Available online at http://ijdea.srbiau.ac.ir

Int. J. Data Envelopment Analysis (ISSN 2345-458X)

Vol. 9, No. 4, Year 2021 Article ID IJDEA-00422, Pages 89-102
Research Article

International Journal of Data Envelopment Analysis

Science and Research Branch (IAU)

Input Estimation in Two-Stage Systems with Undesirable Outputs Based on Cost Efficiency

Zahra Shiri Daryani[1], Shabnam Razavyan[*] [2,1]

[1] Department of Mathematics, South Tehran Branch, Islamic Azad University Tehran, Iran

[2] Research Center for Modeling and Optimization in Science and Engineering, South Tehran Branch, Islamic Azad University, Tehran, Iran

.

Received 9 July 2021, Accepted 13 October 2021

Abstract

In the Inverse Data Envelopment Analysis (InvDEA) models, inputs and outputs of Decision Making Units (DMUs) are estimated while their relative efficiency scores remain unchanged. But, in some cases, the input cost information is available. This paper uses the inputs cost information and generalized the InvDEA concept in two-stage network structures with undesirable output in the second stage. To this end, it proposes a four-stage method to deal with the InvDEA concept for estimating the inputs of the DMUs with a two-stage network structure method, while the allocative efficiency scores of all units remain stable. Eventually, an empirical example is presented to illustrate the capability of the presented method.

Keywords: Inverse DEA, Network DEA, Two-stage network, Cost efficiency, Input/output estimation, Undesirable output.

[*] Corresponding author: Email: sh_razavyan@azad.ac.ir

1. Introduction

The classical DEA methods use inputs and outputs and estimate the efficiency score of the DMU under evaluation. From another point of view, the InvDEA technique which was put forward by [1] aims at answering this question: if among a group of DMUs, we increase certain inputs to a particular unit and assume that the DMU maintains its current efficiency level with respect to other units, how many more outputs could the unit produce? Or if the outputs need to be increased to a certain level and the efficiency of the unit remains unchanged, how many more inputs should be provided for the unit? Many studies have been conducted on the InvDEA concept. The InvDEA problem with preference cone constraints discussed in [2]. Then [3] developed the presented method by [2] by using InvDEA to estimate output levels of a DMU, when some or all of its input entities are increased and its current efficiency level is improved. In addition, [4] suggested a method in which the increase in some inputs (outputs) and the decrease due to some of the other inputs (outputs) are taken into account at the same time. Furthermore, [5] developed an inverse BCC (InvBCC) model that can preserve relative efficiency values of all DMUs in a new production possibility set (PPS) composed of all DMUs and perturbed DMU with new input and output values. [6] discussed the InvDEA with the preference cone constraints in a way that in the DMUs, the undesirable inputs and outputs exist simultaneously. In addition, [7] developed a theoretical background of the InvDEA with pollution generating technology that is capable of dealing with undesirable outputs. Then, [8] proposed two-stage inverse data envelopment analysis models with undesirable outputs to formulate resource plans for 16 Chinese listed commercial banks whose outputs are increased and overall efficiency is kept unchanged in the short term.

Since the existing radial based DEA models neglect slacks while evaluating the overall efficiency level of DMUs, [9] proposed a model for a situation where the investigated DMU has no slack. The revised model can preserve radial efficiency index as well as eliminating all slacks. Moreover, [10] deal with the inverse DEA using the non-radial Enhanced Russell (ER)-measure in the presence of fuzzy data. In addition, [11] proposed the concept of invDEA which is called inverse non-radial DEA. They constructed the mathematical formula of inverse slack based model (SBM) which can overcome the error caused by ignoring slacks.

Since the symptoms of climate changes become more prevalent, [12] introduced an InvDEA method for allocation of CO_2 emissions reduction goal into different two-digit manufacturing industries and different regions. They addressed the CO_2 emission reduction in a three-stage phases. Moreover, a new InvDEA model for optimizing greenhouse gas emission (GHG) was introduced by [13]. The proposed model minimizes the overall GHG emissions by a set of DMUs in order to produce a certain level of outputs given that the DMUs maintain at least their existing performance statuses. In addition, [14] established a new measurement method by employing an InvDEA with frontier changes to evaluate China`s CO_2 emissions reduction.

InvDEA can be applied in managerial environment. [15], for example, used InvDEA problem for merging banks. They suggested a novel application of InvDEA in strategic decision making about mergers and acquisition in banking. Because in some cases of the suggested method, the merger may drop out of the PPS, [16] proposed a method to anticipate whether a merger in a market is generating a minor or major consolidation. Moreover, [17] introduced a new InvDEA method for mergers with negative data. Then, [18] presented two input oriented and output oriented inverse semi-oriented radial models which are applied to determine resource allocation and investment strategies for assessing sustainability of countries. Their proposed models can deal with both positive and negative data. [19] generalized a model in terms of return to scale properties which evaluates the efficiency of Chinese commercial banks after the global financial crisis, where negative outputs existed. Then, [20] proposed an InvDEA-R model for merger analysis that can deal with negative data. Moreover, the problem of target setting in a merger has been addressed by [21]. They considered the InvDEA method as a multi-objective problem and then utilized the goal programming (GP) approach for M&A problem, when there is a preference for saving the specific problem. Then, [22] developed two-stage InvDEA models for estimating potential gains from bank mergers for the top US commercial banks. Recently, [23] proposed a method based on common set of weights for studying multiple scenarios of merger and acquisitions.

