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# Ranking Efficient Decision Making Units in Data Envelopment Analysis based on Changing Reference Set

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**Abstract**

One of the drawbacks of Data Envelopment Analysis (DEA) is the problem of lack of discrimination among efficient Decision Making Units (DMUs). A method for removing this difficulty is called changing reference set proposed by Jahanshahloo [1]. The method has some drawbacks. In this paper a modified method and new method to overcome these problems are suggested. The main advantage of this method is minimizing coefficient of variation t that has crucial role in ranking efficient DMUs. Numerical example for illustration suggested method are given. To validate new methods, the author compared the obtained result from new suggested method with Norm 1 which is efficient methods for ranking DMus.

**Keywords:** Data envelopment analysis, Coefficient of Variation, Efficiency, Changing Reference Set, Ranking.

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## 1. Introduction

Data Envelopment Analysis (DEA) is a fractional programming technique that was developed by Charnes [2,3]. It is used to measure the productive efficiency of decision making units (DMUs) and evaluate their relative efficiencies. This analysis determines the productivities of DMUs, specified as the ratio of the weighted sum of outputs to the weighted sum of inputs, comparing them to each other and determining the most efficient DMUs.

Ranking efficient DMUs is one of the problem which attracted researcher since 1978. Different Methods for this propose have been suggested, see paper by Adler [4]. Most of these methods are not allow to rank non extreme efficient points. The method suggested by Sexton called cross efficiency rank all kind of efficient DMUs. The modified methods of Sexton method are one of the most used for this propose. Changing reference set for ranking efficient DMUs was suggested by Jahanshahloo [1] the main drawbacks in this Method are as follows:

1) In case of having no inefficient DMUs the method cannot be used.

2) The main idea for ranking efficient is average deviation of inefficient DMUs from the original score, which seems to be not fair.

The first difficulty is not discussed in the paper, but we propose a method for removing second problem.

The rest of the paper contains the following by subject: In section 2 we explain DEA methods for measuring efficiency of DMUs, in section 3 Jahanshahloo [1] method will be discussed. Section 4 contains the proposed method. Numerical example for comparing the methods and illustration are given in Section 5. The last section summaries and concludes.

## 2. Data Envelopment Analysis (DEA) Background

The most basic Data Envelopment Analysis (DEA) model is the CRS (Constant Return to Scale) which was proposed by Charnes [2,3]. The basic idea of the CRS model is the following: The efficiency of an observed DMU (Decision Making Unit) which is the organization to be evaluated, can be measured by the ratio output per input, i.e., how well DMU can convert its inputs into its outputs. As we usually work in situations where we face multiples inputs and outputs, we are going to form a unique virtual output and a unique virtual input, for the observed $DMU_p$, by the yet unknown weights $v_i$ and $u_r$. By using Linear Programming (LP), we can find the weights that maximize the ratio output per input through the model:

$$
\begin{aligned}
&Max \quad \sum_{r=1}^{s} u_r y_{rp} \\
&S.t \quad \sum_{r=1}^{s} u_r y_{rj} - \sum_{i=1}^{m} v_i x_{ij} \le 0 \\
&\qquad j = 1,2,\dots,n \\
&\qquad \sum_{i=1}^{m} v_i x_{ip} = 1 \qquad (1) \\
&\qquad u_r \ge \varepsilon \qquad r = 1,2,\dots,s \\
&\qquad v_i \ge \varepsilon \qquad i = 1,2,\dots,m
\end{aligned}
$$

Where $x_{ij}$ is the data of input $i$ on the $DMU_j$, $y_{rj}$ is the data of the output $r$ on the $DMU_j$, $v_i$ is the weight of the input $i$ and $u_r$ is the weight of the output $r$. The dual form of Eq. (1) is:

$$
\begin{aligned}
&Min \quad \mu = \theta - \varepsilon\left(\sum_{i=1}^{m} s_i^- + \sum_{r=1}^{s} s_r^+\right) \\
&S.t \quad \sum_{j=1}^{n} \lambda_j x_{ij} + s_i^- = \theta x_{ip}, \\
&\qquad i = 1,2,\dots,m \\
&\qquad \sum_{j=1}^{n} \lambda_j y_{rj} - s_r^+ = y_{rp} \quad (2) \\
&\qquad r = 1,2,\dots,s \\
&\qquad \lambda_j \ge 0 \qquad j = 1,2,\dots,n \\
&\qquad s_i^- \ge 0 \qquad i = 1,2,\dots,m \\
&\qquad s_r^+ \ge 0 \qquad r = 1,2,\dots,s \\
&\qquad \theta \quad free
\end{aligned}
$$

