IJDEA

International Journal of Data Envelopment Analysis

Science and Research Branch (IAU)

Robust Portfolio Optimization and Performance Evaluation by mGH Distribution

M. Robatjazi[*1], Sh. Banihashemi[2], N. Modarresi

[1,2,3] Department of Mathematics, Faculty of Mathematical Sciences and Computer, Allameh Tabataba'i University, Tehran, Iran

Abstract
Financial returns exhibit stylized facts such as leptokurtosis, skewness and heavy-tailness. Regarding this behavior, in this paper, we apply multivariate generalized hyperbolic (mGH) distribution for portfolio modeling and performance evaluation, using conditional value at risk (CVaR) as a risk measure and allocating best weights for portfolio selection. Moreover, a robust portfolio optimization and performance evaluation modeling in mGH framework are developed, using worst case CVaR (WCVaR) as a risk measure. Due to the fact that expected returns can take negative values, the introduced model is inspired by Range Directional Measure model. Finally, real data in Iran stock market are given to illustrate the effectiveness of the model.

Keywords: Portfolio optimization, Multivariate generalized hyperbolic distribution, Efficiency, Worst case conditional value at risk.

---

*. Corresponding author: Email: Mortezarobatjazi.77@gmail.com

1. Introduction

Portfolio optimization and portfolio management are the most important problems from the past that have attracted the attention of investors. To solve this problem, Markowitz proposed his model that was named mean-variance (MV) model. He believed that all investors are interested to get maximum return with minimum risk in their investment. The optimal portfolio selection problem is a major issue in the financial field in which distribution of returns is usually non-Gaussian. In fact, financial returns have skewness and excess kurtosis. In recent years, several viable alternatives to the Gaussian distribution, capable of capturing commonly observed empirical features have been proposed for financial modeling. For example, Madan and Senata [10] suggested the Variance Gamma distribution; Eberlieen and Keller [6] advocated the use of the hyperbolic distribution and Eberlien [5] applied the generalized hyperbolic distribution. Helmich and Kassberger [8] showed that multivariate generalized hyperbolic (mGH) as a class of non-Gaussian distributions have a natural multivariate structure and are well-fitted to a mean-risk portfolio optimization problem. Furthermore, they applied the Monte Carlo simulation for portfolio selection. Since portfolio is a collection of assets, investors typically try to allocate their capital appropriately to earn higher return with risk management. These results help to evaluate the portfolio performance. One technique to consider the efficiency of portfolio performance is Data Envelopment Analysis (DEA) which introduced by Charnes et al. [4]. Morey and Morey [11] employed the mean-variance framework of Markowitz theory by considering a quadratic constrained non-linear DEA approach. They assumed that returns of assets are normally distributed but as noted by Fama [7], the distributions of returns are asymmetric and usually exhibit fat tail in practice. Furthermore, many studies show that the investors prefer positive skewed portfolios. So, Joro and Na [9] presented a non-linear DEA-like model to evaluate a portfolio efficiency which is based on mean-variance-skewness framework. Majority of DEA models cannot be used for the case in which DMUs include both negative and positive inputs/outputs. Portela et al. [13] represented a DEA model by name Range Directional Measure (RDM) model which can be used in cases where input/output data take positive and negative values. Banihashemi et al. [2] proposed a non-linear mean-variance and modified mean-variance-skewness based on RDM model for portfolio performance evaluation. They replaced variance by value at risk and tried to decrease it in a mean-value at risk model with negative data. Since, value at risk (VaR) as a risk measure is not always sub-additive nor convex, Rockafeller and Uryasev [14] defined an alternative risk measure named the conditional VaR (CVaR). Pflug [12] showed that the CVaR satisfies the requirements of the so-called coherent risk measures which is established by Artzner et al. [1]. These measures have four basic properties: translation invariance, positive homogeneity, subadditivity and monotonicity. Rockafellar and Uryasev [15] transformed mean-CVaR portfolio optimization problem into linear programming problem, based on generated scenarios.

In this paper, we consider the distributions of return by mGH and extend the mean-risk problem of the mGH distribution and utilize DEA technique into evaluating the portfolio efficiency. In addition to measuring the risk we use CVaR and WCVaR measures.

We propose our mean-risk models by mGH distribution. Also performance evaluation models are proposed which are based on RDM model. If the under evaluation asset is not located on the efficient frontier, we call it inefficient asset, then the model shows maximal proportionate reduction in risk and the same proportional maximization in the mean of return. The main advantage of the introduced models is that the mGH distribution that shows skewness and kurtosis of the return distribution is considered. Regarding to this model we can allocate better weights to portfolio and produce more accurate efficiency.

The rest of the paper is organized as follows. In section 2, we present coherent risk measures, CVaR, robust optimization, WCVaR and mGH distribution. In section 3, we propose our models of efficiency measurement based on mean-risk framework under mGH framework. Section 4 is devoted to a real application in Iran stock market. Finally, the conclusion and some remarks is presented in section 5.

2. Preliminary

In this section, we present some definitions which are needed in the following sections.