As was mentioned before, classical DEA models use inputs and outputs data in order to assess the efficiency scores of DMUs, which are considered as black boxes. But, in some cases DMUs may have intermediate products. Two-stage network systems which consist of two divisions are connected together with intermediate measures. A two-stage network system consumes the exogenous inputs to produce outputs of the first stage, called intermediate products. Then the intermediate products are used as the inputs of the second stage to produce the output of the second stage which are also the final outputs of the whole system. There are many studies on the two-stage network concept. [24] used the independent model to assess the efficiency scores of the first stage, the second stage and the whole system of 55 U.S commercial banks. The connections and relationships between stages were not considered in their study. Moreover, [25] investigated the efficiency decomposition in a two-stage network system by taking series relationships of the two sub-processes into account in measuring the efficiencies. Furthermore, [26] provided an examination of the monotonicity of the decomposition weights in a two-stage DEA model with shared resource flows and found that the weight in such a model was not biased towards the second stage. The usage of constant weights in such a model is able to improve the discrimination of the efficient DMUs. Recently, [27] developed a high-tech industrial evaluation framework of technological innovation efficiency based on two-stage network DEA is constructed with shared inputs, additional intermediate inputs, and free intermediate outputs.

In some cases, the inputs/outputs costs information is available and the main issue is to minimize the overall costs of outputs. For example, [28] presented some models for computing technical, scale, cost and allocative efficiency scores in homogenous networks of processes. In addition, [29] proposed a two-stage cost efficiency DEA model that minimizes the cost of the hypothetical DMU, while maintaining the overall merger efficiency by comparing its minimal total cost with its actual cost. Furthermore, [20] incorporated the concepts of cost efficiency and InvDEA. He proposed a model that deals with the InvDEA problem when price information is available. The proposed model is based on cost efficiency problem and preserves technical and cost efficiency scores of DMUs unchanged, simultaneously. As a matter of fact, the allocative efficiency of all DMUs would stay unchanged. Moreover, [30] extended the proposed method by [31] to estimate input/output levels in the presence of input/output cost/price information in the two-stage network systems. In addition, [32] introduced the new models which are based on InvDEA for preserving cost and revenue efficiency, when data are changed. Moreover, [33] proposed a ranking system based on InvDEA in which enables the researcher to rank the efficient DMUs in an appropriate manner.

There are some approaches that incorporate the concepts of InvDEA and network DEA. For instance, a network-dynamic input-oriented RAM model and its inverse for assessing sustainability of supply chains were developed by [34]. The proposed model changes both inputs and outputs of DMUs so that the efficiency scores of DMUs would remain unchanged. Furthermore, [35] proposed a network-dynamic DEA model to assess the sustainability of supply chains in multiple periods. Then, they introduced an inverse network DEA model in a dynamic context. This paper incorporates the inverse two-stage network DEA and allocative efficiencies concept. Then the InvDEA concept is generalized to the two-stage structures with undesirable outputs in the second stage in the presence of inputs cost information. To this end, this paper uses the method proposed by [30] and suggests inverse cost efficiency DEA models for inputs estimation in two-stage network systems with undesirable outputs. The proposed method would like to answer the following inverse DEA question:

If among a group of comparable DMUs with two-stage network structure, the desirable and undesirable output levels of a unit increase to a certain level, how much more inputs are required with respect to unchanged technical and cost efficiency scores of all DMUs?

Eventually, an empirical example is presented to illustrate the capability of the presented method.

The rest of this paper is outlined as follows: in section 2, we review some basic concepts of DEA, InvDEA, the two-stage networks with undesirable outputs in the second stage and efficiency models. The inverse cost efficiency DEA models in two-stage network systems with undesirable outputs in the second stage are presented in section 3. Finally, to examine the proposed model, a case study is presented in section 4.

2. Preliminary

In this section the concepts of DEA, InvDEA, the basic two-stage networks, cost and revenue efficiency are reviewed.

2.1 DEA

DEA as a mathematical approach to evaluate the performance of DMUs with multiple inputs and outputs was proposed by [36]. Assume there are n DMUs to evaluate $\left(DMU_j, j = 1,\dots,n\right)$, which consume m inputs $\left(x_{ij}, i = 1,\dots,m\right)$ to produce s outputs $\left(y_{rj}, r = 1,\dots,s\right)$. The unit under evaluation $\left(x_o, y_o\right)$ is called

DMU_o.The input oriented DEA model was proposed by Charnes, Cooper and Rhodes and was called CCR and is presented as follows:

$$\theta_o^* = \min \quad \theta_o \qquad (1)$$
$$s.t. \quad \sum_{j=1}^{n} \lambda_j x_{ij} \le \theta_o x_{io}, \quad i=1,\dots,m$$
$$\sum_{j=1}^{n} \lambda_j y_{rj} \ge y_{ro}, \quad r=1,\dots,s$$
$$\lambda_j \ge 0, \quad j=1,\dots,n.$$

In the optimal solution of model (1), if $\theta_o^* = 1$, then DMU_o is called CCR efficient, otherwise DMU_o is inefficient.

