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Estimating Production Function under Endogeneity: A Model Based on Data Envelopment Analysis

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Abstract

Endogeneity and its impact on estimating economic models can be seen in many economic studies. Data envelopment analysis is one of the most common non-parametric methods in which different axioms are used to estimate the production function (efficient frontier). However, the issue of endogeneity and its impact on estimating the efficient frontier is less considered. Cordero et al (2016) indicated that standard models of data envelopment analysis do not perform well in the presence of positive and high endogeneity. In this article, a model based on relaxing convexity axiom is presented in which the Cobb-Douglas function is considered as a real production function. Then, the efficiency of the proposed model is compared with the standard models of the data envelopment analysis and Cobb-Douglas function under positive and high endogeneity. The results show that the proposed model outperforms the counterparts. In addition, by comparing the models in different modes of return to scale, it is observed that the type of return to scale is also effective in determining the efficiency and efficiency of the proposed model compared to its economic counterpart.

Keywords: RDM model; Ideal and Anti Ideal Point; Inefficiency; Data Envelopment Analysis (DEA)

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1. Introduction

Efficiency measurement is one of the important issues in organizations that is used to improve their productivity and efficiency. The concept and method of measuring efficiency based on the production function is the maximum amount of output obtained from a certain amount of input. Thus, we need a production function, which is not available, to measure efficiency. Therefore, we are always looking for methods to estimate the production function. Estimation of the production function has been considered in economics significantly [1]. In economics, functions such as Cobb-Douglas, translog, etc. are considered as production functions, which the parameters of these functions are determined by observations and methods such as regression. Data Envelopment Analysis (DEA) is one of the non-parametric methods that uses standard models such as BCC (Bunker et al. [2]), CCR (Charnes et al. [3]) etc. to estimate the production function. In these models, a coverage curve that encompasses all true observations is constructed and then is used as a benchmark for calculating efficiency. In other words, the frontier created by the DEA models is an estimate for the production function. Choosing the Appropriate input and output can be effective in evaluating the correct measure of efficiency (Nataraja et al. [4] and Gattoufi et al. [5]). Numerous articles have also estimated the frontier of the production function, (Carlos et al. [6] and Sashuti et al. [7]). On the other hand, there is another important issue called endogeneity in the production process, the effects of which on the production function have been extensively studied in economics. Various factors such as measurement error and loss of a number of variables can cause endogeneity. In the statistical context, endogeneity occurs when there is no assumption that there is no correlation coefficient between inputs or variables [8]. Although, DEA is one of the most widely used nonparametric methods in estimating the production function, the study of endogeneity impact has received less attention from the users of DEA. In the scope of technical measurement of efficiency using non-parametric methods, the concept of endogeneity means the dependence between input and efficiency (Peyrach Vokli [8]). Cordero et al (2016) [9] examined the effect of endogeneity factor on production functions estimated by DEA and showed that the BCC model does not act accurately under positive and high endogeneity. They assumed the Cobb-Douglas and Translog economic functions as real functions and showed that the measure of efficiency obtained by the CCR and BCC models under endogeneity is not exact. Therefore, proposing an appropriate model in the framework DEA to reduce the impact of such factors in estimating the appropriate production function is important. However, an important point to consider in implementing this non-parametric technique is the assumption that the measure of efficiency is independent of inputs and outputs. According to studies conducted by Ruggiero [10,11], Bifolco and Bretschneider [12,13], Cordero et al. [9] found that the existence of endogeneity between efficiency and one of the inputs can affect the estimates DEA. Mayston [14] proposed a potential method for correcting estimates under endogeneity between input and output. Wilson [15] showed that in large units, because there is access to better management (more efficiency) than smaller units, they produce more output by consuming more input, thus there is a dependency between technical and artificial efficiency of the inputs. Schlotter et al. [16-17] also showed that the education section is a good example of endogeneity. Santin et al. [18]