Definition 1. Assume (Ω, F, P) to be the probability space and $I(\Omega, F)$ to be the set of random variables of one dimensional on the space. The function $\rho : I(\Omega, F) \to R$ is a coherent risk measure whenever it satisfies following axioms for all $X, Y \in I(\Omega, F)$, X and Y are random variables:

a) Monotonicity: If $X \leq Y$, then $\rho(Y) \leq \rho(X)$;

b) Subadditivity: $\rho(X + Y) \leq \rho(X) + \rho(Y)$;

c) Translation Invariance: For all $\alpha \in R$, $\rho(X + \alpha) = \rho(X) - \alpha$;

d) Positive homogeneity: for all $\lambda \geq 0$, $\rho(\lambda X) = \lambda \rho(X)$.

Value at Risk (VaR) is a benchmark standard for firm-wide measures of risk.

Definition 2. For a given time horizon and confidence level $\beta \in (0,1)$, the VaR of a portfolio is the loss in the portfolio's market value over the horizon time that is exceeded with probability $1 - \beta$. In other words, VaR is defined according to $X \in I(\Omega, F)$ whose distribution is continuous

$$VaR_\beta(X) = \inf \{ x \in R \mid P(X \leq x) > \beta \}.$$

An alternative risk measure to VaR is Conditional Value at Risk (CVaR) which is also known as expected shortfall.

Definition 3. For a given time horizon and confidence level $\beta \in (0,1)$, CVaR is the conditional expectation of the losses exceeding VaR for the time horizon and the confidence level β which is defined as follows

$$CVaR_\beta(X) = E\left[X \mid X \geq VaR_\beta(X) \right].$$

CVaR can be reduced to a linear programming problem. We consider vector $w = (w_1, w_2, ..., w_n)$ which represents the position of each of n financial assets in a portfolio. $Y = (Y^1, Y^2, ..., Y^n)$ is the vector of assets mean returns. The loss function is defined as

$$f(w, Y) = -(w_1 Y^1 + w_2 Y^2 + ... + w_n Y^n) = -w^T Y.$$

The CVaR is specified as

$$CVaR_\beta(w) =$$

$$E\left[f(w,Y)\big|f(w,Y) \geq VaR_\beta(w)\right],$$

$$CVaR_\beta(x) =$$

$$VaR_\beta(x) + \frac{1}{1-\beta}E[(X - VaR_\beta(x))^+]$$

Then we have the following approximation function

$$\tilde{F}_\beta(w,x) =$$

$$x + \frac{1}{(1-\beta)Q}\sum_{q=1}^{Q}(f(w,Y_q) - x)^+ =$$

$$x + \frac{1}{(1-\beta)Q}\sum_{q=1}^{Q}(-w^T Y_q - x)^+,$$

Where $(Z)^+ = \max\{Z,0\}$ and Q represents the scenarios of assets log-return $Y_1, Y_2, ..., Y_Q$, where each elements $Y_q (q=1,2,...,Q)$ is a vector in R^n. Therefore, $CVaR_\beta(w)$ has an equivalent definition as follows

$$\min CVaR_\beta(w) = \min_{x \in R}\tilde{F}_\beta(w,x).$$

Definition 4. A random variable $T \in \mathbb{R}_+$ is said to have a Generalized Inverse Gaussian (GIG) distribution with parameters λ, χ and ψ, denoted by $T \sim GIG(\lambda, \chi, \psi)$ if its density is given by

$$f_{GIG}(y;\lambda,\chi,\psi) =$$

$$\begin{cases} \dfrac{\chi^{-\lambda}(\chi\psi)^{\lambda/2}}{2k_\lambda(\sqrt{\chi\psi})}y^{\lambda-1}\exp\left(-\dfrac{\chi y^{-1}+\psi y}{2}\right) & \lambda > 0 \\ 0 & \lambda \leq 0 \end{cases}$$

Where for $x > 0$, $k_\lambda(x)$ is the modified Bessel function of the third kind with index λ

$$k_\lambda(x) =$$

$$\frac{1}{2}\int_0^\infty y^{\lambda-1}\exp\left(-\frac{x(y+y^{-1})}{2}\right)dy$$

Definition 5. A random vector $Y \in \mathbb{R}^n$ is said to follow a d-dimensional mGH distribution with parameters $\lambda, \chi, \psi, \mu, \gamma$ and Σ, denoted by $Y \sim GH_n(\lambda, \chi, \psi, \boldsymbol{\mu}, \boldsymbol{\gamma}, \Sigma)$ if

$$Y = \boldsymbol{\mu} + T\boldsymbol{\gamma} + \sqrt{T}AZ$$

Where

1. $\boldsymbol{\mu}, \boldsymbol{\gamma} \in \mathbb{R}^n$ are deterministic.
2. $Z \sim N_k(0, \boldsymbol{I}_k)$ follows a k-dimensional normal distribution.
3. $T \sim GIG(\lambda, \chi, \psi)$ is a positive, scalar random variable independent of Z.
4. $A \in \mathbb{R}^{n \times k}$ denotes a matrix $n \times k$ and $\Sigma = A \times A'$.