2.2. Inverse DEA

InvDEA is a useful method for inputs/outputs estimation of a DMU. It was firstly proposed by [1] to solve this problem: if among a group of comparable DMUs, the output (input) levels of DMU_o increase, how much more inputs (outputs) should the unit consume (produce) in order that the efficiency score of the unit, θ_o^* would remain unchanged? To solve this problem, suppose the outputs of DMU_o are changed from $y_{ro}(r=1,\dots,s)$ to $\beta_{ro} = y_{ro} + \Delta y_{ro} \quad (r=1,\dots,s)$. We need to estimate the input vector $\alpha_i = x_i + \Delta x_i \quad (i=1,\dots,m)$ in order that the efficiency score of DMU_o would still be θ_o^* that obtained from model (1). The InvDEA model is:

$$\min \quad (\alpha_1, \alpha_2, \dots, \alpha_m) \qquad (2)$$
$$s.t. \quad \sum_{i=1}^{m} \lambda_j x_{ij} \le \theta_o^* \alpha_i \quad , \; i=1,\dots,m$$
$$\sum_{r=1}^{s} \lambda_j y_{rj} \ge \beta_{ro} \quad , \; r=1,\dots,s$$
$$\lambda_j \ge 0 \; , \; j=1,\dots,n$$
$$\alpha_i \ge 0 \; , \; i=1,\dots,m,$$

where all, $y_{rj}\ (r=1,\dots,s)$ and $\beta_{ro}\ (r=1,\dots,s)$ are given and we need to obtain $\alpha_i\ (i=1,\dots,\mathrm{m})$s.

Now assume that (λ,α) is a feasible solution of model (2). If there is no feasible solution $(\overline{\lambda},\overline{\alpha})$ such that for all $i=1,\dots,m$, $\overline{\alpha_i} \le \alpha_i$, then we say that (λ,α) is a weak efficient solution of model (3). It has been proven by [37] that if a revision for DMU_o from (x_o, y_o) to (α_o, β_o) is considered and it is assumed that (λ_o, α_o) is a weak efficient solution of model (3), then the efficiency scores of all DMUs would stay unchanged after revision.

There are several methods to solve the MOLP model (2) ([38]). Assume all inputs are weighed (priced) and the weights (values) are known. Let $w_i(i=1,\dots,m)$ be the value weight for i th input. To solve model (2), the weighed sum method is considered:

$$\min \sum_{i=1}^{m} w_i \alpha_i \qquad (3)$$
$$s.t. \quad \sum_{i=1}^{m} \lambda_j x_{ij} \le \theta_o^* \alpha_i \quad , \ i=1,\ldots,m$$
$$\sum_{r=1}^{s} \lambda_j y_{ij} \ge \beta_{ro} \quad , \ r=1,\ldots,s$$
$$\lambda_j \ge 0 \ , \ j=1,\ldots,n$$
$$\alpha_i \ge 0 \ , \ i=1,\ldots,m.$$

By solving the above single objective programming model, we can get new input levels.

2.3. Two-Stage Network with Undesirable Output

Suppose that there n $DMUs$ with two-stage network structure with undesirable outputs, which is shown in Fig1. For each $DMU_j \left(j=1,\ldots,n\right)$, stage 1 consumes m inputs $x_{ij} \left(i=1,\ldots,m\right)$ and produces h outputs $z_{gj} \left(g=1,\ldots,\mathrm{h}\right)$, which are called intermediate measures. Then h intermediate measures are treated as inputs in stage 2, which produces s desirable outputs $y_{rj} \left(r=1,\ldots,s\right)$ and K undesirable outputs $u_{kj} \left(k=1,\ldots,\mathrm{K}\right)$. [39] proposed the following input oriented DEA model to measure the performance of the two stage system with undesirable outputs, which is depicted in Figure1:

$$E_o^* = \min \ E_o \qquad (4)$$
$$s.t. \quad \sum_{i=1}^{m} \lambda_j x_{ij} \le E_o x_{io} \quad , \ i=1,\ldots,m$$
$$\sum_{g=1}^{h} \lambda_j z_{gj} \ge \tilde{z}_{go} \quad , \ g=1,\ldots,h$$
$$\sum_{g=1}^{h} \pi_j z_{gj} \le \tilde{z}_{go} \quad , \ g=1,\ldots,h$$
$$\sum_{r=1}^{s} \pi_j y_{rj} \ge y_{ro} \quad , \ r=1,\ldots,s$$
$$\sum_{k=1}^{K} \pi_j u_{kj} = u_{ko} \quad , \ k=1,\ldots,K$$
$$\tilde{z}_{go} \ge 0 \ , \ g=1,\ldots,h$$
$$\lambda_j \ge 0 \ , \ j=1,\ldots,n$$
$$\pi_j \ge 0 \ , \ j=1,\ldots,n;$$

Which optimal E_o^* represents the overall efficiency score of the two-stage system with undesirable outputs.

Definition1. If the cost efficiency score E_o^* of $\mathrm{DMU_o}$ is equal to 1, then $\mathrm{DMU_o}$ is overall efficient, otherwise $\mathrm{DMU_o}$ is inefficient.

2.4. Cost Efficiency DEA model

Traditional DEA models use inputs and outputs of DMUs to assess the efficiency score of units. But, in some cases, the prices or weights of inputs are known and we need to estimate the minimum cost of inputs.