Where μ is the efficiency measure and $\varepsilon$ is a non archimedean small and positive number so that the Eq. (1) is feasible and consequently objective function of (2) is bounded. We know that $DMU_p$is CRS-efficient if in Eq. (2). $\theta^* = 1$, $s_i^-$ = 0 and

$s_r^+= 0$, otherwise $DMU_p$ is CRS-inefficient. In order to determine the CRS-efficient DMUs, the DEA computer code can use a two-phase LP problem, which may be formalized as follows:

**Phase 1**. solve $\theta^* = \theta$ subject to (2).
**Phase 2**. incorporates this value $\theta^*$ instead of $\theta$ in with a new objective function: $Max\ \{\sum_{i=1}^{m} s_i^- + \sum_{r=1}^{s} s_r^+\}$.
For further details in DEA solving procedures readers are referred to [5]. It is important to note that $DMU_p$ is extreme efficient if and only if Eq. (2) has unique optimal solution:
$(\lambda_p^* = 1, \lambda_j^* = 0 \quad j = 1,2,\dots,p-1,p+1,\dots,n\ , s^+ = 0, s^- = 0)$.

## 3. DEA ranking system based on changing the reference set

A DMU that is strong efficiency by CRS or VRS model will be denoted by SE (Strong Efficient). The non-SE DMUs should be re-evaluated through (for more details see Jahanshahloo [1]):

$$\begin{aligned}
&Min\ \partial_{a,b} = \theta - \varepsilon\left(\sum_{i=1}^{m} s_i^- + \sum_{r=1}^{s} s_r^+\right) \\
&S.t\ -\sum_{j\in J-\{b\}} \lambda_j x_{ij} - s_i^- + \theta x_{ia} = 0 \\
&\qquad i = 1,2,\dots,m \\
&\qquad \sum_{j\in J-\{b\}} \lambda_j y_{rj} - s_r^+ = y_{ra} \\
&\qquad r = 1,2,\dots,s \qquad\qquad (3) \\
&\qquad \lambda_j \geq 0 \qquad\qquad j \in J-\{b\} \\
&\qquad s_i^- \geq 0 \qquad\qquad i = 1,2,\dots,m \\
&\qquad s_r^+ \geq 0 \qquad\qquad r = 1,2,\dots,s \\
&\qquad \theta \quad free
\end{aligned}$$

Where J = {1, 2, ..., n}, $a \in J_n,\ b \in J_{e,}\ J_n$ is the set of non-SE DMUs and $J_{e,}$ is the set of SE DMUs. After calculating the efficiency measure $\partial$ for all the non-SE DMUs, the efficiency of SE DMUs will be denoted by $\Omega$ and will be given by:

$$\Omega_b = \frac{\sum_{a\in J_n} \partial_{a,b}}{\tilde{n}} \qquad (4)$$

Where b is the evaluated SE DMU and en is the number of non-SE DMUs. The dual form of Eq. (5) is as follows:

$$\begin{aligned}
&Max \qquad \sum_{r=1}^{s} u_r y_{ra} \\
&S.t \qquad \sum_{r=1}^{s} u_r y_{rj} - \sum_{i=1}^{m} v_i x_{ij} \leq 0 \\
&\qquad\qquad j \in J-\{b\} \qquad\qquad (5) \\
&\qquad\qquad \sum_{i=1}^{m} v_i x_{ia} = 1 \\
&\qquad\qquad u_r \geq \varepsilon \qquad r = 1,2,\dots,s \\
&\qquad\qquad v_i \geq \varepsilon \qquad i = 1,2,\dots,m
\end{aligned}$$

And $\Omega$ is criteria for ranking of SE DMUs.