Definition 6. Robust optimization is a field of optimization theory that deals with optimization problems in which a certain measure of robustness is sought against uncertainty that can be represented as deterministic variability in the value of the parameters of the problem itself and/or its solution.

Definition 7. Let ρ be a class of multivariate asset return distributions, let Y^p be a random vector of asset returns with distribution $p \in \rho$, and let $w \in \chi$ be a vector of portfolio weights. The WCVaR of a portfolio with weights w at level $\beta \in (0,1)$ is defined as

$$WCVaR_\beta^\rho(\boldsymbol{w}) = \sup_{p \in \rho} CVaR(-w^T Y^p)$$

Definition 8. Assume that $\rho = \{GH_d(\lambda, \chi, \psi, \mu, \gamma, \Sigma); (\mu, \gamma, \Sigma) \in M\}$

is a family of mGH distributions with λ, χ, ψ fixed. (μ, γ, Σ) is assumed to be

an element of a separable polyhedral uncertainty set $M = I_\mu \times I_\gamma \times I_\Sigma$ With

$$I_\mu = \left\{ \boldsymbol{\mu} \in \mathbb{R}^n ; \boldsymbol{\mu}_L \leq \boldsymbol{\mu} \leq \boldsymbol{\mu}_U \right\}$$

$$I_\gamma = \left\{ \boldsymbol{\gamma} \in \mathbb{R}^n ; \boldsymbol{\gamma}_L \leq \boldsymbol{\gamma} \leq \boldsymbol{\gamma}_U \right\}$$

$$I_\Sigma = \left\{ \begin{matrix} \boldsymbol{\Sigma} \in \mathbb{R}^{n \times n} ; \boldsymbol{\Sigma}_L \leq \boldsymbol{\Sigma} \leq \boldsymbol{\Sigma}_U , \\ \boldsymbol{\Sigma} \text{ positive definite} \end{matrix} \right\}$$

Compact intervals. All inequalities in the set definitions are to be understood component-wise. Since M is compact, WCVaR in the mGH distribution framework is as follows [6]

$$WCVaR_\beta^\rho \left(\boldsymbol{w} \right) =$$

$$\sup_{p \in \rho} CVaR_\beta \left(-\boldsymbol{w}^T Y^p \right) =$$

$$\max_{p \in \rho} CVaR_\beta \left(-\boldsymbol{w}^T Y^p \right)$$

3. Proposed Models under mGH distribution

Now, we present the return-risk model under mGH distribution. The mean return of asset and the risk measure of each asset is CVaR or WCVaR that they are computed by mGH distribution parameters, these items are simulated by Monte Carlo method. First, we introduce return-CVaR model under mGH distribution. By this model, we can obtain best weights for potfolio selection. The model determines as following

$$\min \quad x + \frac{1}{(1-\beta)Q} \sum_{q=1}^{Q} (-w^T Y_q - x)^+$$

$$s.t. \quad E(Y(w)) \geq r_f \tag{1}$$

$$Y_q \sim GH_n(\lambda, \chi, \psi, \mu, \gamma, \Sigma)$$

$$ew = 1$$

Where $Y(w) = \sum_{j=1}^{n} w_j Y^j$ For convenience, assume that $z_q = -w^T Y_q - x$. In this case, the model is rewritten as following:

$$\min \quad x + \frac{1}{(1-\beta)Q} \sum_{q=1}^{Q} z_q$$

$$s.t.$$

$$z_q \geq -w^T Y_q - x$$

$$z_q \geq 0 \tag{2}$$

$$E(Y(w)) \geq r_f$$

$$Y_q \sim GH_n(\lambda, \chi, \psi, \mu, \gamma, \Sigma)$$

$$ew = 1$$

The application of optimization methods to real world problems is not only dependent on numerical tractability, but also due to its power to analyze real problems. It should be noted that the smallest changes in input data are affected by optimization results. The main idea in robust optimization problems is to consider uncertainty sets in place of point estimates of unknown parameters. In this paper, we use the WCVaR in the mGH distribution framework.

According to the calculations performed for CVaR, the above equality can be rewritten as follows

$$WCVaR_\beta^\rho \left(\boldsymbol{w} \right) =$$

$$\max_{(\boldsymbol{\mu},\boldsymbol{\gamma},\boldsymbol{\Sigma}) \in M} \min_{x \in \mathbb{R}} \tilde{F}_\beta \left(\boldsymbol{w}, x; \lambda, \chi, \psi, \boldsymbol{\mu}, \boldsymbol{\gamma}, \boldsymbol{\Sigma} \right) \tag{a}$$

where

$$\tilde{F}_\beta(\boldsymbol{w}, x; \lambda, \chi, \psi, \mu, \gamma, \Sigma) =$$

$$x + \frac{1}{(1-\beta)Q} \sum_{q=1}^{Q} (-w^T Y_q - x)^+$$

Now, for the simplicity, the following proposition is presented.