presented a method for determining the correlation coefficient and endogeneity between input and efficiency measure. They also reduced the effect of endogeneity on estimating the production function by replacing the endogeneity input with a combination of exogenous inputs. In nonparametric technique DEA, different frontiers are defined based on different axioms. In other words, by manipulating axioms, different models with different frontiers are made. The convexity axiom is one of the axioms that various models have been presented by relaxing it. Maleki et al. [19] presented a model by relaxing the convexity axiom in which the obtained efficiency is closer to the real efficiency (assuming the Cobb-Douglas function is considered a real function) and under positive and high endogeneity conditions compared to the standard BCC the proposed model outperform the counterparts. Return to scale is another economic concept related to efficiency of DEA that determining the type and reviewing it has been highly regarded by researchers. Qianzhi et al. [20] first considered return to scale in cost allocation issues then investigated its effect. Considering the endogeneity impacts in the economic production process, and the inaccurate efficiency of nonparametric DEA models, in this paper, by relaxing the convexity axiom and considering the Cobb-Douglas function as a real function, a model for estimating the production function is presented that is closer to the real frontier (Cobb-Douglas frontier) than the BCC model. Then, the efficiency of Cobb-Douglas function in two modes of efficiency on a constant and variable return to scale is compared with the efficiency of standard models and the proposed model in the form of a practical example. The article is organized as follows. Section two reviews the basic concepts. In the third section, the proposed model (two-stage model) for measuring efficiency is discussed. In the fourth section, the efficiency of the proposed model is examined by providing numerical examples, and also the effect of the type of efficiency on a scale on the efficiency measurement of the models and its comparison with the Cobb-Douglas efficiency measurement is examined. The results section is the last section of the article.

2. Review of basic concepts

2.1. Standard DEA models

Many scientific activities have been done in order to evaluate the efficiency of units and sections. Certainly, the relationship between efficiency and influencing factors is a function as in which input produces output. The production function is a function that gives the maximum output for each combination of inputs. This function is very important in economics. Since it can be determined whether a unit is acting well or not. The production function is not available due to the complexity of the production process, changes in production technology and multiple quantities. The Cobb-Douglas function is one of the well-known economic functions used in many articles. The general form of the Cobb-Douglas function with three inputs and one output is as follows:

$$\ln \mathrm{y_i} = \alpha_1 \mathrm{lnx_{1i}} + \alpha_2 \mathrm{lnx_{2i}} + \alpha_3 \mathrm{lnx_{3i}} \quad (1)$$

Where $\mathrm{y_i}$ is the $\mathrm{i^{th}}$ output and x_{i1}, x_{i2}, x_{i3} are the observed input values and α_1, α_2, α_3 are the weights of the inputs (parameters). In this function, for $\sum \alpha_i = 1$ it is constant return to scale and for $\sum \alpha_i > 1$, it is variable return to scale. In parametric methods, it is tried to estimate this function using a series of observations and data. The curve fitting process is used for this purpose. However, obtaining the production function using this process has its drawbacks, including:

1) The relationships between inputs and outputs are considered arbitrary. In economics, for example, the relation $Q = x_{\circ} A_1^{x_1} ... A_n^{x_n}$ is considered for the production function, in which $A_1, ..., A_n$ are the inputs, Q is output, and $x_1, ..., x_n$ are parameters of the function that should be determined.

2) If the output vector dimension is more than one, this method cannot be used for problems that have only one output.

3) The obtained curve has a central inclination and it should be eliminated.

The above drawbacks are among the most basic drawbacks of the parametric method. In 1957, Farrell [21] introduced a non-parametric method that was considered the base of the work of Charnes et al. [3] and Bunker et al. [4]. In the non-parametric method, a set of possible activities or the production possibility set is stated as follows:

$T = \{(x, y) \in R^{m+n}$: The output vector $y \geq 0$ can be generated by the input vector $x \geq 0$ } (2)

Constructing production possibility set (PPS) is obtaining a frontier to estimate the production function, which gives the relative efficiency of the set of observations. Standard DEA models each depend on a set of unique production possibilities, and the production possibility set is uniquely constructed by a set of certain assumptions and axioms. Assuming we have an n observation as $(x_j, y_j)(j = 1, ..., n)$ that the input vector $x_j \geq 0 \circ, x_j \neq 0$ produces the output vector $y_j \geq 0$, $y_j \neq 0$. A T set is satisfying in the following axioms:

Axiom 1: (subject to observations): All of the observed activities belong to T. In other words:

$$(x_j, y_j) \in T \quad , \quad j = 1, ..., n$$

Axiom 2: Convexity

$$\forall (x,y), (\bar{x},\bar{y}), \forall \lambda \in [\circ,1]:$$
$$[(x,y), (\bar{x}, \bar{y}) \in T \Rightarrow$$
$$\lambda(x,y) + (1-\lambda)(\bar{x}, \bar{y}) \in T]$$

