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Finding Closest Target in Network DEA

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Abstract
As an important concept in data envelopment analysis (DEA), closest target has wide theoretical and practical applications. By considering the data of closest target and utilizing it appropriately, the decision-making unit (DMU) under evaluation determines how to transform its inputs and outputs to become efficient. Also, traditional DEA models only utilize the external inputs to produce the final outputs in evaluating the relative efficiency of decision-making units (DMUs), and internal operations are not be considered. Therefore, traditional models can not accurately determine the source of inefficiency inside the structures. To overcome this problem, different authors proposed various network DEA models (NDEA). This paper is an attempt to find the closest target in various scenarios of network DEA. The study concerns about different existed scenarios in network DEA models and proposes specific models to find closest targets in each scenario. Also, an empirical example has been presented to illustrate the proposed models.

Keywords: Network data envelopment analysis, constant returns to scale technology, projection point, closest target.

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1. Introduction

Data envelopment analysis (DEA) is a useful method based on linear programming to assess the relative efficiency of peer decision making units (DMUs). There are two basic models CCR (Charnes, Cooper, Rhodes, 1978) and BCC (Banker, Charnes & Cooper, 1984), in which the constant and variable returns to scale technology are considered. The CCR and BCC models do not consider internal structures of DMUs and DMUs are treated as black boxes. With considering the internal structures of DMUs, we will be able to take a look in to the internal different stages efficiencies which are useful to improve the internal structure of DMUs (see Cook et al., 2010, Du et al., 2011, Kao et al., 2008, Sexton and Lewis, 2003, Tone and Tsutsui, 2009). Also, Hassanzadeh and Mostafaee, 2019, have been extended the existed different scenarios with a sophisticated definition of production possibility set (PPS). They considered six scenarios based on the concept of link control for intermediate products through different stages. Divisional network DEA efficiencies and along with the overall efficiency may be found in Tone and Tsutsui, 2009. Also, dynamic slack-based measure has been investigated by Tone and Tsutsui, 2010. Zhou et al., 2013, introduced a bargaining game model for efficiency decomposition in two-stage systems. Liang et al., 2008, utilized the concept of game theory for two-stages processes.

Also, finding the closest targets is an important issue in DEA literature. The Euclidean distance has been used to find the closest target by Frei and Harker, 1999. Some radial models have been utilized to propose a multi-stage method to find the closest targets by Coelli, 1998. Cherchye and Van Puyenbroeck, 2001, used oriented measures and least distance combination. Lozano and Villa, 2005, proposed a method which determines a sequence of targets. Also, Razipour-Ghalehjough et al., 2020, proposed a model for finding closest targets in the presence of weight restrictions.

This paper presents some models to find closest targets in network DEA. In particular, we utilize the mixed-integer linear programming model presented in Aparicio et al., 2007 to obtain closest targets of different scenarios of network DEA proposed by Hassanzah and Mostafaee, 2019. We propose models for finding closest targets in scenarios 1, 2, 4 and 5. Also, an empirical example has been provided to shed lights on the usefulness of models.

The paper includes five sections: After the introduction Section, the Preliminaries are presented in Section 2. We can see the proposed model for finding the closest target of network DEA in Section 3. An empirical example has been presented in Section 4. The conclusion of the paper is provided in Section 5.

2. Preliminaries

With considering the Tone and Tsutsui, 2009, and Hassanzadeh and Mostafaee, 2019 notation, we deal with n DMUs $j=1,...,n$ consisting of K stages $k=1,...,K$. Let m^k, r^k and $\tau^{(k,h)}$ be the numbers of inputs, outputs and intermediate products from stage k to stage h for stage k, respectively. The link leading from stage k to stage h is denoted by (k,h) and the set of all links is denoted by L. Let $x_{ij}^k, i=1,...,m^k$, $j=1,...,n$, $k=1,...,K$ be the input resource i of DMU_j for stage k. Also, let x_{rj}^k, $r=1,...,r^k$, $j=1,...,n$, $k=1,...,K$ be the output product r of DMU_j for stage k and $z_{Lj}^{(k,h)}, l=1,...,\tau^{(k,h)}$, $j=1,...,n$ be the intermediate product l of DMU_j from stage k to stage h. Also, let β^k, γ^k and w^k be the amount of control over link excesses for stage k, the amount of control over link shortfalls for stage k and the user-specified weights of stage k such that $\sum_{k=1}^{K} w^k = 1$, respectively. The potential decreases (excesses) of inputs for stage k, potential increases (shortfalls) of outputs for stage k and potential decrease or increase of links for stage k to stage h are denoted by s^{k-}, s^{k+} and $s_z^k = s_z^{k-} - s_z^{k+}$, respectively. The production possibility set (PPS) $T_G = \left\{ \left(x^k, y^k, z^{(k,h)} \right) \right\}$ is defined by:

$$
\begin{aligned}
& x^k - s^{k-} = \sum_{j=1}^{n} x_j^k \lambda_j^k, \quad k = 1,\dots,K \qquad (1)\\
& y^k + s^{k+} = \sum_{j=1}^{n} y_j^k \lambda_j^k, \quad k = 1,\dots,K\\
& z^{(k,h)} + s_z^{k+} - s_z^{k-} = \sum_{j=1}^{n} z_j^{(k,h)} \lambda_j^k, \quad \forall (k,h)\\
& z^{(k,h)} + s_z^{h+} - s_z^{h-} = \sum_{j=1}^{n} z_j^{(k,h)} \lambda_j^h, \quad \forall (k,h)\\
& s_z^{k-} \le \beta^k z^{(k,h)}, s_z^{k+} \le \gamma^k z^{(k,h)}, \quad \forall (k,h)\\
& s_z^{h-} \le \beta^k z^{(k,h)}, s_z^{h+} \le \gamma^k z^{(k,h)}, \quad \forall (k,h)\\
& s_z^{k+} - s_z^{k-} \ge s_z^{h+} - s_z^{h-}, \quad \forall (k,h)\\
& s^{k-}, s^{k+}, s_z^{k-}, s_z^{k+}, s_z^{h-}, s_z^{h+} \ge 0, \ k,h = 1,\dots,K\\
& \lambda_j^k \ge 0, j = 1,\dots,n, k = 1,\dots,K
\end{aligned}
$$