Proposition 1. Let $\mathcal{X} \subseteq \mathbb{R}_+^d$ be a convex set.

(a) The $\tilde{F}_\beta(w, x; \lambda, \chi, \psi, \mu, \gamma, \Sigma)$ is component-wise monotonically

decreasing in μ and γ also component-wise monotonically increasing in Σ. In particular, for any $(\boldsymbol{w}, x) \in \mathbb{R} \times \boldsymbol{\mathcal{X}}$

(b) $\tilde{F}_\beta(\boldsymbol{w}, x; \lambda, \chi, \psi, \boldsymbol{\mu}, \boldsymbol{\gamma}, \boldsymbol{\Sigma})$ is convex in $(\boldsymbol{w}, x)$ on $\mathbb{R} \times \boldsymbol{\mathcal{X}}$.

Proof: [8] Therefore, according to the above proposition, we can rewrite the relation (a) as following

$$WCVaR_\beta^\rho(\boldsymbol{w}) =$$

$$\min_{x \in \mathbb{R}} \tilde{F}_\beta(\boldsymbol{w}, x; \lambda, \chi, \psi, \boldsymbol{\mu}_L, \boldsymbol{\gamma}_L, \boldsymbol{\Sigma}_U)$$

The robust model description is as following:

$$\min \quad x + \frac{1}{(1-\beta)Q} \sum_{q=1}^{Q} (-w^T Y_q - x)^+$$

$$s.t. \quad E(Y(w)) \geq r_f \qquad (3)$$

$$Y_q \sim GH_n(\lambda, \chi, \psi, \mu_L, \gamma_L, \Sigma_U)$$

$$ew = 1$$

For convenience, assume that $z_q = -w^T Y_q - x$. In this case, the model is rewritten as follows:

$$\min \quad x + \frac{1}{(1-\beta)Q} \sum_{q=1}^{Q} z_q$$

$$s.t.$$

$$z_q \geq -w^T Y_q - x$$

$$z_q \geq 0 \qquad (4)$$

$$E(Y(w)) \geq r_f$$

$$Y_q \sim GH_n(\lambda, \chi, \psi, \mu_L, \gamma_L, \Sigma_U)$$

$$ew = 1$$

First, we estimate the mGH distribution parameters by using the EM algorithm in models (1) and (2) and find the uncertain intervals for the parameters μ, γ, Σ. Then by using the Monte Carlo simulation for

$GH_d(\lambda, \chi, \psi, \boldsymbol{\mu}_L, \boldsymbol{\gamma}_L, \boldsymbol{\Sigma}_U)$, we generate Y_q $(1 \leq q \leq Q)$. So, by solving the model, optimum weights are obtained.

Another purpose in this paper is performance evaluation assets that can be done by below models. In real applications we deal with assets which they may have negative mean return. So we cannot utilize the conventional DEA models in assessing a portfolio efficiency. Then we apply the Range Directional Measure (RDM) model in our proposed models. In this section, we present our models based on RDM model that they are in DEA-like framework with mGH distribution. Following Banihashemi et al. [3]

$$g = (R_{E(Y^o)}, R_{CVaR_\beta(Y^o)})$$

$$\in [0, +\infty) \times [0, +\infty) \qquad (b)$$

is a vector shows a direction in which α is going to be maximized. So, we have this function

$$\xi : R^2 \to (0,1]$$

$$\xi(y) = \sup\{\alpha : y_o + \alpha g \in T \mid \alpha \in R_+\} \qquad (c)$$

Such that $y_o = (E(Y^o), CVaR_\beta(Y^o))$ is an under evaluation asset. Based on g and mentioned set of α, it is clear that the purpose is to increase mean of return and reduce CVaR as a risk measure of under evaluation asset in direction of vector g, simultaneously. Vector of direction could be chosen as

$$g = \begin{pmatrix} (\max(E(Y^j): \\ \quad j = 1,...,n) - E(Y^o)) = R_{E(Y^o)} \\ (CVaR_\beta(Y^o) - \min(CVaR_\beta(Y^j): \\ \quad j = 1,...,n)) = R_{CVaR_\beta(Y^o)} \end{pmatrix} \qquad (d)$$

Definition 9. A specified direction

$g = (R_{E(Y^o)}, R_{CVaR_\beta(Y^o)})$ is considered and an under evaluation financial instrument is $y_o = (E(Y^o), CVaR_\beta(Y^o))$, in which $CVaR_\beta$ is supposed as an input since it should be decreased and mean of return of a financial instrument is assumed as an output because it should be increased. In addition, we assume $Y(w) = \sum_{j=1}^{n} w_j Y^j$ and

$Y^j \sim GH_n(\lambda, \chi, \psi, \mu, \gamma, \Sigma)$. So, we solve following optimization model

$$\max \quad \alpha$$
$$st. \quad E(Y(w)) \geq E(Y^o) + \alpha R_{E(Y^o)}$$
$$CVaR(Y(w)) \leq CVaR_\beta(Y^o) - \alpha R_{CVaR_\beta(Y^o)} \quad (5)$$
$$ew = 1$$
$$w \geq 0.$$

The process of calculating α in model (5) is similar to RDM model. This model maximizes proportional reduction in CVaR as a risk measure, while it is maximizing return in the same proportion. This proportion is the inefficiency of under evaluation financial instrument. The vector $w = (w_1, w_2, ..., w_n)$ is of decision variables, and β is a probability level.