Axiom 3: (Infinity of radiation or efficiency at a constant scale).

$$\forall (x,y), \forall \lambda \geq \circ:$$
$$[(x,y) \in T \Rightarrow (\lambda x, \lambda y) \in T]$$

Axiom 4: Possibility

$$\forall (x,y) \in T, \forall \bar{x}, \bar{y}:$$
$$[(x,y) \in T, \bar{x} \geq x, \bar{y} \leq y \Rightarrow (\bar{x},\bar{y}) \in T]$$

Axiom 5: Extrapolation minimum: T is the smallest set that is true in the first to fourth axioms.

The set that applies to the above axioms is denoted by T_C or T_{CCR} as follows:

$$T_C = \left\{ (x,y) \left| \begin{array}{l} x \geq \sum_{j=1}^{n} \lambda_j x_j \ \& \\ y \leq \sum_{j=1}^{n} \lambda_j y_j, \lambda_j \geq 0, j = 1, ..., n \end{array} \right. \right\} \quad (3)$$

Using the above PPS and to evaluate the relative efficiency of homogeneous decision-making unit (DMU) that produces the input vector using the output vector, a model called was introduced by Charnes et al. in 1978 [3]:

$$\begin{array}{lll} \text{Min} & \theta & \\ \text{s} \cdot \text{t:} & \sum_{j=1}^{n} \lambda_j x_{ij} \leq \theta \, x_{ip}, & i = 1, ..., m \\ & \sum_{j=1}^{n} \lambda_j y_{rj} \geq y_{rp} & , \ r = 1, ..., s \\ & \lambda_j \geq 0 \circ & , \ j = 1, ..., n \end{array} \quad (4)$$

In this model, parameter θ is the minimum value that $(\theta x_o, y_o)$ will be on the T_C frontier. That is, the goal is to reduce the input level by a ratio θ so that at least the same output can be produced. The above model is called in input oriented and if $\theta^* = 1$, the unit is evaluated for efficiency, otherwise; it is considered inefficient.

Model (1) is also called a model with a constant return to scale. By accepting the axioms of inclusion of observations, convexity, possibility, and the minimum of interpolation, a set is constructed that is capable of production of T_V or T_{BCC}.

The only difference between this set with T_C is addition of a constraint $\sum_{j=1}^{n} \lambda_j = 1$. Based on this technology, Bunker et al. [2] proposed a model called BCC model. The envelopment form of this model is following linear programming model in the nature of input:

$$
\begin{aligned}
\text{Min}\ & \theta_B \\
\text{s.t.}\ & \sum_{j=1}^{n} \lambda_j x_{ij} \le \theta_B x_{iP}\ , \quad i = 1,\dots,m \\
& \sum_{j=1}^{n} \lambda_j y_{rj} \ge y_{rP}\ , \ r = 1,\dots,s \\
& \sum_{j=1}^{n} \lambda_j = 1 \\
& \lambda_j \ge 0 \quad , \qquad j = 1\ ,\ ,n
\end{aligned} \tag{5}
$$

It can be seen that this model is the same as the form of the CCR model to which the constraint $\sum_{j=1}^{n} \lambda_j = 1$ has been added. This model is also always possible and has a finite optimal answer. We always have $0 < \theta^* \le 1$ in optimality. The assumption of variable return to scale is also considered in this model.

$$
T_V = \left\{ \begin{array}{l} (x, y) \middle| x \ge \sum_{j=1}^{n} \lambda_j x_j, \\ y \le \sum_{j=1}^{n} \lambda_j y_j\ ,\ \sum_{j=1}^{n} \lambda_j = 1\ , \\ \lambda_j \ge \circ,\ j = 1,\dots,n \end{array} \right\} \tag{6}
$$