Moreover, Aparicio et al., 2007 presented the following model for finding the closest target in traditional DEA models. The set "E" stands for efficient DMUs:

$$
\begin{aligned}
Min \quad & \sum_{i=1}^{m} s_{io}^- + \sum_{r=1}^{s} s_{ro}^+ \qquad (2)\\
s.t \quad & \sum_{j\in E} \lambda_j x_{ij} = x_{io} - s_i^-, \quad i = 1,\dots,m,\\
& \sum_{j\in E} \lambda_j y_{rj} = y_{ro} + s_r^+, \quad r = 1,\dots,s,\\
& \sum_{r=1}^{s} u_r y_{rj} - \sum_{i=1}^{m} v_i x_{ij} + d_j = 0, \ j \in E,\\
& u_r, v_i \ge 1, \quad r = 1,\dots,s, i = 1,\dots,m,\\
& d_j \le M\gamma_j, \lambda_j \le M\left(1-\gamma_j\right), \gamma_j \in \{0,1\}, j \in E,\\
& \lambda, d, s^-, s^+ \ge 0
\end{aligned}
$$

The above-mentioned model identifies the closest target for DMU_o.

3. Proposed model

According to T_G in the case of scenario 1 of Hassanzadeh and mostafaee, 2019, in which the link is only the output under the control of the previous stage, one can easily define the following slack based measure model to characterize whether the DMU under assessment is overall efficient or not.

$$
\begin{aligned}
\rho_o^{s1} = Min & \frac{\sum_{k=1}^{K} w^k \left\{ 1 - \frac{1}{m^k} \left\{ \sum_{i=1}^{m^k} \frac{s_i^{k-}}{x_{io}^k} \right\} \right\}}{\sum_{k=1}^{K} w^k \left\{ 1 + \frac{1}{r^k + \tau^{(k,h)}} \left\{ \sum_{r=1}^{r^k} \frac{s_r^{k+}}{y_{ro}^k} + \sum_{l=1}^{\tau^{(k,h)}} \frac{s_{lz}^{k+}}{z_{lo}^{(k,h)}} \right\} \right\}}\\
s.t \quad & \sum_{j\in J} \lambda_j^k x_{ij}^k = x_{io}^k - s_i^{k-}, \ i = 1,\dots,m^k, k = 1,\dots,K\\
& \sum_{j\in J} \lambda_j^k y_{ro}^k = y_{ro}^k + s_r^{k+}, \ r = 1,\dots,r^k, k = 1,\dots,K\\
& \sum_{j\in J} \lambda_j^k z_{lj}^{(k,h)} = z_{lo}^{(k,h)} + s_{lz}^{k+}, \ l = 1,\dots,\tau^{(k,h)}, \forall (k,h)\\
& \sum_{j\in J} \lambda_j^h z_{lj}^{(k,h)} = z_{lo}^{(k,h)} + s_{lz}^{h+}, \ l = 1,\dots,\tau^{(k,h)}, \forall (k,h)\\
& s_{lz}^{k+} = s_{lz}^{h+}, \forall (k,h) \qquad (3)\\
& \lambda_j^k \ge 0, \lambda_j^h \ge 0, \quad j \in J, \ h,k = 1,\dots,K\\
& s_i^{k-}, s_r^{k+} \ge 0, i = 1,\dots,m^k, r = 1,\dots,r^k, k = 1,\dots,K\\
& s_{lz}^{k+}, s_{lz}^{h+} \ge 0, \ l = 1,\dots,\tau^{(k,h)}h, \ k = 1,\dots,K
\end{aligned}
$$

Where $w^k, k = 1,\dots,K$ are super-specified positive weights such that $\sum_{k=1}^{K} w^k = 1$. Superscript S1 means scenario 1. It can easily be shown that $0 \le \rho_o^{s1} \le 1$ DMU_o is called overall efficient if $\rho_o^{s1} = 1$ Otherwise, is called overall inefficient. The stage-k efficiency score of DMU_o may be found as follows:

$$
\rho_o^{k,s1} = \frac{1 - \frac{1}{m^k} \left\{ \sum_{i=1}^{m^k} \frac{s_i^{k-*}}{x_{io}^k} \right\}}{1 + \frac{1}{r^k + \tau^{(k,h)}} \left\{ \sum_{r=1}^{r^k} \frac{s_r^{k+*}}{y_{ro}^k} + \sum_{l=1}^{\tau^{(k,h)}} \frac{s_{lz}^{k+*}}{z_{lo}^{(k,h)}} \right\}}, \qquad (4)
$$

The superscript "*" stands for optimality. Stage k of DMU_o is efficient if $\rho_o^{k,s1} = 1$. The set of all overall efficient DMUs is denoted by "E".