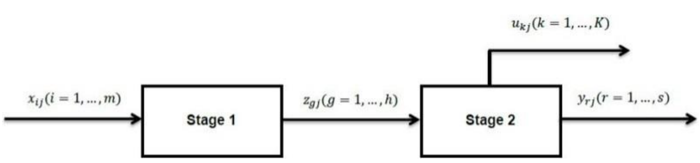

Figure1: Two-Stage Network System with Undesirable Outputs

Assume $c_i \in \mathbb{R}^+ \left(i=1,\ldots,m\right)$ is the input price (weight). Then, the observed cost of $\mathrm{DMU_o}$ with input-output vector

$\left(x_o, y_o\right)$ can be computed by $c^t x_o = \sum_{i=1}^{m} c_i x_{io}$. In order to obtain the cost efficiency score of $\left(x_o, y_o\right)$, first the following model is solved ([39]):

$$c^t x^* = \min \sum_{i=1}^{m} c_i x_i \qquad (5)$$

$$\begin{aligned} s.t. \quad & \sum_{j=1}^{n} \lambda_j x_{ij} \le x_i, \quad i = 1,\dots,m \\ & \sum_{j=1}^{n} \lambda_j y_{rj} \ge y_{ro}, \quad r = 1,\dots,s \\ & \lambda_j \ge 0, \quad j = 1,\dots,n \\ & x_i \ge 0, \quad i = 1,\dots,m. \end{aligned}$$

Then, the cost efficiency score of $\left(x_o, y_o\right)$ which is the ratio of optimal cost to actual cost can be obtained by:

$$CE_o = \frac{c^t x^*}{c^t x_o} = \frac{\sum_{i=1}^{m} c_i x_i^*}{\sum_{i=1}^{m} c_i x_{io}}. \qquad (6)$$

Definition2. If the cost efficiency score $\left(CE_o\right)$ of $\mathrm{DMU_o}$ is equal to 1, then $\mathrm{DMU_o}$ is called cost efficient, otherwise $\mathrm{DMU_o}$ is cost inefficient.

Definition3. The ratio of cost efficiency to technical efficiency is called allocative efficiency, that is $AE_o = \frac{CE_o}{\theta_o}$. If AE_o is equal to 1, $\mathrm{DMU_o}$ is called allocative efficient, otherwise $\mathrm{DMU_o}$ is allocative inefficient.

3. The Inverse Cost Efficiency Model in Two Stage Network Systems with Undesirable outputs

In this section, we are attempting to answer this question: if among a group of comparable DMUs with two-stage network structure, we increase output levels of DMUs, how many inputs are required in order that the technical and cost efficiency scores of DMUs stay unchanged? To answer this question, we follow the undergoing steps:

Step 1: Solve model (4) and suppose that $\left(E_o^*, \lambda^*, \pi^*, \tilde{z}^*\right)$ is the optimal solution.

Step 2: Assume $c_i \in \mathbb{R}^+ \left(i = 1,\dots,m\right)$ are input prices of the whole system (also input prices of stage1). Then, find the cost efficiency score of $\mathrm{DMU_o}$ before perturbation using the following two-stage network cost efficiency model:

$$c^t x^* = \min \sum_{i=1}^{m} c_i x_i \qquad (7)$$

$$\begin{aligned} s.t. \quad & \sum_{i=1}^{m} \lambda_j x_{ij} \le x_i \ , \ i = 1,\dots,m \\ & \sum_{g=1}^{h} \lambda_j z_{gj} \ge \tilde{z}_{go} \ , \ g = 1,\dots,h \\ & \sum_{g=1}^{h} \pi_j z_{gj} \le \tilde{z}_{go} \ , \ g = 1,\dots,h \\ & \sum_{r=1}^{s} \pi_j y_{rj} \ge y_{ro} \ , \ r = 1,\dots,s \\ & \sum_{k=1}^{K} \pi_j u_{kj} = u_{ko} \ , \ k = 1,\dots,K \\ & \tilde{z}_{go} \ge 0 \ , \ g = 1,\dots,h \\ & \lambda_j \ge 0 \ , \ j = 1,\dots,n \\ & \pi_j \ge 0 \ , \ j = 1,\dots,n; \end{aligned}$$

Using $\left(x_i^*, \lambda^*, \pi^*, \tilde{z}^*\right)$ as an optimal solution of model (7), the cost efficiency score of $\mathrm{DMU_o}$ is calculated by (6).

Step 3: Assume the desirable and undesirable output level of DMU_o perturbs from y_o to $\beta_o = y_o + \Delta y_o$ and from u_o to $\mu_o = u_o + \Delta u_o$, respectively. Also $(\gamma_{1o}, \gamma_{2o}, ..., \gamma_{ho})$ is the new intermediate measures of perturbed DMU_o. Solve the following cost efficiency model with new desirable and undesirable output levels:

$$
\begin{aligned}
c^t \tilde{x} = \quad & \min \sum_{i=1}^{m} c_i x_i && (8)\\
s.t. \quad & \sum_{i=1}^{m} \lambda_j x_{ij} \le x_i \quad , \quad i = 1,...,m\\
& \sum_{g=1}^{h} \lambda_j z_{gj} \ge \gamma_{go} \quad , \quad g = 1,...,h\\
& \sum_{g=1}^{h} \pi_j z_{gj} \le \gamma_{go} \quad , \quad g = 1,...,h\\
& \sum_{r=1}^{s} \pi_j y_{rj} \ge \beta_{ro} \quad , \quad r = 1,...,s\\
& \sum_{k=1}^{K} \pi_j u_{kj} = \mu_{ko} \quad , \quad k = 1,...,K\\
& \gamma_{go} \ge 0 \quad , \quad g = 1,...,h\\
& \lambda_j \ge 0 \quad , \quad j = 1,...,n\\
& \pi_j \ge 0 \quad , \quad j = 1,...,n;
\end{aligned}
$$

Assume that $(\tilde{x}_i, \tilde{\lambda}^*, \tilde{\mu}^*, \gamma^*)$ is an optimal solution of model (8).