2.2. Relaxing the convexity axiom

The BCC model is presented on the basis of the assumption of variable return to scale, which makes a very accurate analysis by calculating the technical efficiency in terms of the values of efficiency due to scale and efficiency resulted from management. The BCC model can be expanded by relaxing the convexity axiom $\sum_{j=1}^{n} \lambda_j = 1$ as $L \le \sum_{j=1}^{n} \lambda_j \le U$. Where $L(0 \le L \le 1)$ and $U(U \ge 1)$ are the lower and upper bounds of the sets, respectively. It is noteworthy that $L = 0$ and $U = \infty$ corresponds to the CCR model and $U = L = 1$ corresponds to the BCC model. Given that the axiom of convexity is one of the basic assumptions in estimating the efficiency frontier, changing or extending the axiom of convexity can lead to the creation of different models. Maleki et al. [19] examined this issue with a two-step method for evaluating the efficiency of a unit based on the proposed efficiency frontier. In other words, instead of using a relation $\sum_{j=1}^{n} \lambda_j = 1$ in the BCC model, it is considered as an interval with variable frontiers. Without losing the whole issue, the lower and upper bounds can be defined as $[(1-\alpha),(1+\alpha)]$. So the relationship $\sum_{j=1}^{n} \lambda_j = 1$ is written as $1-\alpha \le \sum_{j=1}^{n} \lambda_j \le 1+\alpha$. Where we have $\sum_{j=1}^{n} \lambda_j = 1$ by $\alpha = 0$. The main goal of this work is to find the best value of α. To achieve this goal, a two-step method is proposed. In the first stage, the programming model is presented in the form of model (7) by adding a condition $1-\alpha \leq \sum_{j=1}^{n} \lambda_j \leq 1+\alpha$ to the standard BCC model and with a different objective function.

$$
\begin{aligned}
&\text{Min} \quad \theta + \varepsilon\alpha \\
&\text{s.t.} \quad \sum_{j=1}^{n} \lambda_j x_{ij} \leq x_{io}\theta \ , \quad i=1,...,m \\
&\qquad \sum_{j=1}^{n} \lambda_j y_{rj} \geq y_{ro} \ , \ r=1,...,s \qquad (7) \\
&\qquad 1-\alpha \leq \sum_{j=1}^{n} \lambda_j \leq 1+\alpha \ , \quad \lambda \geq 0
\end{aligned}
$$

It is easily proved that model (7) is always possible and finite. Assuming that the optimal answer of model (7) is $\alpha_j (j=1,...,n)$. By defining $\alpha' = \min\{\alpha_j, j=1,...,n\}$ we have the following model in the second step:

$$
\begin{aligned}
&\text{Min} \quad \theta \\
&\text{s.t.} \quad \sum_{j=1}^{n} \lambda_j x_{ij} \leq x_{io}\theta \ , \ i=1,...,m \\
&\qquad \sum_{j=1}^{n} \lambda_j y_{rj} \geq y_{ro} \ , \ \ r=1,...,s \qquad (8) \\
&\qquad 1-\alpha' \leq \sum_{j=1}^{n} \lambda_j \leq 1+\alpha', \ \lambda \geq 0
\end{aligned}
$$

The proposed frontier in model (8) is located between the two standard and known frontiers of standard CCR and BCC models .The proposed two-stage model can differentiate more accurately than the standard variable return to scale model or BCC.

3. The proposed model

By performing BCC and CCR models for random data with positive and negative endogeneity, Cordero et al. (2016) concluded that the frontier estimated by the BCC model under positive and high endogeneity has a distance with the real frontier (in Cordero's paper, Cobb-Douglas and Translog economic functions are considered real frontier) and the CCR model that has constant return to scale performs better than the BCC model [9]. In this section, a model is presented by modifying the BCC model in order to bring the estimated frontier to the economic frontier and reduce the difference between economic efficiency and the BCC model under positive and high endogeneity. The proposed model consists of two steps in which the Cobb-Douglas function is assumed to be the real frontier. It is based on the method proposed by Maleki et al. [19].

Step 1: In the first step, the constraint $\theta_{\text{COB}} \leq \theta$ is added to model (7) and the new model will be as follows:

$$
\begin{aligned}
&\text{Min} \quad \theta - \varepsilon\alpha \\
&\text{s.t.} \quad \sum_{j=1}^{n} \lambda_j x_{ij} \leq \theta x_{io}, \ i=1,...,m \\
&\qquad \sum_{j=1}^{n} \lambda_j y_{rj} \geq y_{ro} \ , \ r=1,...,s \qquad (9) \\
&\qquad 1-\alpha \leq \sum_{j=1}^{n} \lambda_j \leq 1+\alpha \\
&\qquad \theta_{\text{COB}} \leq \theta \\
&\qquad \lambda, \alpha \geq 0
\end{aligned}
$$