Theorem 1 Suppose that D_o indicates Pareto-efficient points of T_G in scenario 1 which dominates, DMU_o. Then $\left(x^k, y^k, z^{(k,h)}\right) \in D_o$ if and only if there exists:

$$
\begin{array}{ll}
u_r^k, v_i^k \geq 1, & i = 1,\ldots,m^k, r = 1,\ldots,r^k, k = 1,\ldots,K \\
t_l^k, t_l^h \geq 1, & l = 1,\ldots,\tau^{(k,h)}, \quad h,k = 1,\ldots,K \quad (5) \\
y_j^h \in \{0,1\}, \quad \lambda_j^k, d_j^k, \lambda_j^h \geq 0, & j \in E, \; k,h = 1,\ldots,K \\
s_i^{k-}, s_r^{k+} \geq 0, & i = 1,\ldots,m^k, \quad r = 1,\ldots,r^k, \quad k = 1,\ldots,K \\
s_{lz}^{k+}, s_{lz}^{h+} \geq 0, & l = 1,\ldots,\tau^{(k,h)}, \quad h,k = 1,\ldots,K
\end{array}
$$

Such that

$$
\begin{array}{l}
\sum_{j \in E} \lambda_j^k x_{ij}^k = x_i^k, \; i = 1,\ldots,m^k, k = 1,\ldots,K, \\
\sum_{j \in E} \lambda_j^k y_{rj}^k = y_r^k, \; r = 1,\ldots,r^k, k = 1,\ldots,K, \\
\sum_{j \in E} \lambda_j^k z_{lj}^{(k,h)} = z_l^{(k,h)}, \; l = 1,\ldots,\tau^{(k,h)}, \forall (k,h), \\
\sum_{j \in E} \lambda_j^h z_{lj}^{(k,h)} = z_l^{(k,h)}, \; l = 1,\ldots,\tau^{(k,h)}, \forall (k,h), \\
\sum_{j \in E} \lambda_j^k x_{ij}^k = x_{io}^k - s_i^{k-}, \; i = 1,\ldots,m^k, k = 1,\ldots,K, \\
\sum_{j \in E} \lambda_j^k y_{rj}^k = y_{ro}^k + s_i^{k+}, \; r = 1,\ldots,r^k, k = 1,\ldots,K, \\
\sum_{j \in E} \lambda_j^k z_{lj}^{(k,h)} = z_{lo}^{(k,h)} + s_{lz}^{k+}, \; l = 1,\ldots,\tau^{(k,h)}, \forall (k,h), \\
\sum_{j \in E} \lambda_j^h z_{lj}^{(k,h)} = z_{lo}^{(k,h)} + s_{lz}^{h+}, \; l = 1,\ldots,\tau^{(k,h)}, \forall (k,h), \\
\sum_{r=1}^{r^k} u_r^k y_{rj}^k + \sum_{l=1}^{\tau^{(k,h)}} t_l^k z_{lj}^{(k,h)} - \sum_{l=1}^{\tau^{(k,h)}} t_l^h z_{lj}^{(k,h)} - \sum_{i=1}^{m} v_j^k x_{ij}^k \\
+ d_j^k = 0, \; j \in J, k = 1,\ldots,K \\
d_j^k \leq M^k \gamma_j^k, \lambda_j^k \leq M^k \left(1 - \gamma_j^k\right), \; j \in J, k = 1,\ldots,K \\
s_{lz}^{h+}, s_{lz}^{k+} \geq 0, \quad l = 1,\ldots,\tau^{(k,h)}, \quad h,k = 1,\ldots,K
\end{array}
$$

Where $M^k, k = 1,\ldots,K$ are sufficiently large positive numbers.

Proof. The proof of Theorem 1 of Razipour-ghalehjough et al., 2020, can easily be extended in the case of this theorem. Therefore, is omitted.

Considering the Pareto-efficient points which dominate DMU_o, we present some models for finding the closest targets of network DEA in different scenarios (1,2,4 and 5) presented by Hassanzadeh A. and Mostafaee A., 2019. By applying the L_1-distance norm in scenario 1 of Hassanzadeh A. and Mostafaee A., 2019, in which the intermediate products are considered as the outputs under the control of previous stage, we propose the following Mixed-Integer linear programming problem for finding the closest target of DMU_o:

$$
\begin{array}{ll}
Min & \sum_{k=1}^{K} \sum_{i=1}^{m^k} s_i^{k-} + \sum_{k=1}^{K} \sum_{r=1}^{r^k} s_r^{k+} + \sum_{k=1}^{K} \sum_{l=1}^{\tau^{(k,h)}} s_{lz}^{k+} \qquad (6) \\
s.t & \sum_{j \in E} \lambda_j^k x_{ij}^k = x_{io}^k - s_i^{k-}, \; i = 1,\ldots,m^k, k = 1,\ldots,K \\
& \sum_{j \in E} \lambda_j^k y_{rj}^k = y_{ro}^k + s_r^{k+}, \; r = 1,\ldots,r^k, k = 1,\ldots,K \\
& \sum_{j \in E} \lambda_j^k z_{lj}^{(k,h)} = z_{lo}^{(k,h)} + s_{lz}^{k+}, \; l = 1,\ldots,\tau^{(k,h)}, \forall (k,h) \\
& \sum_{j \in E} \lambda_j^h z_{lj}^{(k,h)} = z_{lo}^{(k,h)} + s_{lz}^{h+}, \; l = 1,\ldots,\tau^{(k,h)}, \forall (k,h) \\
& s_{lz}^{k+} = s_{lz}^{h+}, \forall (k,h), \\
& \begin{cases} \sum_{r=1}^{r^k} u_r^k y_{rj}^k + \sum_{l=1}^{\tau^{(k,h)}} t_l^k z_{lj}^{(k,h)} - \sum_{l=1}^{\tau^{(k,h)}} t_l^h z_{lj}^{(k,h)} - \sum_{i=1}^{m} v_j^k x_{ij}^k \\ + d_j^k = 0, \; j \in J, k = 1,\ldots,K \end{cases} \\
& u_r^k, v_i^k \geq 1, \; i = 1,\ldots,m^k, r = 1,\ldots,r^k, k = 1,\ldots,K \\
& \lambda_j^h \geq 0, \; j \in E, h = 1,\ldots,K \\
& s_r^{k+}, s_i^{k-} \geq 0, i = 1,\ldots,m^k, r = 1,\ldots,r^k, k = 1,\ldots,K \\
& s_{lz}^{k+}, s_{lz}^{h+} \geq 0, \quad l = 1,\ldots,\tau^{(k,h)}, \qquad h,k = 1,\ldots,K \\
& \lambda_j^k \geq 0, \lambda_j^h \geq 0, \qquad j \in J, \; h,k = 1,\ldots,K \\
& s_i^{k-}, s_r^{k+} \geq 0, i = 1,\ldots,m^k, r = 1,\ldots,r^k, k = 1,\ldots,K \\
& s_{lz}^{k+}, s_{lz}^{h+} \geq 0, \quad l = 1,\ldots,\tau^{(k,h)} h, \quad k = 1,\ldots,K \\
& \begin{cases} d_j^k, \lambda_j^k \geq 0, d_j^k \leq M^k \gamma_j^k, \lambda_j^k \leq M^k \left(1 - \gamma_j^k\right), \\ \lambda_j^k \in \{0,1\}, \; j \in J, k = 1,\ldots,K \end{cases}
\end{array}
$$

Also, the closest target in scenario 1 may be considered as follows:

$$
\begin{array}{l}
\hat{x}_{io}^k = x_{io}^k - s_i^{k-*}, i = 1,\ldots,m^k, k = 1,\ldots,K \qquad (7) \\
\hat{y}_{ro}^k = y_{ro}^k + s_r^{k+*}, \quad r = 1,\ldots,r^k, k = 1,\ldots,K \\
\hat{z}_{lo}^{(k,h)} = z_{lo}^{(k,h)} + s_{lz}^{k+*}, \quad l = 1,\ldots,\tau^{(k,h)}, \forall (k,h)
\end{array}
$$

According to T_G in the case of scenario 2 of Hassanzadeh and mostafaee, 2019, one can easily define the following slack based measure model to characterize whether the DMU under assessment is overall efficient or not.

$$\rho_o^{s2} = Min \frac{\sum_{k=1}^{K} w^k = \left\{1 - \frac{1}{m^k + \tau^{(k,h)}} \left\{ \sum_{i=1}^{m^k} \frac{s_i^{k-}}{x_{io}^k} + \sum_{l=1}^{\tau^{(k,h)}} \frac{s_{lz}^{k-}}{z_{lo}^{(k,h)}} \right\}\right\}}{\sum_{k=1}^{K} w^k \left\{1 + \frac{1}{r^k} \left\{ \sum_{r=1}^{r^k} \frac{s_r^{k+}}{y_{ro}^k} \right\}\right\}}$$

$$\begin{aligned} s.t \quad & \sum_{j\in J} \lambda_j^k x_{ij}^k = x_{io}^k - s_i^{k-}, \ i = 1,...,m^k, k = 1,...,K \\ & \sum_{j\in J} \lambda_j^k y_{rj}^k = y_{ro}^k + s_r^{k+}, \ r = 1,...,r^k, k = 1,...,K \\ & \sum_{j\in J} \lambda_j^k z_{lj}^{(k,h)} = z_{lo}^{(k,h)} - s_{lz}^{k-}, \ l = 1,...,\tau^{(k,h)}, \forall (k,h) \\ & \sum_{j\in J} \lambda_j^h z_{lj}^{(k,h)} = z_{lo}^{(k,h)} - s_{lz}^{h-}, \ l = 1,...,\tau^{(k,h)}, \forall (k,h) \\ & s_{lz}^{k-} = s_{lz}^{h-}, \forall (k,h), \qquad (8) \\ & \lambda_j^k \ge 0, \lambda_j^h \ge 0, \ j \in J, \ h,k = 1,...,K \\ & s_r^{k+}, s_i^{k-} \ge 0, i = 1,...,m^k, r = 1,...,r^k, k = 1,...,K \\ & s_{lz}^{k-}, s_{lz}^{h-} \ge 0, \ l = 1,...,\tau^{(k,h)}, \quad h,k = 1,...,K \end{aligned}$$

Superscript S2 means scenario 2. One can easily find that $0 \le \rho_o^{s2} \le 1$ DMU_o is called overall efficient if $\rho_o^{s2} = 1$ Otherwise, DMU_o is called overall inefficient. The stage-k efficiency score of DMU_o may be found as follows:

$$\rho_o^{k,s2} = Min \frac{1 - \frac{1}{m^k + \tau^{(k,h)}} \left\{ \sum_{i=1}^{m^k} \frac{s_i^{k-*}}{x_{io}^k} + \sum_{l=1}^{\tau^{(k,h)}} \frac{s_{lz}^{k-*}}{z_{lo}^{(k,h)}} \right\}}{1 + \frac{1}{r^k} \left\{ \sum_{r=1}^{r^k} \frac{s_r^{k+*}}{y_{ro}^k} \right\}} \qquad (9)$$