Step 4: Now, we are going to find the minimum levels of inputs, in which the cost efficiency scores of DMUs remain unchanged by solving the following model:

$$
\begin{aligned}
\min \quad & (\alpha_1, \alpha_2, ..., \alpha_m) && (9)\\
s.t. \quad & \sum_{i=1}^{m} \lambda_j x_{ij} \le E_o^* \alpha_{io} \quad , \quad i = 1,...,m\\
& \sum_{g=1}^{h} \lambda_j z_{gj} \ge \eta_{go} \quad , \quad g = 1,...,h\\
& \sum_{g=1}^{h} \pi_j z_{gj} \le \eta_{go} \quad , \quad g = 1,...,h\\
& \sum_{r=1}^{s} \pi_j y_{rj} \ge \beta_{ro} \quad , \quad r = 1,...,s\\
& \sum_{k=1}^{K} \pi_j u_{kj} = \mu_{ko} \quad , \quad k = 1,...,K\\
& c^t \alpha = \frac{c^t \tilde{x}}{CE_o}\\
& \alpha_i \ge 0 \quad , \quad i = 1,...,m\\
& \eta_{go} \ge 0 \quad , \quad g = 1,...,h\\
& \lambda_j \ge 0 \quad , \quad j = 1,...,n\\
& \pi_j \ge 0 \quad , \quad j = 1,...,n.
\end{aligned}
$$

In model (9), the first set of constraint guarantees that the efficiency score of the DMU under evaluation stays unchanged. Also, the first five constraint sets ensure production possibility. Finally, the sixth constraint set ensures that the cost efficiency of DMU_o remains unchanged. Note that model (9) is an MOLP and can be inverted to single objective form to solve.

Theorem1. Assume DMU_o with two-stage network structure perturbs its desirable and undesirable output levels from y_o to $\beta_o = y_o + \Delta y_o$ and from u_o to $\mu_o = u_o + \Delta u_o$, respectively. If E_o^* is the optimal value of model (4) and $(\lambda, \pi, \alpha, \eta)$ is a weak efficient solution of the MOLP model (9); then, technical and cost efficiency of all DMUs remain unchanged after changing the inputs and outputs of DMU_o.

Proof. Consider the following problem:

$$E_o' = \min \ E_o \qquad (10)$$

$$\begin{aligned}
s.t. \quad & \sum_{i=1}^{m} \lambda_j x_{ij} \le E_o x_{io} \ , \ i = 1,...,m \\
& \sum_{g=1}^{h} \lambda_j z_{gj} \ge \tilde{z}_{go} \ , \ g = 1,...,h \\
& \sum_{g=1}^{h} \pi_j z_{gj} \le \tilde{z}_{go} \ , \ g = 1,...,h \\
& \sum_{r=1}^{s} \pi_j y_{rj} \ge y_{ro} \ , \ r = 1,...,s \\
& \sum_{k=1}^{K} \pi_j u_{kj} = u_{ko} \ , \ k = 1,...,K \\
& \tilde{z}_{go} \ge 0 \ , \ g = 1,...,h \\
& \lambda_j \ge 0 \ , \ j = 1,...,n \\
& \pi_j \ge 0 \ , \ j = 1,...,n;
\end{aligned}$$

We want to show that $E_o' = E_o^*$. Since $(\lambda, \pi, \alpha, \eta)$ is a weak efficient solution of the MOLP model (9), it satisfies the following conditions:

$$\begin{aligned}
& \sum_{i=1}^{m} \lambda_j x_{ij} \le E_o^* \alpha_{io} \ , \ i = 1,...,m \qquad (11) \\
& \sum_{g=1}^{h} \lambda_j z_{gj} \ge \eta_{go} \ , \ g = 1,...,h \\
& \sum_{g=1}^{h} \pi_j z_{gj} \le \eta_{go} \ , \ g = 1,...,h \\
& \sum_{r=1}^{s} \pi_j y_{rj} \ge \beta_{ro} \ , \ r = 1,...,s \\
& \sum_{k=1}^{K} \pi_j u_{kj} = \mu_{ko} \ , \ k = 1,...,K \\
& \eta_{go} \ge 0 \ , \ g = 1,...,h \\
& \lambda_j \ge 0 \ , \ j = 1,...,n \\
& \pi_j \ge 0 \ , \ j = 1,...,n.
\end{aligned}$$