The added constraint $\theta_{\text{COB}} \leq \theta$ gives the lowest value θ in such a way that the Cobb-Douglas (θ_{COB}) efficiency is less than the shrinkage coefficient θ. In other words, the Cobb-Douglas frontier is high or tangential to the frontier defined in Model (6). The smaller the distance, the better the efficiency of the model. The objective function is also obtained in this maximum value α for each unit. By obtaining optimal answers of model (9) and determining the minimum value $\alpha_j : j=1,...,n$ or

$\alpha' = Min\{\alpha_j : j = 1,...,n\ ,\alpha_j \neq 0\}$ we go to the second stage.

Second stage: with $\alpha' = Min\{\alpha_j : j = 1,...,n\ ,\alpha_j \neq 0\}$ from the first stage, the proposed model of the second stage will be as follows:

$$\begin{aligned} &\text{Min} \quad \theta \\ &\text{s.t.} \quad \sum_{j=1}^{n} \lambda_j x_{ij} \le \theta x_{io}\ ,\ \ i = 1,...,m \\ &\qquad \sum_{j=1}^{n} \lambda_j y_{rj} \ge y_{ro}\ ,\quad r = 1,...,s \qquad (10) \\ &\qquad 1-\alpha' \le \sum_{j=1}^{n} \lambda_j \le 1+\alpha' \\ &\qquad \lambda_j \ge 0 \qquad\qquad j = 1,...,n \end{aligned}$$

It is observed that this model is the same as model (8). It is proved that this model is always possible and the amount of efficiency obtained from this model is always less than or equal to the amount of efficiency obtained from the BCC model [21].

Theorem: The new frontier is the best linear frontier that is close to the Cobb-Douglas function. This is equivalent to saying that α which we found is the best one, according to which the new frontier has the shortest distance from the Cobb-Douglas frontier.

Proof: Assume that $(\theta_k^*, \lambda_k^*, \alpha_k^*)$ is an optimal solution for model (9) for each DMU_k and α_k^* is maximum value of α that holds for the problem DMU_k.

a) Suppose that $\alpha' < \alpha_k^*$ exists that is satisfies in the conditions of model (9) in this case:

$$1-\alpha' \le \sum_{j=1}^{n} \lambda_j \le 1+\alpha'$$

Since $\alpha' < \alpha_k^*$, we have:

$$\sum_{j=1}^{n} \lambda_j \le 1+\alpha' \le 1+\alpha_k^*$$

$$\sum_{j=1}^{n} \lambda_j \ge 1-\alpha' \le 1-\alpha_k^*$$

Since S' is feasible region with α' and S is feasible region in model (9) according to α_k^*, we have:

$$S' \subseteq S$$

It means that the feasible region is reduced and the answer is not being optimal.

b) Suppose $\alpha'' > \alpha_k^*$ is existed that is hold in the problem condition, in this case:

$$1-\alpha'' \le \sum_{j=1}^{n} \lambda_j \le 1+\alpha''$$

Since, $\alpha'' > \alpha_k^*$ we have:

$$\sum_{j=1}^{n} \lambda_j \le 1+\alpha_k^* \le 1+\alpha''$$

$$\sum_{j=1}^{n} \lambda_j \ge 1-\alpha_k^* \le 1-\alpha''$$

If S' is the feasible region of the problem with α'' and S is the feasible region of model (9) based on α_k^* we have:

$$S \subseteq S'$$

That is, the feasible region is increased and the optimal value is not worsen:

$$\theta_{s'}^* < \theta_k^*$$

And this contradicts being optimal θ_k^*.

In the next section, we examine the efficiency of the proposed model by implementing it on 30 DMU and considering the Cobb-Douglas function as a real function under positive and high endogeneity. Also, considering that return to scale is one of the important axioms in DEA models (which is different in different models, for example, standard models BCC and CCR are defined by variable and constant return to scale, respectively) and the Cobb-Douglas function is defined by the variable or constant return to scale based on the considered parameters. With a numerical example, we examine the effect of the type of return to scale on the efficiency of models.