The superscript "*" stands for optimality. Stage k of DMU_o is efficient if $\rho_o^{k,s2} = 1$. By utilizing the L_1-distance norm in scenario 2 of Hassanzadeh A. and Mostafaee A., 2019, in which the intermediate products are considered as the inputs under the control of next stage, the following Mixed-Integer linear programming problem is proposed to obtain the closest target:

$$\begin{aligned} Min \quad & \sum_{k=1}^{K} \sum_{i=1}^{m^k} s_i^{k-} + \sum_{k=1}^{K} \sum_{r=1}^{r^k} s_i^{k+} + \sum_{k=1}^{K} \sum_{l=1}^{\tau^{(k,h)}} s_{lz}^{h-} \qquad (10) \\ s.t \quad & \sum_{j\in E} \lambda_j^k x_{ij}^k = x_{io}^k - s_i^{k-}, \ i = 1,...,m^k, k = 1,...,K \\ & \sum_{j\in E} \lambda_j^k y_{rj}^k = y_{ro}^k + s_r^{k+}, \ r = 1,...,r^k, k = 1,...,K \\ & \sum_{j\in E} \lambda_j^k z_{lj}^{(k,h)} = z_{lo}^{(k,h)} - s_{lz}^{k-}, \ l = 1,...,\tau^{(k,h)}, \forall (k,h) \\ & \sum_{j\in E} \lambda_j^h z_{lj}^{(k,h)} = z_{lo}^{(k,h)} - s_{lz}^{h-}, \ l = 1,...,\tau^{(k,h)}, \forall (k,h) \\ & s_{lz}^{k-} = s_{lz}^{h-}, \forall (k,h), \\ & u_r^k, v_i^k \ge 1, \ i = 1,...,m^k, r = 1,...,r^k, k = 1,...,K \\ & t_l^k, t_l^h \ge 1, \qquad l = 1,...,\tau^{(k,h)}, \quad h,k = 1,...,K \\ & \begin{cases} d_j^k, \lambda_j^k \ge 0, d_j^k \le M^k \gamma_j^k, \lambda_j^k \le M^k \left(1 - \gamma_j^k\right), \\ \gamma_j^k \in \{0,1\}, j \in J, k = 1,...,K \end{cases} \\ & \lambda_j^h \ge 0, j \in E, h = 1,...,K \\ & s_r^{k+}, s_i^{k-} \ge 0, i = 1,...,m^k, r = 1,...,r^k, k = 1,...,K \\ & s_{lz}^{k-}, s_{lz}^{h-} \ge 0, \ l = 1,...,\tau^{(k,h)}, \quad h,k = 1,...,K \end{aligned}$$

Also, the closest target in scenario 2 may be considered as follows:

$$\begin{aligned} & \hat{x}_{io}^k = x_{io}^k - s_i^{k-*}, i = 1,...,m^k, k = 1,...,K \qquad (11) \\ & \hat{y}_{ro}^k = y_{ro}^k + s_r^{k+*}, \ r = 1,...,r^k, k = 1,...,K \\ & \hat{z}_{lo}^{(k,h)} = z_{lo}^{(k,h)} - s_{lz}^{k-*}, \ l = 1,...,\tau^{(k,h)}, \forall (k,h) \end{aligned}$$

According to T_G in the case of scenario 4 of Hassanzadeh and mostafaee, (2019) one can easily define the following slack based measure model to characterize whether the DMU under assessment is overall efficient or not.

$$\rho_o^{s4} = Min \frac{\sum_{k=1}^{K} w^k \left\{1 - \frac{1}{m^k + \tau^{(k,h)}} \left\{ \sum_{i=1}^{m^k} \frac{s_i^{k-}}{x_{io}^k} + \sum_{l=1}^{\tau^{(k,h)}} \frac{s_{lz}^{k-}}{z_{lo}^{(k,h)}} \right\}\right\}}{\sum_{k=1}^{K} w^k \left\{1 - \frac{1}{r^k + \tau^{(k,h)}} \left\{ \sum_{r=1}^{r^k} \frac{s_r^{k+}}{y_{ro}^k} + \sum_{l=1}^{\tau^{(k,h)}} \frac{s_{lz}^{k+}}{z_{lo}^{(k,h)}} \right\}\right\}}$$

$$\begin{aligned} s.t \quad & \sum_{j\in J} \lambda_j^k x_{ij}^k = x_{io}^k - s_i^{k-}, \ i = 1,...,m^k, k = 1,...,K \\ & \sum_{j\in J} \lambda_j^k y_{rj}^k = y_{ro}^k + s_r^{k+}, \ r = 1,...,r^k, k = 1,...,K \\ & \sum_{j\in J} \lambda_j^k z_{lj}^{(k,h)} = z_{lo}^{(k,h)} + s_{lz}^{k+} - s_{lz}^{k-}, \ l = 1,...,\tau^{(k,h)}, \forall (k,h) \\ & \sum_{j\in J} \lambda_j^h z_{lj}^{(k,h)} = z_{lo}^{(k,h)} + s_{lz}^{h+} - s_{lz}^{h-}, \ l = 1,...,\tau^{(k,h)}, \forall (k,h) \\ & s_{lz}^{k+} = s_{lz}^{h+}, s_{lz}^{k-} = s_{lz}^{h-}, l = 1,...,\tau^{(k,h)}, \forall (k,h), \qquad (12) \\ & \lambda_j^k \ge 0, \lambda_j^h \ge 0, \ j \in J, \ h,k = 1,...,K \\ & s_r^{k+}, s_i^{k-} \ge 0, i = 1,...,m^k, r = 1,...,r^k, k = 1,...,K \\ & s_{lz}^{h-}, s_{lz}^{h+} \ge 0, \ l = 1,...,\tau^{(k,h)}, \quad h,k = 1,...,K \end{aligned}$$