From which it seems that E_o' is a feasible solution of model (10), and hence we have $E_o' \le E_o^*$. So, it has been shown that $E_o' \not< E_o^*$. Assume $E_o' < E_o^*$; then, the following condition is held:

$$E_o' = tE_o^* \ , \ 0 \le t \le 1. \qquad (12)$$

So, we have the subsequent constraints:

$$\begin{aligned}
& \sum_{i=1}^{m} \lambda_j x_{ij} \le E_o' \alpha_{io} = tE_o^* \alpha_{io} \ , \ i = 1,...,m \qquad (13) \\
& \sum_{g=1}^{h} \lambda_j z_{gj} \ge \eta_{go} \ , \ g = 1,...,h \\
& \sum_{g=1}^{h} \pi_j z_{gj} \le \eta_{go} \ , \ g = 1,...,h \\
& \sum_{r=1}^{s} \pi_j y_{rj} \ge \beta_{ro} \ , \ r = 1,...,s \\
& \sum_{k=1}^{K} \pi_j u_{kj} = \mu_{ko} \ , \ k = 1,...,K \\
& \eta_{go} \ge 0 \ , \ g = 1,...,h \\
& \lambda_j \ge 0 \ , \ j = 1,...,n \\
& \pi_j \ge 0 \ , \ j = 1,...,n.
\end{aligned}$$

It means that $(\lambda, \pi, t\alpha_o, \eta)$ is a feasible solution for the MOLP model (9) and this contradicts the assumption that $(\lambda, \pi, \alpha_o, \eta)$ is a weak efficient solution for the MOLP model (11). Then, the technical efficiency score of perturbed $\mathrm{DMU_o}$ is equal to the technical efficiency score of the original $\mathrm{DMU_o}$ and we have $E_o' = E_o^*$.

Following the proof of theorem by [20], we see that the cost efficiency of $\mathrm{DMU_o}$ stays unchanged. □

Remark 1. Since the technical and cost efficiency scores of $\mathrm{DMU_o}$ stay unchanged, the allocative efficiency of $\mathrm{DMU_o}$ which is the ratio of cost efficiency to technical efficiency, stays unchanged after $\mathrm{DMU_o}$ perturbs to new input-output levels.

4. Case Study

In this section, we examine our proposed model for a data set of 16 Chinese commercial banks from [41]. The data are given in Table 1. Operation cost (x_1), Interest expense (x_2) and the Labor (x_3) are three indicators which are considered as inputs (second, third and fourth column) to produce deposits (z_1) as an intermediate product (fifth column). Interest income (y_1) and Non-interest income (y_2) are the desirable outputs (columns six and seven). Non-performing loan balance (u_1) which is defined as non-performing loans at the end of the year is the undesirable output (column eight).

Table 1: Data set of [28]

Banks	Operation cost (100 million RMB)	Interest expense (100 million RMB)	Labor (Persons)	Deposits (100 million RMB)	Profit (100 million RMB)	Non-Interest Income (100 million RMB)	Non-Performing Loan Balance (100 million RMB)
	(x_1)	(x_2)	(x_3)	(z_1)	(y_1)	(y_2)	(u_1)
1	1768.29	3237.76	441902	146208.3	7671.11	1477.93	191.14
2	1693.97	2371.82	473766	118114.1	6133.84	875.85	19.33
3	1478.42	2354.1	251617	100977.9	5189.95	1245.09	78.23
4	1557.79	2567.09	368410	122230.4	6462.53	1208.98	106.46
5	538.12	1286.34	99919	41578.33	2592.92	341.7	73.15
6	458.96	745.82	51667	27752.76	1734.95	342.05	66.38
7	328.45	776.47	38803	26516.78	1633.35	191.34	77.11
8	266.05	926.27	38976	24196.96	1778.04	151.64	41.39
9	291.9	1037.57	33134	21703.45	1896.02	236.25	50.45
10	380.9	991.21	53064	21466.89	1821.54	332.01	28.81
11	212.79	524.14	28369	12170.02	931.02	115.86	6.75
12	176.23	373.51	25043	11775.92	762.53	63.62	11.04
13	207.81	692.2	31464	16052.78	1200.82	145.8	24.16
14	78.41	315.96	9193	8344.8	578.81	44.32	8.44
15	32.55	116.72	4357	2601.49	207.68	14.29	2.64
16	44.5	122.36	6310	2339.38	234.95	14.16	4.17

Assume that $(c_1,c_2,c_3)=(2,3,4)$ is the inputs price and let us increase both the desirable and undesirable output levels of all DMUs by 10 percent. Now we are interested in finding the required input levels of all DMUs in order to keep the cost efficiency score unchanged (the results are shown in columns eight to ten in Table 2. First, we find the technical and cost efficiencies of all units by solving models (4) and (6), respectively. The technical and cost efficiencies of all 16 units are depicted in the second and third columns of Table 2. As is seen in Table 2, DMU7 is allocative efficient.

