 0.0001$)

DMU	$\mathbf{X}_1$	$\mathbf{X}_2$	$\mathbf{Y}$	β_o^* (32)	The right RTS	α_o^* (33)	The left RTS
1	1.1	1.1	1	1.000122	IRS	1.000000	IRS
2	1.4	1.3	0.95	-----	-----	-----	-----
3	2	2	2	1.000100	CRS	0.999910	IRS
4	2.1	2.2	1.9	-----	-----	-----	-----
5	3	3	3	1.000057	DRS	0.999900	CRS
6	4.2	5	4	1.000081	DRS	0.999886	DRS
7	5.5	6.5	5	1.000065	DRS	0.999882	DRS
8	7	8.5	6	1.000000	DRS	0.999871	DRS
9	5	5	3.9	-----	-----	-----	-----

Table 2
Comparisons to Golany and Yu's method.

DMU	Results of our proposed method		Results of Golany and Yu's method	
	The right RTS	The left RTS	The right RTS	The left RTS
1	IRS	IRS	IRS	N.F.S. [1]
2	-----	-----	inefficient	inefficient
3	CRS	IRS	CRS	IRS
4	-----	-----	inefficient	inefficient
5	DRS	CRS	DRS	CRS
6	DRS	DRS	DRS	DRS
7	DRS	DRS	DRS	DRS
8	DRS	DRS	N.F.S.	DRS
9	-----	-----	inefficient	inefficient

[1] No Feasible Solution.

As presented in Table 2, in our RTS method, IRS prevails at DMU_1 and furthermore DRS prevails at DMUs 6, 7, and 8 on both the right and left RTSs, while Golany and Yu's method has no feasible solution to estimate the right and left RTSs of DMU_8 and DMU_1, respectively. Moreover, our new RTS method respectively indicates that the right and left RTSs of DMU_3 are constant and increasing and also it indicates DRS and CRS on the right and left RTSs of DMU_5, respectively.

Now, in order to determine RTS of inefficient DMUs (DMUs 2, 4, and 9), we first project each of them twice to the BCC-efficient frontier. In turn, the input-oriented BCC model is applied then the output-oriented BCC model is used to project inefficient DMUs onto the efficient frontier. The input and output values of the projected units and also the obtained results from models (32) and (33) are given in Table 3.

Table 3
The obtained results from models (32) and (33) for projections of inefficient DMUs. ($\delta = 0.0001$)

DMU	$\mathbf{X}_1$	$\mathbf{X}_2$	$\mathbf{Y}$	β_o^* (32)	The right RTS	α_o^* (33)	The left RTS
Input-oriented BCC model:							
2	1.1	1.1	1	1.000122	IRS	1.000000	IRS
4	1.91	1.91	1.9	1.000075	DRS	0.999910	IRS
9	4.125	4.575	3.9	1.000000	DRS	0.999895	DRS
Output-oriented BCC model:							
2	1.3	1.3	1.22222	1.000117	IRS	1.000000	IRS
4	2.1	2.1	2.1	1.000070	DRS	0.999909	IRS
9	4.43	5	4.142857	1.000000	DRS	0.999893	DRS

Table 3 indicates the following analysis for all six projected units:

- Both the right and left of the projections of DMU_2 are increasing.
- Both the right and left of the projections of DMU_9 are decreasing.
- The right and left RTSs of the projections of DMU_4 are decreasing and increasing, respectively.

Therefore according to the above analysis, although DMUs 2 and 9 are inefficient, we can associate them with a single RTS condition.

5. Conclusion and future extensions

The economic concept of RTS and approaches to estimate it have been widely investigated within the framework DEA and this has, in turn, further extended the applicability DEA.
In this current study, a new RTS method is proposed to estimate the right and left RTSs of the frontier DMUs in DEA by solving two variants of the BCC model.
The main aim of this study is to modify Golany and Yu’s method to estimate the right and left RTSs of DMUs. In other words, using our RTS procedure in order to estimate the right and left RTSs of DMUs, we always can determine RTSs. However, Golany and Yu’s method is always incapable of estimating RTSs of DMUs.

It is worth stressing that, since inefficient DMUs have more than one projection on the empirical frontier function, hence by applying our new RTS method, different right and left RTSs can be determined for their projections. We suggest a further analysis in this work for future researches in computing the magnitude the right and left RTSs and furthermore in determining the right and left RTSs of BCC-efficient DMUs for special data e.g., integer, stochastic, interval data, and etc.

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