Superscript S4 means scenario 4. One can easily find that $0 \le \rho_o^{S4} \le 1$. DMU_o is called overall efficient if $\rho_o^{S4} = 1$. Otherwise, DMU_o is called overall inefficient. The stage-k efficiency score of DMU_o may be found as follows:

$$\rho_o^{k,S4} = \frac{1 - \frac{1}{m^k + \tau^{(k,h)}} \left\{ \sum_{i=1}^{m^k} \frac{s_i^{k-*}}{x_{io}^k} + \sum_{l=1}^{\tau^{(k,h)}} \frac{s_{lz}^{k-*}}{z_{lo}^{(k,h)}} \right\}}{1 + \frac{1}{r^k + \tau^{(k,h)}} \left\{ \sum_{r=1}^{r^k} \frac{s_r^{k+*}}{y_{ro}^k} + \sum_{l=1}^{\tau^{(k,h)}} \frac{s_{lz}^{k+*}}{z_{lo}^{(k,h)}} \right\}} \quad (13)$$

Stage k of DMU_o is efficient if $\rho_o^{k,s4} = 1$.

By utilizing the L_1-distance norm in scenario 4 of Hassanzadeh A. and Mostafaee A., 2019, in which the intermediate products are considered as the outputs under the control of previous stage and simultaneously as the inputs of next stage, the following Mixed-Integer linear programming problem may be considered to achieve the closest target:

$$\begin{aligned}
& Min \sum_{k=1}^{K}\sum_{i=1}^{m^k} s_i^{k-} + \sum_{k=1}^{K}\sum_{r=1}^{r^k} s_i^{k+} + \sum_{k=1}^{K}\sum_{l=1}^{\tau^{(k,h)}} s_{lz}^{k-} + \sum_{k=1}^{K}\sum_{l=1}^{\tau^{(k,h)}} s_{lz}^{k-} \\
& s.t \sum_{j\in E} \lambda_j^k x_{ij}^k = x_{io}^k - s_i^{k-}, \ i=1,\dots,m^k, k=1,\dots,K \\
& \sum_{j\in E} \lambda_j^k y_{rj}^k = y_{ro}^k + s_r^{k+}, \ \ r=1,\dots,r^k, k=1,\dots,K \\
& \sum_{j\in E} \lambda_j^k z_{lj}^{(k,h)} = z_{lo}^{(k,h)} + s_{lz}^{k+} - s_{lz}^{k-}, \ \ l=1,\dots,\tau^{(k,h)}, k=1,\dots,K \\
& \sum_{j\in E} \lambda_j^h z_{lj}^{(k,h)} = z_{lo}^{(k,h)} + s_{lz}^{h+} - s_{lz}^{h-}, \ \ l=1,\dots,\tau^{(k,h)}, k=1,\dots,K \\
& s_{lz}^{k+} = s_{lz}^{h+}, s_{lz}^{k-} = s_{lz}^{h-}, \forall (k,h), \qquad (14) \\
& \sum_{r=1}^{r^k} u_r^k y_{rj}^k + \sum_{l=1}^{\tau^{(k,h)}} t_l^k z_{lj}^{(k,h)} - \sum_{l=1}^{\tau^{(k,h)}} t_l^h z_{lj}^{(k,h)} \sum_{i=1}^{m^k} v_j^k x_{ij}^k + d_j^k = 0, \ j\in E, \forall (k,h) \\
& u_r^k, v_i^k \ge 1, \ i=1,\dots,m^k, r=1,\dots,r^k, k=1,\dots,K \\
& \qquad t_l^k, t_l^h \ge 1, \qquad l=1,\dots,\tau^{(k,h)}, \quad h,k=1,\dots,K \\
& \qquad \begin{cases} d_j^k, \lambda_j^k \ge 0, d_j^k \le M^k \gamma_j^k, \lambda_j^k \le M^k \left(1-\gamma_j^k\right), \\ \gamma_j^k \in \{0,1\}, \ j\in J, k=1,\dots,K \end{cases} \\
& \qquad \lambda_j^h \ge 0, \ j\in E, h=1,\dots,K \\
& \qquad s_r^{k+}, s_i^{k-} \ge 0, i=1,\dots,m^k, r=1,\dots,r^k, k=1,\dots,K \\
& \qquad s_{lz}^{h+}, s_{lz}^{h-} \ge 0, \ \ l=1,\dots,\tau^{(k,h)}, \qquad h,k=1,\dots,K
\end{aligned}$$

Also, the closest target in scenario 4 may be considered as follows:

$$\begin{aligned}
& \hat{x}_{io}^k = x_{io}^k - s_i^{k-*}, i=1,\dots,m^k, k=1,\dots,K \quad (15) \\
& \hat{y}_{ro}^k = y_{ro}^k + s_r^{k+*}, \ \ r=1,\dots,r^k, k=1,\dots,K \\
& \hat{z}_{lo}^{(k,h)} = z_{lo}^{(k,h)} + s_{lz}^{k+*}, \ \ l=1,\dots,\tau^{(k,h)}, \forall (k,h)
\end{aligned}$$