3.1. Evaluation of the efficiency of the proposed model under high and positive endogeneity conditions

In order to evaluate the efficiency of the proposed model under high endogeneity conditions, the Cobb-Douglas function of Eq. (1) is considered as a real function. Where y_i is the ith output obtained by replacing the inputs of Table 1 with x_{1i}, x_{2i}, x_{3i}. Here, the assigned input weight $\alpha_1 = 0/3$, $\alpha_2 = 0/35$ and $\alpha_3 = 0/35$ respectively, assuming a constant return to scale, and accordingly 30 DMU are obtained with three inputs and one output and endogeneity 0.65. Although this form of operation is commonly used in economics and operational research, the assumption of constant input and output tensions is likely to be a significant constraint on real-world estimates, meaning that the marginal effects of inputs on outputs are the same regardless of production scale. Table 1 shows the data for 30 DMU.

Table 1. Data related to 30 decision-making units

o1	i1	i2	i3	DMU
8.47062	34.42154	9.117443	6.594056	D01
16.99164	38.71092	23.23919	7.279092	D02
12.48751	31.24336	9.718081	7.977206	D03
15.27609	38.30145	10.05278	10.12688	D04
13.62349	15.56721	40.29926	10.12769	D05
18.13896	38.07309	18.12066	15.27945	D06
20.20783	48.67693	32.159	15.33655	D07
24.22406	44.01186	48.39902	15.36346	D08
13.34721	8.880554	24.46182	17.39441	D09
23.15318	21.48965	36.26385	17.68191	D10
19.421	21.61395	39.11447	18.99152	D11
18.0572	35.82628	24.4689	19.6544	D12
27.36067	31.90737	34.49741	19.8591	D13
16.81166	40.52138	9.938977	23.34786	D14
19.88782	21.54438	47.01919	24.99487	D15
13.75144	14.27125	13.43574	25.17925	D16
16.39591	8.899995	16.97805	25.97908	D17
27.91087	39.73703	40.90236	28.12167	D18
21.02067	14.25535	26.94217	31.09482	D19
19.9722	22.47222	39.60312	31.99456	D20
24.53788	29.83003	22.8203	32.14203	D21
19.90583	15.3029	17.28225	32.14834	D22
13.31615	33.88733	6.675559	33.35923	D23
23.02412	26.80162	35.29827	34.33987	D24
17.40338	11.83305	24.3304	39.01614	D25
31.89034	40.18694	25.32827	40.24697	D26
24.15679	9.527285	32.44357	43.18685	D27

15.28041	18.23299	7.673148	44.60298	D28
17.45992	15.68179	19.21151	45.05163	D29
29.96419	28.88925	39.7725	49.03537	D30

Table 2. Efficiency size obtained from the implementation of models

θ_{COB}	θ_{NEW}	θ_{BCC}	θ_{CCR}	DMU
0.705389	0.96	1	0.78	D01
0.975226	1	1	1	D02
0.975707	0.99	1	0.98	D03
0.985745	1	1	1	D04
0.760039	0.99	1	0.97	D05
0.854772	0.89	0.9	0.89	D06
0.735206	0.77	0.78	0.77	D07
0.796457	0.96	1	0.92	D08
0.841756	1	1	0.99	D09
0.981407	1	1	1	D10
0.78064	0.83	0.85	0.82	D11
0.71535	0.77	0.77	0.76	D12
0.99852	1	1	1	D13
0.802088	1	1	1	D14
0.679635	0.76	0.77	0.76	D15
0.791716	0.95	1	0.86	D16
0.997235	1	1	1	D17
0.793633	0.81	0.82	0.81	D18
0.894321	0.95	0.96	0.93	D19
0.648231	0.68	0.68	0.68	D20
0.871145	0.95	0.95	0.95	D21
0.943617	1	1	1	D22
0.664828	1	1	1	D23
0.715896	0.76	0.77	0.75	D24
0.742103	0.77	0.77	0.77	D25
0.919351	1	1	1	D26
0.964614	1	1	1	D27
0.786915	1	1	1	D28
0.698673	0.81	0.81	0.81	D29
0.765811	0.91	1	0.81	D30
0.8262	0.917	0.92767	0.90033	AVE
0.01225	0.01006	0.01035	0.0108	VAR
0.1107	0.10031	0.10174	0.10391	STDEV

According to the results of Table 2, the efficiency of the proposed model (θ_{NEW}) is closer to that of Cobb-Douglas. The D30, D16, D08, D01, D03, and D05 units identified in the efficient BCC model are inefficient in the proposed and Cobb-Douglas models. Also, according to the results, it is observed that the efficiency of the proposed BCC model is less than or equal to the efficiency of the BCC model and is greater than or equal to the efficiency of the model, and the efficiency of the Cobb-Douglas function is smaller than all of them. The production frontier of the proposed model is between or tangential BCC and CCR frontiers.

