Financial Risk Modeling with Markova Chain

Receipt: 19 , 1 , 2012   Acceptance: 8 , 5 , 2012

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Abstract

Investors use different approaches to select optimal portfolio. so, Optimal investment choices according to return can be interpreted in different models. The traditional approach to allocate portfolio selection called a mean - variance explains._Another approach is Markov chain. Markov chain is a random process without memory. This means that the conditional probability distribution of the next state depends only on the current state and not related to earlier events. This type of memory is called the Markov property._Based on proposed approach, the possibility of testing the assumption of independence of the intervals selected a portfolio of distribution of a relationship between these values there. The presence of this dependency, consider a model based on Markov chain makes it possible._In this paper, assuming that independent portfolios can be modeled by a Markov chain model to describe different portfolio selection, Value at risk (VaR) and Conditional Value at Risk (CVaR)._In fact, the portfolio return is selected, the ranges are divided into n range, each interval of a discrete Markov chains, we consider the situation._Finally, the results of this study indicate that the optimal portfolio selection based on Markov models arehigh performance but complex.

Keywords: Optimal portfolio, Markovian Chain, Transition Probability Matrics, Value at Risk, Conditional Value at Risk.

1. Introduction

In this paper we pursue two objectives. We first propose different markovian models that may be used to determine optimal portfolio strategies and to value opportunely the risk of a given portfolio. Then we compare portfolio selection strategies obtained either by modeling the return distributions with a Markov chain or by using a mean–variance analysis. Following the methodology proposed by Christoffersen [3], it is possible to test the null hypothesis that the intervals of the distributional support of a given portfolio are independent against the hypothesis that the intervals follow a Markov chain. Several empirical analyses, carried out by considering both different distributional hypotheses for many return portfolios (Gaussian, Stable Paretian, Student's *t*, and semi-parametric), and different percentiles θ, have shown that we cannot reject the markovian hypothesis. Therefore, the sequence of intervals of the distributional support are significantly dependent along time.

Accordingly, in this paper we assume that interval dependence of portfolios can be characterized by a Markov chain so that we can describe different portfolio selections, VaR and CVaR models. As a matter of fact, given a portfolio of gross returns, we share the support of the portfolio in N intervals and each interval is assumed to be a state of a Markov chain. Then, we build up the transition matrix and maximize the expected logarithmic utility function by assuming that in each interval the return is given by the middle point.

2. Literature review

Three major model paradigms have been developed in the literature: common factor models (Bluhm et al. 2001), mixed binomial models (Frey and McNeil 2003, SchÄonbucher and Schubert 2001) and dependent lifetime models (Li 2000). Let us brie°y discuss these approaches. The *common factor approach* originates on Merton's ¯rm value model (Merton 1974), developed by Va- sicek (1987) and Bluhm et al. (2001). The ¯nancial viability of debtor *i* is described by *vi = p ½c+ p* 1 ¡ *½ai*. Here *c* is a random variable common to all debtors and *ai* is an individual variable, which is independent of *c*. Debtor *i* will default in the next period, if *vi* · *v*, where *v* is a critical threshold. Typically *c* and *ai* are assumed to be normally distributed. However, Hull and White (2004) use *t*-distributions for both *c* and *ai*. The *mixed binomial model* assumes that the probability of default for an individual debtor, *q*, is a random variable. Then, even if conditional on *q*, the default events of debtors *i* and *j* are independent, the unconditional events are dependent, if the distribution of *q* is non-degenerate. A typical distribution for *q* is beta, and in multivariate cases Dirichlet. In the *dependent lifetime model* the time until default for debtor *i* is modelled by an exponential distribution with parameter ¸*i*, but the distributions for di®erent *i* are made dependent using a normal copula for the logarithms of the default times.

These models su®er from some drawbacks. They use distributional assumptions, which are di±cult to verify. Often there is no explicit correspondence between the

correlations (of portfolio components) which may be observed empirically and the

numerical parameters determining, typically via copulas, interdependence of assets

involved in a model. Also, some of these models are not in accordance with the transition matrices used in rating agencies. For attempts in harmonizing these

approaches see Koylouglu and Hickman (1998), and Bluhm et al. (2001).

3. Portfolio Selection with Homogeneous Markov Chains

Portfolio choice problem by describing the behavior of portfolios through a homogeneous Markov chain.

Let us consider $n + 1$ assets: n of these assets are risky with gross returns[1] z_{t+1} = [$z_{1,t+1}, \ldots, z_{n,t+1}$]_ and the ($n$+1)-th asset is characterized by a risk-free gross return $z_{0,t+1}$. If we denote with x_0 the weight of the riskless asset and with x = [$x_1, \ldots, x_n$]_ the vector of the positions taken in the n assets forming the risky portfolio, then the return portfolio during the period [t, t + 1] is given by

$$z_{(x),t+1} = \sum_{i=1}^{n} x_i z_{i,t+1} + x_0 z_{0,t+1}$$

Let us assume that the portfolio of gross returns has support on the interval (min*k* *z*(*x*),*k*;max*k* *z*(*x*),*k*), where *z*(*x*),*k* is the *k*-th past observation of the portfolio *z*(*x*). We first share the portfolio support (min*k* *z*(*x*),*k*;max*k* *z*(*x*),*k*) in *N* intervals (*a*(*x*),*i*; *a*(*x*),*i*+1) where *a*(*x*),*i* = _max*k* *z*(*x*),*k*