According to T_G in the case of scenario 5 of Hassanzadeh and mostafaee, 2019, one can easily define the following slack based measure model to characterize whether the DMU under assessment is efficient or not.

$$\rho_o^{s5} = Min \frac{\sum_{k=1}^{K} w^k = \left\{ 1 - \frac{1}{m^k} \left\{ \sum_{i=1}^{m^k} \frac{s_i^{k-}}{x_{io}^k} \right\} \right\}}{\sum_{k=1}^{K} w^k = \left\{ 1 + \frac{1}{r^k} \left\{ \sum_{r=1}^{r^k} \frac{s_r^{k+}}{y_{ro}^k} \right\} \right\}} \qquad (16)$$

$$\begin{aligned}
s.t \qquad & \sum_{j\in J} \lambda_j^k x_{ij}^k = x_{io}^k - s_i^{k-}, \ i=1,\dots,m^k, k=1,\dots,K \\
& \sum_{j\in J} \lambda_j^k y_{rj}^k = y_{ro}^k + s_r^{k+}, \ \ r=1,\dots,r^k, k=1,\dots,K \\
& \sum_{j\in J} \lambda_j^k z_{lj}^{(k,h)} = z_{lo}^{(k,h)}, \ \ l=1,\dots,\tau^{(k,h)}, \forall (k,h) \\
& \sum_{j\in J} \lambda_j^h z_{lj}^{(k,h)} = z_{lo}^{(k,h)}, \ \ l=1,\dots,\tau^{(k,h)}, \forall (k,h) \\
& \lambda_j^k \ge 0, \lambda_j^h \ge 0, \ j\in J, \ \ h,k=1,\dots,K \\
& s_r^{k+}, s_i^{k-} \ge 0, i=1,\dots,m^k, r=1,\dots,r^k, k=1,\dots,K
\end{aligned}$$

One can easily find that $0 \le \rho_o^{s5} \le 1$ DMU_o is called overall efficient if $\rho_o^{s5} = 1$ Otherwise, DMU_o is called overall inefficient.

By applying the L_1-distance norm in scenario 5 of Hassanzadeh A. and Mostafaee A., 2019, in which the intermediate products are considered neither the outputs under the control of previous stage nor the inputs under the control of next stage, we present the following Mixed-Integer linear programming problem for obtaining the closest target:

$$\begin{aligned}
& Min \sum_{k=1}^{K}\sum_{i=1}^{m^k} s_i^{k-} + \sum_{k=1}^{K}\sum_{r=1}^{r^k} s_i^{k+} \qquad (17) \\
& s.t \ \sum_{j\in E} \lambda_j^k x_{ij}^k = x_{io}^k - s_i^{k-}, \ i=1,\dots,m^k, k=1,\dots,K \\
& \quad \sum_{j\in E} \lambda_j^k y_{rj}^k = y_{ro}^k + s_r^{k+}, \ \ r=1,\dots,r^k, k=1,\dots,K \\
& \quad \sum_{j\in E} \lambda_j^k z_{lj}^{(k,h)} = z_{lo}^{(k,h)}, \ \ l=1,\dots,\tau^{(k,h)}, k=1,\dots,K \\
& \quad \sum_{j\in E} \lambda_j^h z_{lj}^{(k,h)} = z_{lo}^{(k,h)}, \ \ l=1,\dots,\tau^{(k,h)}, k=1,\dots,K
\end{aligned}$$

$$\begin{cases}\sum_{r=1}^{r^k} u_r^k y_{rj}^k + \sum_{l=1}^{\tau^{(k,h)}} t_l^k z_{lj}^{(k,h)} - \sum_{l=1}^{\tau^{(k,h)}} t_l^h z_{lj}^{(k,h)} \\ -\sum_{i=1}^{m^k} v_j^k x_{ij}^k + d_j^k = 0,\ j \in E, \forall (k,h)\end{cases}$$
$$u_r^k, v_i^k \geq 1,\ \ i = 1,\dots,m^k, r = 1,\dots,r^k, k = 1,\dots,K$$
$$t_l^k, t_l^h \geq 1,\ \ l = 1,\dots,\tau^{(k,h)},\ \ \ h,k = 1,\dots,K$$
$$\begin{cases} d_j^k, \lambda_j^k \geq 0,\ \ d_j^k \leq M^k \gamma_j^k,\ \ \lambda_j^k \leq M^k \left(1-\gamma_j^k\right), \\ \gamma_j^k \in \{0,1\}, j \in J, k = 1,\dots,K\end{cases}$$
$$\lambda_j^h \geq 0, j \in E, h = 1,\dots,K$$
$$s_r^{k+}, s_i^{k-} \geq 0, i = 1,\dots,m^k, r = 1,\dots,r^k, k = 1,\dots,K$$

Also, the closest target in scenario 5 may be considered as follows:

$$\hat{x}_{io}^k = x_{io}^k - s_i^{k-*}, i = 1,\dots,m^k, k = 1,\dots,K$$
$$\hat{y}_{ro}^k = y_{ro}^k + s_r^{k+*},\ \ r = 1,\dots,r^k, k = 1,\dots,K$$
$$\hat{z}_{lo}^{(k,h)} = z_{lo}^{(k,h)},\ \ l = 1,\dots,\tau^{(k,h)}, \forall (k,h) \quad (18)$$

4. Numerical example

Consider a supply chains 3-stage structure company with eight branches. The inputs, outputs and intermediate products of all stages are introduced as follows:

- Stage 1:
  - Inputs: $X_1, X_2 and\ X_3$
  - Intermediate products: $z_1^{(1,2)} and\ z_2^{(1,2)}$
- Stage 2:
  - Intermediate products: $z_1^{(2,3)} and\ z_2^{(2,3)}$
- Stage 3:
  - Outputs: $Y_1 and\ Y_2$

The data set is given in Table 1.