Table 2: Technical, cost and revenue efficiencies and inputs changes of 27 banks based on inverse two-stage cost and revenue efficiency models

Banks	Technical Efficiency $\left(E_o^*\right)$	Cost Efficiency $\left(CE_o\right)$	Allocative Efficiency $\left(AE_o\right)$	Operation Cost changes $\left(\Delta x_1\right)$	Interest Expense changes $\left(\Delta x_2\right)$	Labor changes $\left(\Delta x_3\right)$	α_1	α_2	α_3
1	0.802	0.790	0.985	-0.044	0.221	0.072	1768.246	3237.981	441902.072
2	0.807	0.760	0.941	-0.077	0.256	-0.030	1693.893	2372.076	473765.970
3	0.808	0.772	0.955	-0.145	0.111	0.163	1478.275	2354.211	251617.163
4	0.802	0.766	0.956	-0.044	0.221	0.005	1557.746	2567.311	368410.005
5	0.800	0.765	0.956	-0.044	0.022	0.005	538.076	1286.362	99919.005
6	0.924	0.791	0.856	-0.234	0.260	-0.168	458.726	746.080	51666.832
7	1.000	1.000	1.000	-0.157	0.153	0.010	328.293	776.623	38803.010
8	0.934	0.918	0.983	0.068	0.087	0.071	266.118	926.357	38976.071
9	1.000	0.976	0.976	-0.016	-0.021	0.181	291.884	1037.549	33134.181
10	0.802	0.766	0.956	-0.044	0.221	0.005	380.856	991.431	53064.005
11	0.732	0.663	0.906	-0.097	0.397	0.048	212.693	524.537	28369.048
12	0.838	0.667	0.796	-0.272	0.196	-0.191	175.958	373.706	25042.809
13	0.802	0.766	0.956	-0.044	0.221	0.005	207.766	692.421	31464.005
14	0.802	0.766	0.956	-0.044	0.221	0.005	78.366	316.181	9193.005
15	0.889	0.865	0.973	0.035	0.026	0.115	32.585	116.746	4357.115
16	0.802	0.766	0.956	-0.044	0.221	0.005	44.456	122.581	6310.005

Consider DMU5 with the technical and cost efficiency score of 0.800 and 0.765, respectively. By 10 percent perturbation in its desirable and undesirable output levels, its first input levels will decrease by 0.044 units. Then, its second and third input level would increase by 0.022 and 0.005 units, respectively.
DMU11 has the lowest technical and cost efficiency score of 0.732 and 0.663, respectively. After perturbation in its desirable and undesirable output levels by 10 percent, its first input levels would decrease by 0.097 units. But, its second and third input level would increase by 0.397 and 0.048 units, respectively.
Now, consider DMU7 which is allocative efficient. It would have new input vector (328.293, 776.623, 38803.010) after 10 percent perturbation in its desirable and undesirable output levels.

5. Conclusion

In this paper, the inverse DEA methods was generalized for inputs estimation to the two-stage network systems with undesirable outputs in the presence of cost information. The proposed methods deal with inputs cost information and estimate input levels of the unit under evaluation in order to keep technical and cost efficiency scores unchanged.
In the proposed MOLP model, the decision maker's preferences can be considered in inputs weights in the inputs estimation procedure. The proposed method was applied to an empirical example in the presence of cost information of data.
A stream of future research can extend our framework in presence of desirable and undesirable price information.

References

[1] Wei, Q., Zhang, J., & Zhang, X. (2000). An inverse DEA model for inputs/outputs estimate. European Journal of Operational Research, 121(1), 151–163.

[2] Yan, H., Wei, Q., & Hao, G., (2002). DEA models for resource reallocation and production input/output estimation. European Journal of Operational Research, 136(1), 19–31.

[3] Jahanshahloo, G. R., Lotfi, F. H., Shoja, N., Tohidi, G., & Razavyan, S. (2004a). The outputs estimation of a DMU according to improvement of its efficiency. Applied Mathematics and Computation, 147(2), 409–413.

[4] Hadi-Vencheh, A., Foroughi, A.A., (2006). A generalized DEA model for inputs/outputs estimation, Mathematical and Computer Modeling, 43, 447-457.

[5] Lertworasirikul, S., Charnsethikul, P., & Fang, S. C. (2011). Inverse data envelopment analysis model to preserve relative efficiency values: The case of variable returns to scale. Computers & Industrial Engineering, 61, 1017–1023.

[6] Eyni, M., Tohidi, G., Mehrabian, S., (2016). Applying inverse DEA and cone constraint to sensitivity analysis of DMUs with undesirable inputs and outputs, Journal of the Operational Research Society, 68 (1), 34-40.

[7] Ghiyasi, M., (2017b). Industrial sector environmental planning and energy efficiency of Iranian provinces, 142 (4), 2328-2339.

[8] An, Q., Liu, X., Li, Y., Xiong, B., (2019), Resource planning of Chinese commercial banking systems using two stage, PLoS ONE, 14 (6), e0218214.

[9] Hu, X., Li, J., Li, X., Cui, J., (2020). A revised inverse data envelopment analysis model based on radial models, Mathematics, 8, 803.

[10] Ghobadi, S. (2018). Inverse DEA using enhanced Russell measure in the presence of fuzzy data. Int. J. Ind. Math., 10, 165–180.

[11] Zhang, G., Cui, J., (2020), A general inverse DEA model for non-radial DEA, Computer and Industrial Engineering, 142.

[12] Emrouznejad, A.,Yang, G., Amin, G.R., (2019). A novel inverse DEA model with application to allocate the CO2 emissions quota to different regions in Chinese manufacturing industries, Journal of the Operational Research Society, 70 (7), 1079-1090.

[13] Wegener, M., Amin, G.R., (2019). Minimizing greenhouse gas emissions using inverse DEA with an application in oil and gas, Expert Systems with Applications, 122, 369-375.