Also, According to the proposition 5 we can formulate model (5) by using the WCVaR as the new risk measure by considering $Y^j \sim GH_n(\lambda, \chi, \psi, \mu_L, \gamma_L, \Sigma_U)$.

Then, the robust portfolio optimization model is as follows

$$\max \quad \alpha$$
$$st. \quad E(Y(w)) \geq E(Y^o) + \alpha R_{E(Y^o)}$$
$$WCVaR(Y(w)) \leq \quad (6)$$
$$WCVaR_\beta(Y^o) - \alpha R_{WCVaR_\beta(Y^o)}$$
$$ew = 1$$
$$w \geq 0.$$

Optimal objective value of the model indicates different maximum proportionally changes in mean return and risk of the asset and tries to maximize α in directions of mean return and risk measure, separately. When an asset is located on the efficient frontier we call it efficient asset.

3. Numerical Example

In this section, we present a numerical example, based on empirical data. We collect the stock 's price of the 5 Iranian stock companies, namely Irankhodro (Khodro), Mallat Bank(Vabemellat), Esfahan oil refining(Shapna), tamin petroleum and Petrochemical investment co(Tapico) and Dana insurance(Dana). This dataset is selected from 2015/01/03 still 2019/01/03. According to Table 1, we conclude that because the data have skewness and kurtosis we can't use normal distribution for describing the returns of financial assets.

Table 1: Mean, Skewness and Kurtosis of companies

	mean return	skewness	kurtosis
khodro	-0.0013	0.5317	5.1683
vabmellat	0.0005	-0.5538	9.5007
Shapna	0.0062	0.9529	4.2040
tapico	-0.0015	-3.3425	37.8409
dana	0.0008	0.0617	4.1585

We calibrate an mGH model to 156 weekly returns of these stocks observed, Using the EM algorithm for calibration. Estimated parameters are obtained for the joint return distribution, where the order of the elements in the following vectors and matrices corresponds to the order in the above enumeration $\lambda = 1.7$, $\chi = 3.3514$, $\psi = 6.5396$

$$\mu = 10^{-3} * \begin{bmatrix} -20.7418 \\ -7.5893 \\ -14.1292 \\ -0.7789 \\ -16.5655 \end{bmatrix}, \gamma = 10^{-3} * \begin{bmatrix} 20.2789 \\ 6.5168 \\ 18.6986 \\ 2.0916 \\ 14.4954 \end{bmatrix}$$

$\Sigma =$

$$10^{-4} * \begin{bmatrix} 35.0773 & 0.1973 & -1.1073 & -0.3374 & -0.7095 \\ 0.1973 & 18.9592 & 6.2403 & 2.1108 & 1.3375 \\ -1.1073 & 6.2403 & 26.1478 & 4.9254 & 6.7039 \\ -0.3374 & 2.1108 & 4.9254 & 32.8908 & -1.4277 \\ -0.7095 & 1.3375 & 6.7039 & -1.4277 & 43.5559 \end{bmatrix}$$

X follows an mGH distribution with the above parameters. In this case, by using the Monte Carlo simulation for $GH_d(\lambda, \chi, \psi, \mu, \gamma, \Sigma)$, we generate $\{x_q; 1 \leq q \leq 1000\}$. Then

$$E[X] = 10^{-3} * \begin{bmatrix} -0.2560 \\ -0.5965 \\ 3.2731 \\ 2.6653 \\ -1.4370 \end{bmatrix}$$

$cov(X) =$

$$10^{-4} * \begin{bmatrix} 39.5679 & 0.4016 & -1.5121 & -1.3426 & -1.6536 \\ 0.4016 & 20.040 & 7.5745 & 2.3316 & 1.6427 \\ -1.5121 & 7.5745 & 30.795811 & 5.4967 & 8.2873 \\ -0.3406 & 2.3316 & 5.4967 & 35.0563 & -0.1341 \\ -1.6536 & 1.6427 & 8.2873 & -0.1341 & 43.8290 \end{bmatrix}$$

$corr(X) =$

$$10^{-4} * \begin{bmatrix} 1 & 0.0143 & -0.0433 & 0.0360 & -0.0397 \\ 0.0143 & 1 & 0.3049 & 0.0880 & 0.0554 \\ -0.0433 & 0.3049 & 1 & 0.1673 & 0.2256 \\ 0.0360 & 0.0880 & 0.1673 & 1 & -0.0034 \\ -0.0397 & 0.0554 & 0.2256 & -0.0034 & 1 \end{bmatrix}$$

$$skewness(X) = 10^{-3} * \begin{bmatrix} 0.130 \\ 0.0340 \\ 0.0997 \\ -0.0470 \\ 0.3011 \end{bmatrix},$$

$$kurtosis(X) = 10^{-3} * \begin{bmatrix} 3.2802 \\ 3.7286 \\ 3.1375 \\ 3.2504 \\ 3.5418 \end{bmatrix}$$

Apparently, the shapna features the highest expected return (0.3273 percentage points per week), while Dana has the lowest expected return. The volatility of Vabmellat is lowest and Dana can be seen to be substantially more volatile than the other stocks. Tapico exhibits only moderate negative skewness and all stocks excess kurtosis when observed on a weekly basis.