Finally, the results of the models are analyzed statistically. The last three rows of Table 2, show the mean, variance and standard deviation, respectively. The average efficiency of the proposed model is less than the BCC model and larger than the CCR model. That is, this mean is inserted between the mean of the two known models. In fact, it can be claimed that the proposed model is able to distribute efficiency. The variance of the proposed model is smaller than the variance of the BCC model. Also, both are smaller than the amount of CCR variance. In addition, the standard deviation of the proposed model is very small. It claims that the amount of efficiency obtained by the proposed model is close to the average. In practice, the dispersion of the efficiency measure in the proposed model is much less than the CCR model. In other words, the low standard deviation indicates that the efficiency size is spread over a more limited range. The results also confirm that the proposed model, increases the discriminatory power of models with variable return to scale.

4. Evaluation of return to scale on the efficiency of models

In this section, the effect of the type of return to scale in determining the measure of efficiency is examined with a numerical example, in which the efficiency of the proposed model is compared with BCC and CCR models in two modes of Cobb-Douglas function with constant and variable return to scale. For this purpose, the Cobb-Douglas function (1) is considered. In the case of constant return to scale, it is sufficient $\sum_{j=1}^{n}\lambda_j = 1$, and in the case of variable return to scale, it is $\sum_{j=1}^{n}\lambda_j > 1$. First, the input weights are assumed to be constant return to scale, and then the input weights $\alpha_1 = -0/1, \alpha_2 = 1/45, \alpha_3 = -0/35$ and then by assuming variable return to scale, the input weights $\alpha_1 = 0/6$, $\alpha_2 = 1/9$ and $\alpha_3 = 0/1$, for 25 bank branches with considered three input x_1, x_2 and x_3 the output of 25 branches are calculated. The results of implementing the models are given in Table 3.

Table 3. Efficiency measure obtained from the implementation of models in the case of efficiency in different scales

θ_{BCC}	$\theta_{NEW-VRS}$	$\theta_{COB-VRS}$	θ_{CCR}	$\theta_{NEW-CRS}$	$\theta_{COB-CRS}$	DMU
0.68	0.65	0.102085	0.26	0.41	0.149113	D01
0.73	0.69	0.109752	0.26	0.41	0.144913	D02
1	0.94	0.061643	0.14	0.48	0.076016	D03
1	0.96	0.089941	0.47	0.64	0.118607	D04
1	1	0.071071	1	1	0.096599	D05

0.61	0.57	0.037149	0.1	0.27	0.048911	D06
0.87	0.84	0.211211	0.54	0.62	0.309817	D07
0.66	0.62	0.05472	0.15	0.32	0.075694	D08
0.59	0.58	0.191019	0.52	0.54	0.283771	D09
0.44	0.43	0.117784	0.33	0.35	0.166141	D10
1	0.99	0.335681	0.88	0.89	0.4184	D11
1	0.98	0.331603	0.8	0.87	0.455247	D12
0.48	0.47	0.131756	0.42	0.42	0.184709	D13
1	0.95	0.156374	0.35	0.61	0.20525	D14
1	1	0.334897	1	1	0.476078	D15
0.49	0.46	0.020081	0.06	0.22	0.027255	D16
0.24	0.24	0.055321	0.22	0.22	0.087086	D17
0.61	0.58	0.096968	0.4	0.43	0.128666	D18
0.52	0.5	0.141425	0.38	0.41	0.196873	D19
0.46	0.43	0.01975	0.1	0.21	0.026909	D20
0.68	0.64	0.058062	0.22	0.36	0.077832	D21
0.56	0.52	0.050921	0.13	0.26	0.068402	D22
0.57	0.53	0.010738	0.06	0.25	0.014124	D23
0.52	0.48	0.05327	0.12	0.25	0.061433	D24
0.9	0.89	0.307677	0.84	0.85	0.420018	D25

The efficiency measurements of the Cobb-Douglas function are given in Table 3 for constant and variable return to scale $\theta_{COB-CRS}$ and $\theta_{COB-VRS}$ respectively. Also, the results obtained from the implementation of the proposed model for Cobb-Douglas function modes with variable and constant return to scale are recorded as $\theta_{NEW-VRS}$, $\theta_{NEW-CRS}$ respectively. And θ_{CCR}, θ_{BCC} are the measures of the efficiency of the CCR and BCC models, respectively. Figure 1 shows the results of BCC model implementation, the proposed model, and the Cobb-Douglas function in the variable return to scale efficiency mode.