[14] Chen, Y., Chen, M., Li, T., (2021). China's CO2 emissions reduction potential: A novel inverse DEA model with frontier changes and comparable value. Energy Strategy Reviews, 38, 100762.

[15] Gattoufi, S., Amin, G. R., & Emrouznejad, A., (2014). A new inverse DEA method for merging banks. IMA Journal of Management Mathematics, 25(1), 73–87.

[16] Amin, G.R., Emrouznejad, A., Gattoufi, S., (2017). Minor and major consolidations in inverse DEA: Definition and Determination, Computers & Industrial Engineering 103, 193–200.

[17] Amin, G.R., Al-Muharrami, S., (2016). A new inverse data envelopment

analysis model for mergers with negative data, MA Journal of Management Mathematics, 29 (2), 137–149.

[18] Hassanzadeh, A., Yousefi., S., Farzipour Saen, R., Seyed Hosseininia, S, S., (2018). How to assess sustainability of countries via inverse data envelopment analysis? Clean Technologies and Environmental Policy, 20, 29-40.

[19] Ghiyasi, M., Zhu, N., (2020). An inverse semi-oriented radial data envelopment analysis measure for dealing with negative data. IMA Journal of Management Mathematics, 31(4), 505-516.

[20] Soltanifar, M., Ghiyasi, M., Sharafi, H. (2022a). Inverse DEA-R models for merger analysis with negative data. IMA Journal of Management Mathematics, 00, 1-20.

[21] Amin, G.R., Al-Muharrami, S., Mehdi Toloo, M., (2019). A combined goal programming and inverse DEA method for target setting in mergers, Expert Systems with Applications, 115, 412-417.

[22] Amin, G. R., & Ibn Boamah, M. (2021). A two-stage inverse data envelopment analysis approach for estimating potential merger gains in the US banking sector. Managerial and Decision Economics, 42(6), 1454-1465.

[23] Soltanifar, M., Ghiyasi, M., Emrouznejad, A., Sharafie, H., (2022b). A Novel Model for Merger Analysis and Target Setting: A Csw-Inverse DEA Approach. Available at SSRN 4115552.

[24] Seiford, L.M., Zhu, J., (1999). Profitability and Marketability of the Top 55 US Commercial Banks.”Management Science, 45, 1270-1288.

[25] Farrell, M.J., (1957). The measurement of productive efficiency, Journal of the Royal Statistical Society Series A, 120 (3), 253 – 281.

[26] Wang, Q., Wu, Z., Chen, X., (2019). Decomposition weights and overall efficiency in a two-stage DEA model with shared resources, Computers & Industrial Engineering, 136, 135-148.

[27] Wang, Y., Pan, J.F., Pie, R.M., Yi, B. W., Yang, G. L., (2020). Assessing the technological innovation efficiency of China’s high-tech industries with a two-stage network DEA approach, Socio-Economic Planning Sciences, In Press.

[28] Lozano, S., (2011), Scale and cost efficiency analysis of networks of processes, Expert Systems with Applications, 38, 6612–6617.

[29] Shi, X., Li, Y., Emrouznejhad, A., Xie, J., Liang. L., (2017). Estimation of potential gains from bank mergers: a novel two stage cost efficiency DEA model, Journal of the operational research society, 68, 1045-1055.

[30] Shiri Daryani, Z., Tohidi, G., Daneshian, B., Razavyan, Sh., Hosseinzadeh Lotfi, F., (2021). Inverse DEA in Two-Stage Systems Based on Allocative Efficiency, Journal of Intelligent and Fuzzy Systems, 40, 591-603.

[31] Ghiyasi, M., (2017a). Inverse DEA based on cost and revenue efficiency, Computers & Industrial Engineering, 114, 258–263.

[32] Soleimani-Chamkhorami, Kh., Hosseinzade Lotfi, F., Jahanshahloo, G. R., Rostamy-Malkhalifeh, M., (2019a), Preserving cost and revenue efficiency through inverse data envelopment analysis models, Information Systems and Operational Research, 1-18.

[33] Soleimani-Chamkhorami, Kh., Hosseinzade Lotfi, F., Jahanshahloo, G. R., Rostamy-Malkhalifeh, M., (2019b), A ranking system based on inverse data envelopment analysis, Journal of Management Mathematics, 1-19.

[34] Kalantary, M., Farzipoor Saen, R., Toloie Eshlaghy, A., (2018a). Sustainability assessment of supply chains by inverse network dynamic data envelopment analysis, Scientia Iranica, 25(6), 3723-3743.

[35] Kalantary, M., Farzipoor Saen, R., Toloie Eshlaghy, A., (2018b). Assessing Sustainability of Supply Chains: An Inverse Network Dynamic DEA Model, Computers & Industrial Engineering, 135, 1224-1238.

[36] Charnes, A., Cooper, W. W., & Rhodes, E. (1978). Measuring the efficiency of decision making units. European Journal of Operational Research, 2, 429–444.

[37] Ghiyasi, M. (2015). On inverse DEA model: The case of variable returns to scale. Computers & Industrial Engineering, 87, 407–409.

[38] Steuer, R.E., (1986). Multiple Criteria Optimization: Theory, Computation, and Application. Krieger Publishing, Malabar, FL.