Based on these parameters and $\beta = 0.95$, we perform a mean-CVaR optimization. We solve model (2) By inserting x_q $(1 \leq q \leq 1000)$ in model, by using the GAMS software. We considered 100 portfolio with optimal weights. Some of them are shown in Table 2.

Table 2: weights and CVaR and return of portfolio in classical case

	Weight of khodro	Weight of vabMallat	Weight of Shapna	Weight of tapico	Weight of tapico	Risk of portfolio (CVaR)	Return of portfolio
Portfolio 1	0.2225	0.1818	0.1666	0.2147	0.2144	0.057690	$7.5754 * 10^{-4}$
Portfolio 2	0.2097	0.1877	0.1699	0.2212	0.2115	0.057698	$7.8295 * 10^{-4}$
Portfolio 3	0.2085	0.1824	0.1748	0.2233	0.2110	0.057729	$8.0836 * 10^{-4}$
. . .	. . .	. . .	. . .	. . .	. . .	. . .	. . .
Portfolio 49	0.2185	0.0171	0.3728	0.3106	0.0811	0.0660	$0.0019 * 10^{-4}$
Portfolio 50	0.2152	0.0049	0.3836	0.3082	0.0881	0.0663	$0.0020 * 10^{-4}$
Portfolio 51	0.2148	$5.5033*10^{-4}$	0.3864	0.3123	0.0860	0.0666	$0.00202 * 10^{-4}$
. . .	. . .	. . .	. . .	. . .	. . .	. . .	. . .
Portfolio 98	0	$10^{-16} * 2.9998$	0.9164	0.0836	$10^{-16} * 2.9998$	0.1024	$0.003222 * 10^{-4}$
Portfolio 99	$10^{-31} * 1.716$	$10^{-32} * 1.1355$	0.9582	0.0418	0	0.1056	$0.003247 * 10^{-4}$
Portfolio 100	$10^{-8} * 7.9617$	$10^{-8} * 3.8466$	1.0000	$10^{-7} * 1.1780$	$10^{-8} * 1.5876$	0.1092	$0.003273 * 10^{-4}$

Figure 1 presents the compositions of the efficient portfolios, The companies Vabmellat and Dana have the lowest contribution in the efficient portfolios, while the maximum-return portfolio consists solely of a position in Shapna.
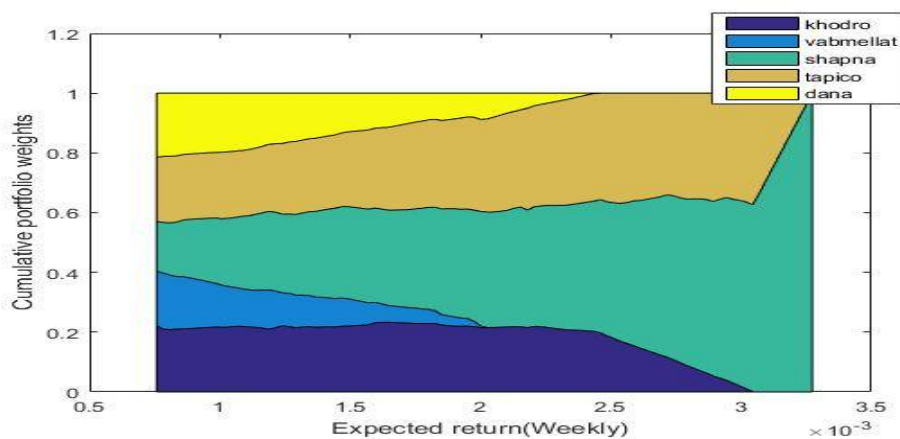

Figure 1: Composition of efficient portfolio

We assume that the uncertainty sets arise from the classical-case parameters presented above, with a shift of the latter either up or down by 10%:

$$\mu_L = \mu - 0.1\mu = 10^{-3} * \begin{bmatrix} -18.6676 \\ -6.8303 \\ -12.7163 \\ -0.7010 \\ -14.9089 \end{bmatrix}, \gamma_L = \gamma - 0.1\gamma = 10^{-3} * \begin{bmatrix} 18.2510 \\ 5.8652 \\ 16.8287 \\ 1.8824 \\ 13.0459 \end{bmatrix}$$

$$\Sigma_U = \Sigma + 0.1\Sigma =$$

$$10^{-4} * \begin{bmatrix} 38.5850 & 0.2170 & -1.2180 & -0.3712 & -0.7805 \\ 0.2170 & 20.8552 & 6.8643 & 2.3219 & 1.4712 \\ -1.2180 & 6.8643 & 28.7626 & 5.4179 & 7.3743 \\ -0.3712 & 2.3219 & 5.4179 & 36.1799 & -1.5704 \\ -0.7805 & 1.4712 & 7.3743 & -1.5704 & 47.9115 \end{bmatrix}$$