The results of proposed models and closest targets in scenario 1 and scenario 2 have been depicted in Table 2 and Table 3, respectively.

As it can be seen in Table 2, in scenario 1 the optimal link value is determined by the previous stage and the next stage has no control in determining the link value in this scenario. Table 2 shows the overall and stage efficiencies and also the closest targets for inefficient DMUS by solving Model (3) and Model (6). In this scenario we have four overall inefficient DMUs. Although DMUs B, E, F and G show overall inefficiency, it does not mean these DMUs are necessarily stage inefficient. It means that at least in one stage, they are inefficient.

Also, Table 3 shows the results in scenario 2. In this scenario the optimal link value is determined by the next stage and the previous stage has no role in determining the link value. As it is obvious in Table 3, we have two inefficient supply chains, A and E. The overall and stage efficiencies and also the closest targets for DMUs A and E have been calculated by solving Model (8) and Model (10). It can easily be seen that closest target in scenarios 4 and 5 are achievable by solving Model (14) and Model (17), respectively.

Table 1. Data set**.**

DMU	X_1	X_2	X_3	$Z_1^{(1,2)}$	$Z_2^{(1,2)}$	$Z_1^{(2,3)}$	$Z_2^{(2,3)}$	Y_1	Y_2
A	3.2	3.6	3.22	12	2.62	5.84	17.46	11.56	2.94
B	6.69	7.4	6.76	25.1	21.28	14.254	55.7	29.6	22.54
C	17.32	19.4	17.52	64.94	38.26	24	79.92	67.38	38.74
D	35.32	39.74	35.72	132.46	74.47	69.84	236.1	128.4	71.34
E	5.83	6.56	5.9	21.86	9.16	20.38	63.15	23.4	9.2
F	10.14	11.4	10.24	38	17.82	30.02	109.04	30.2	15.4
G	6.56	7.38	6.62	24.59	10.6	7.64	40.12	11.62	8.2
H	1.2	1.36	1.22	4.53	9.76	5.34	28.56	7.1	9.78

Table 2. The overall and stage efficiencies scores and closest projected target (scenario 1).

DMU	Overall	St. 1	St. 2	St. 3	$\widehat{X_1}$	$\widehat{X_2}$	$\widehat{X_3}$	$\hat{z}_1^{(1,2)}$	$\hat{z}_2^{(1,2)}$	$\hat{z}_1^{(2,3)}$	$\hat{z}_2^{(2,3)}$	$\widehat{Y_1}$	$\widehat{Y_2}$
A	1	1	1	1	3.2	3.6	3.22	12	2.62	5.84	17.46	11.56	2.94
B	0.92	1	0.86	1	4.34	5.21	4.25	26.2	22.3	15.23	56	31.2	24
C	1	1	1	1	17.32	19.4	17.52	64.94	38.26	24	79.92	67.38	38.74
D	1	1	1	1	35.32	39.74	35.72	132.46	74.47	69.84	236.1	128.4	71.34
E	0.52	1	0.42	0.61	5.23	5.45	4.3	22.4	12	23.34	66	25	11.21
F	0.63	1	1	0.32	9.18	10.5	8.7	42	19.21	32	112	32.32	16.54
G	0.82	1	1	0.65	4.46	7.38	6.62	24.59	11.2	8.42	42	12.34	9.6
H	1	1	1	1	1.2	1.36	1.22	4.53	9.76	5.34	28.56	7.1	9.78

Table 3. The overall and stage efficiencies scores and closest projected target (scenario 2).

DMU	Overall	St. 1	St. 2	St. 3	$\widehat{X_1}$	$\widehat{X_2}$	$\widehat{X_3}$	$\hat{z}_1^{(1,2)}$	$\hat{z}_2^{(1,2)}$	$\hat{z}_1^{(2,3)}$	$\hat{z}_2^{(2,3)}$	$\widehat{Y_1}$	$\widehat{Y_2}$
A	0.96	1	0.87	1	3.2	3.6	3.22	12	2.62	5.46	16.75	12.1	3.11
B	1	1	1	1	4.34	5.21	4.25	26.2	22.3	15.23	56	31.2	24
C	1	1	1	1	17.32	19.4	17.52	64.94	38.26	24	79.92	67.38	38.74
D	1	1	1	1	35.32	39.74	35.72	132.46	74.47	69.84	236.1	128.4	71.34
E	0.67	1	0.42	0.61	5.83	6.56	5.9	21.4	8.45	19.78	61.24	24.3	10.56
F	1	1	1	0.32	9.18	10.5	8.7	42	19.21	32	112	32.32	16.54
G	1	1	1	0.65	4.46	7.38	6.62	24.59	11.2	8.42	42	12.34	9.6
H	1	1	1	1	1.2	1.36	1.22	4.53	9.76	5.34	28.56	7.1	9.78

5. Conclusion

A careful study of different scenarios of network DEA made us think about the projection points and the closest targets. This attempt led to find the closest targets of network DEA in different scenarios of intermediate products (link control value). In particular, considering the intermediate products as outputs for previous stage, inputs for next stage, dual role of outputs and inputs for previous stage and inputs for next stage, respectively and neither the outputs for previous stage and inputs for next stage led to some new network DEA models to obtain the closest targets in each scenario.

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