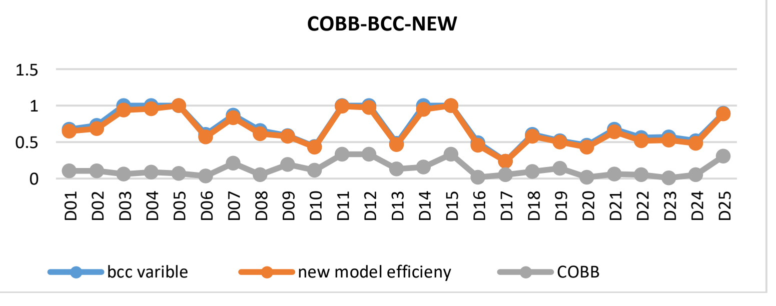

Figure 1: Results of model implementation in variable return to scale efficiency mode

According to Fig. 1, in the case of variable return to scale, efficiency obtained from the proposed model $(\theta_{NEW-VRS})$ is closed of efficiency of BCC. In other words, the BCC model performs well compared to the Cobb-Douglas function (which is considered the real production function). Also, units

D03, D04, D11, D12 and D14, which were identified in the efficient BCC model, are inefficient in the proposed model. These units are also inefficient Cobb-Douglas. In general, in the case of variable return to scale, the results of the model implantation have a shorter distance to efficiency BCC, and this distance is somewhat greater than Cobb-Douglas efficiency. Fig. 2 shows the results of the model implementation, the proposed model and the Cobb-Douglas function in the constant return to scale efficiency mode.

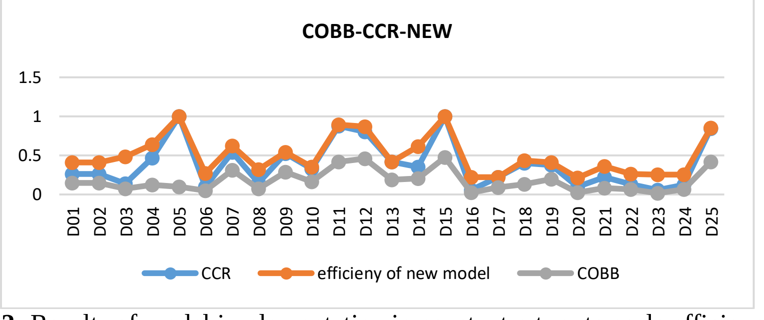

Figure 2: Results of model implementation in constant return to scale efficiency mode

According to Fig. 2, in the case of constant return to scale, units D05 and D15 have been found to be efficient in both CCR model and the proposed model. In these two models, the efficiency measurements of units D09, D10, D11, D12, D18, D19 and D25 are slightly different from each other, and units D13 and D17 have the same efficiency measurement in both models. In general, when the efficiency is constant return to scale, the results of the model implementation have a greater distance to CCR efficiency, and this distance is somewhat smaller than the Cobb-Douglas efficiency.

5. Conclusion

Since the estimated frontiers are constructed by DEA models based on observations and the axioms of a particular theme, real production frontiers (economic frontiers such as Cobb-Douglas, which are considered real frontiers) are always above or tangential to them. It is very important to bring the estimated frontier closer to the real frontier. Considering the importance of endogeneity and its effect in estimation DEA model, in this research, by relaxing the convexity axiom and considering the Cobb-Douglas function as a real production function, a two-stage model for estimating the production function was presented. The results of numerical examples and Theorems show that the proposed model performs better than the variable return to scale efficiency model under endogeneity, and The estimated frontier by the proposed approach is closer to the real production function, hence the evaluated efficiency is more accurate. What's more, in presence of different return to scale axioms, the comparison of estimated frontier via proposed model, standard DEA models and Cobb Douglas method reveals that return to scale axioms will affect the model performance.

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