Let X follow an mGH distribution with the above parameters, in this case, by using the Monte Carlo simulation for $GH_d(\lambda, \chi, \psi, \mu_L, \gamma_L, \Sigma_U)$, we generate $\{x_q; 1 \leq q \leq 1000\}$. Based on these parameters and $\beta = 0.95$, we perform a mean-WCVaR optimization under minimum return constraints, i.e. we solve model (4) By inserting x_q $(1 \leq q \leq 1000)$ in model, by using the GAMS software, we obtained 100 portfolio with optimal weights, some of them are shown in Table 3.

We compare efficient frontiers of classic model and robust model in figure 2. By comparison of the efficient frontiers, we conclude that efficient portfolios are optimal and feasible in robust model.

Table 3: weight, return and WCVaR of portfolio in worst case

|	Weight ofkhodro	Weight of vabmellat	Weight of shapna	Weight of tapico	Weight of dana	Risk of portfolio (WCVaR)	Return of portfolio
Portfolio 1	0.1737	0.2790	0.1827	0.1967	0.1679	0.058317	$15.4073 * 10^{-4}$
Portfolio 2	0.1790	0.2741	0.1927	0.1889	0.1652	0.058326	$16.0998 * 10^{-4}$
Portfolio 3	0.17.36	0.2739	0.2014	0.1885	0.1626	0.058368	$16.7922 * 10^{-4}$
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$
Portfolio 49	0.1797	$0.6.77*10^{-21}$	0.5110	0.1934	0.1159	0.0694	$48.6459 * 10^{-4}$
Portfolio 50	0.1802	0	0.5226	0.1853	0.1118	0.0698	$49.3383 * 10^{-4}$
Portfolio 51	0.1758	$1.2817*10^{-33}$	0.5315	0.1814	0.1113	0.0703	$50.0308 * 10^{-4}$
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$
Portfolio 98	$1.23*10^{-23}$	$1.15 * 10^{-21}$	0.9764	0.0115	0.0120	0.1027	$82.57 * 10^{-4}$
Portfolio 99	0	$4.13*310^{-33}$	0.9868	$2.0309*10^{-32}$	0.0132	0.1035	$83.26 * 10^{-4}$
Portfolio 100	$3.2237*10^{-9}$	$2.7773*10^{-9}$	1.0000	$2.8870*10^{-9}$	$4.7938*10^{-10}$	0.1044	$83.96 * 10^{-4}$
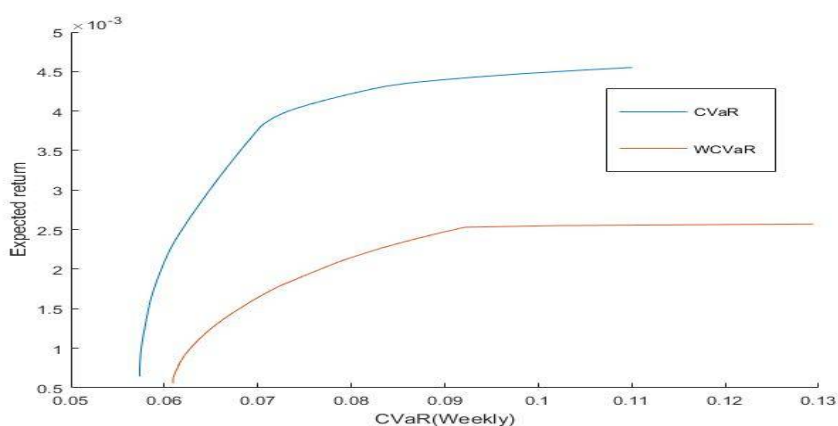
Figure 2: Efficien frontiers

Figure 1 presents the compositions of the efficient portfolios in worst case, comparing the compositions of classical-case and worst-case efficient portfolios (figures 1 and 3, respectively), one recognizes that the weight of the khodro and tapico has decreased throughout the full spectrum of expected returns, while the weight of the khodro, Dana and shapna increased.