## 4. New Methods for ranking of SE DMUs based on Coefficient of Variation (CV)

Since means of non-SE DMUs is not logical criteria for ranking of SE DMUs so we define Coefficient of Variation (CV) for SE DMU, that it's as follows:

$$\sigma_b = \sqrt{\frac{\sum_{a\in J_n} (\partial_{a,b} - \Omega_b)^2/(\tilde{n}-1)}{\Omega_b}} \qquad (6)$$

Therefore $\frac{\Omega_b}{\sigma_b}$ is logical criteria for ranking of SE DMUs.
One of the new models for evaluating efficiency of SE DMUs ($b \in J_e$) based on $\sigma$, $\Omega$ that $\Omega_b$ is output and $\sigma_b$ is input:

$$\begin{aligned}
&\gamma_b = Max \qquad u\Omega_b \\
&S.t \quad u\Omega_j - v\,\sigma_j \leq 0 \qquad j \in J_e \\
&\qquad v\,\sigma_b = 1 \qquad\qquad (7) \\
&\qquad u \geq 0 \\
&\qquad v \geq 0
\end{aligned}$$

The other model suggestion is as follows (without input and two outputs):

$$\begin{aligned}
&\mu_b = Max \qquad u_1\Omega_b + u_2\frac{1}{\sigma_b} \\
&S.t \quad u_1\,\Omega_j - u_2\frac{1}{\sigma_j} \leq 1 \qquad j \in J_e \\
&\qquad u_1, u_2 \geq 0 \qquad\qquad (8)
\end{aligned}$$

$\gamma_b, \mu_b$ are criteria for ranking of SE DMUs. You can see the results of suggestion models in the section of examples.

## 5. Example

**Table 1 (Data of seven nursing homes)**

| Unit | $x_1$ | $x_2$ | $y_1$ |
|---|---|---|---|
| $DMU_1$ | 4 | 3 | 1 |
| $DMU_2$ | 7 | 3 | 1 |
| $DMU_3$ | 8 | 1 | 1 |
| $DMU_4$ | 4 | 2 | 1 |
| $DMU_5$ | 2 | 4 | 1 |
| $DMU_6$ | 10 | 1 | 1 |
| $DMU_7$ | 3 | 7 | 1 |

**Table 2**

| Unit | Efficiency | Super Efficiency (SEF) | Ranking (SEF) | Ranking Norm 1 (N1) |
|---|---|---|---|---|
| $DMU_1$ | 0.8571 | 0.8571 | 5 | 5 |
| $DMU_2$ | 0.6316 | 0.6316 | 7 | 7 |
| $DMU_3$ | 1 | 1.1429 | 3 | 3 |
| $DMU_4$ | 1 | 1.2500 | 2 | 2 |
| $DMU_5$ | 1 | 1.5714 | 1 | 1 |
| $DMU_6$ | 1 | 1 | 4 | 4 |
| $DMU_7$ | 0.6667 | 0.6667 | 6 | 6 |

In this Example: $j = 1,2,\dots,7, J_e = \{DMU_3, DMU_4, DMU_5\}, J_n = \{DMU_1, DMU_2, DMU_6, DMU_7\}$, $\tilde{n} = 4$

**Table 3**

| Unit | $\Omega$ | Ranking based on $\Omega$ | $\sigma$ | $\frac{1}{\sigma}$ | $\frac{\Omega}{\sigma}$ | Ranking based on $\frac{\Omega}{\sigma}$ |
|---|---|---|---|---|---|---|
| $DMU_1$ | - | - | - | - | - | - |
| $DMU_2$ | - | - | - | - | - | - |
| $DMU_3$ | 0.790950 | 3 | 0.19070 | 5.24383 | 4.14761 | 3 |
| $DMU_4$ | 0.858975 | 2 | 0.18151 | 5.50933 | 4.73238 | 2 |
| $DMU_5$ | 0.897025 | 1 | 0.18348 | 5.45018 | 4.88895 | 1 |
| $DMU_6$ | - | - | - | - | - | - |
| $DMU_7$ | - | - | - | - | - | - |

**Table 4**

| Unit | $\gamma$ | Ranking based on $\gamma$ | $\mu$ | Ranking based on $\mu$ |
|---|---|---|---|---|
| $DMU_1$ | - | - | - | - |
| $DMU_2$ | - | - | - | - |
| $DMU_3$ | 0.8484 | 3 | 0.9518 | 3 |
| $DMU_4$ | 0.9680 | 2 | 1.0000 | 2 |
| $DMU_5$ | 1.0000 | 1 | 1.0000 | 1 |
| $DMU_6$ | - | - | - | - |
| $DMU_7$ | - | - | - | - |

## 6. Conclusion

The modified models suggested in this paper are used to rank and evaluate the efficient DMUs. The results seem to be logical and have economic and managerial interpretation. As shown in numerical example the suggested new method could rank efficient DMUs by logical interpretation.

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