The software Matlab was used to calculate CVaR andWCVaR companies by solving model (2) and model (4)

respectively. Also the software GAMS was used to measure the relative efficiency of companies and efficiency of companies in worst case by solving model (5) and model (6), respectively. In this model α shows amount of inefficiency. Therefore, when amount of α for company equal to zero, means that the company is efficient. According to table 4 the tapico is efficiency in classic model, the khodro and the tapico are efficiency in worst case.
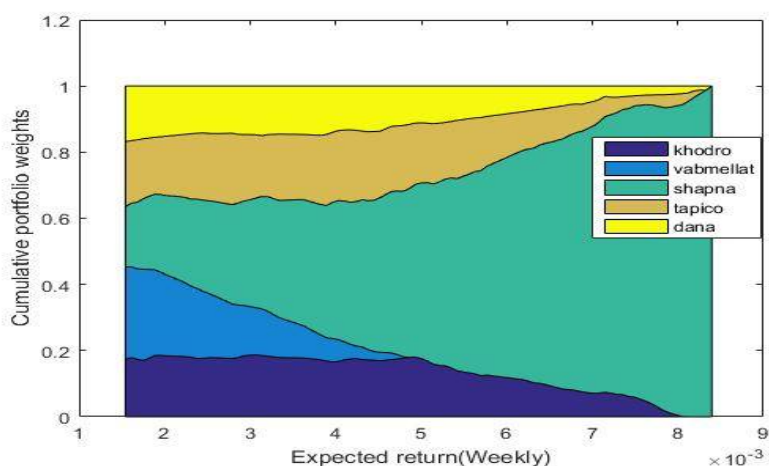

Figure 3: Composition of efficient portfolio in worst case

Table 4: risk,expected return and inefficiency of companies

	CVaR	Expected Return	inefficiency	WCVaR	Worst case Expected Return	Worst case inefficiency
khodro	0.1288	0.0003	0.08	0.1382	0.0038	0.00
vabMellat	0.0972	-0.0006	0.29	0.1000	-0.0021	0.40
Shapna	0.2092	0.0033	0.29	0.1044	0.0046	0.40
Tapico	0.1231	0.0027	0.00	0.1313	0.0025	0.00
Dana	0.1306	-0.0014	0.41	0.1480	-0.0052	0.59

4. Conclusion

The purpose of this paper is to utilize an appropriate distribution for fitting the data to it, applying appropriate risk measures and improving the results of the portfolio optimization. By generalizing the multivariate normal distribution (by randomness mean and variance of distribution), we obtain the family of distributions called the Normal mean-variance mixture. Then we use the mGH distribution that is a specific group of these distributions, to describe the data and evaluate assets. Also, number of efficient companies will increase in worst case. The conclusion that can be derived from this paper is that we need to describe the distribution of return assets because financial returns have skewness and kurtosis.

References

[1] Artzner P. Application of coherent risk measures to capital requirements in insurance. North American Actuarial Journal. 1999 Apr 1; 3(2): 11-25.

[2] Banihashemi S, Sanei M, Azizi M. Portfolio Performance Evaluation in a Modified Mean-Variance-Skewness Framework with Negative Data. International Journal of Data Envelopment Analysis. 2014 Sep 26; 2 (3): 409-21.

[3] Banihashemi , SH., Navidi, S. Portfolio performance evaluation in mean-CVaR framework:A comparison with nonparametric methods Value-at-Risk in mean-VaR analysis. Operations Research Perspective 2017; 4: 21-28.

[4] Banker RD, Charnes A, Cooper WW. Some models for estimating technical and scale inefficiencies in data envelopment analysis. Management science. 1984 Sep; 30 (9): 1078-92.

[5] Eberlein, E., Application of generalized hyperbolic Le´vy motion to finance. In Levy Processes, Theory and Applications, edited by O.E. Barndorff-Nielsen, T. Mikosch, and S. Resnick,. 2001; 319–336,

[6] Eberlein, E. and Keller, U., Hyperbolic distributions in finance. Bernoulli, 1995, 1, 281–299.

[7] Fama, E.F. The behavior of stock market prices. Journal of Business 1965; 38: 34-105.

[8] Hellmich M, Kassberger S. Efficient and robust portfolio optimization in the multivariate generalized hyperbolic framework. Quantitative Finance. 2011 Oct 1; 11(10): 1503-16.

[9] Joro T, Na P. Portfolio performance evaluation in a mean–variance–skewness framework. European Journal of Operational Research. 2006 Nov 16; 175(1): 446-61.

[10] Madan, D. and Seneta, E., The Variance Gamma model for share market returns. J. Business, 1990, 63, 511–524.

[11] Morey MR, Morey RC. Mutual fund performance appraisals: a multi-horizon perspective with endogenous benchmarking. Omega. 1999 Apr 1; 27(2): 241-58.

[12] Pflug GC. Some remarks on the value-at-risk and the conditional value-at-risk. InProbabilistic constrained optimization 2000 (pp. 272-281). Springer, Boston, MA.

[13] Portela, MC, Thanassoulis, E., Simpson, G. "A directional distance approach to deal with negative data in DEA: An application to bank branches. Operational Research Society 2004; 55(10): 1111-1121.

[14] Rockafellar, R.T., Uryasev, S. Conditional Value-at-Risk for general loss distributions. Journal of Banking and Finance 2002; 26(7): 1443-1471.

[15] Rockafellar, R.T., Uryasev, S. Optimization of Conditional Value-at-Risk. Journal of Risk 2000; 2:21-41.