min*k* *z*(*x*),*k* _*i/N* min*k* *z*(*x*),*k*, *i* = 0, 1, . . .,*N*. For simplicity, we assume that on the interval (*a*(*x*),*i*; *a*(*x*),*i*+1) the state of the return is given by the geometric mean of the extremes *z*(*i*) (*x*) *a*(*x*),*ia*(*x*),*i*+1. Moreover, we add an additional state, *z* (*N*+1) (*x*) := *z*0, in the case we assume a fixed riskless return. Secondly, we build the transition matrix ***P****t* = [*pi*,*j*;*t*]1≤*i*,*j*≤*N* valued at

time *t* where the probability *pi*,*j*;*t* points out the probability (valued at time *t*)of a transition of the process between the state *z*(*i*)(*x*) and the state *z*(*j*)(*x*). On the other hand, if we consider an homogeneous Markov chain, the transition matrix is independent of time and it can be denoted simply by ***P***. We observe that the transition probability matrix associated with the Markov chain is usually sparse and this deeply reduces the computational costs. In constructing the approximating Markov chain, we need to choose the length of a time step and the number of states of the process. In portfolio selection problems we assume daily step with the convention that the Markov chain is computed on returns valued with respect to investor's temporal horizon *T* . For instance, if the investor recalibrates the portfolio every month (*T* = 20 working days), we consider monthly returns with daily frequency and compute on the portfolio series the relative

transition matrix. Moreover, for portfolio selection problems, it is better to use a limited number of states since the transition matrix is strictly dependent on the portfolio composition. As the portfolio composition is the variable of the optimization problem, the complexity of the problems becomes relevant when the number of states increases. However this does not excessively compromise the goodness of the investor's choices.

Under these assumptions, the final wealth (after *T* periods (days)) obtained investing *W*0 in the portfolio with composition (*x*0, *x*) is simply given by:

$$S_{(x),t+T} = \prod_{h=1}^{N+1} (z_{(x)}^{(h)})^{\sum_{i=1}^{T} v_{t+i}^{(h)}}$$

Where

v(*h*) (*t*+*i*) =< _1 if at (*t* + *i*)-th period the portfolio return is in the *s*-th state

0 otherwise

As a consequence of the Chapman-Kolmogorov equations, when at *t*-th time the portfolio is in the *m*-th state, the expected value of the logarithm of the final wealth is given by:

$$E_m(\log(S_{(x),t+T})) = \log(W_0) + \sum_{s=1}^{N+1} (\sum_{i=1}^{T} p_{m,s}^{(i)}) \log(z_{(x)}^{(s)})$$

where *p*(*i*)*m*,*s* is the element in position (*m*, *s*) of the *i*-th power of the transitionmatrix ***P****i*. The expected value of the log final wealth is :

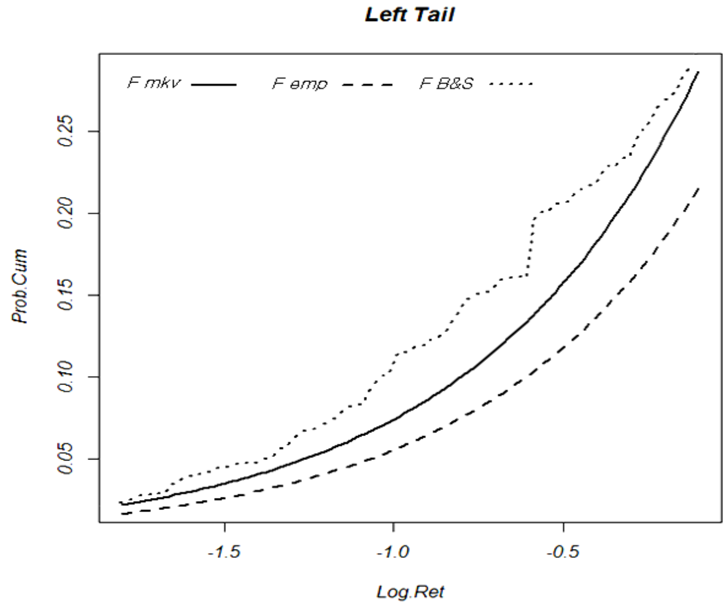
$$E_m(\log(S_{(x),t+T})) = \log(W_0) + \sum_{m=1}^{N+1} p_m \sum_{s=1}^{N+1} (\sum_{i=1}^{T} p_{m,s}^{(i)}) \log(z_{(x)}^{(s)})$$

where *pm* is the probability of being in the state *m*. When no short sales are allowed, an investor with logarithmic utility function and temporal horizon *T* tries to solve the following optimization problem. This fact is a consequence of the discretization process we adopt in building the approximating transiction matrix that depends on the portfolio composition. Thus, the sensitivity of the maximum expected utility respect to the portfolio composition implies that we have many local maximum in the above optimization problem. In order to approximate the optimal solution of portfolio problem (5), we consider two procedures.

Procedure 1

First we look for a local optimum near a potential optimal point. To verify our model, we consider the optimal allocation amongst 24 assets: 23 of these assets are risky and the 24-th is risk-free with annual rate 20%. Our dataset consists of monthly gross returns (20 working days).

Figure 1 reports a comparison between the markovian approach and the mean–variance one. In both cases the initial wealth is one and the portfolio is calibrated 60 times according to the procedure proposed in Leccadito et al., [9].



Procdure 2

VaR and CVaR Models with Markov Chains

In this section we propose some alternative models to compute Value at Risk (VaR) and Conditional Value at Risk (CVaR) with an homogeneous Markov chain. If we denote with τ the investor's temporal horizon, with $Wt+\tau - Wt$ the profit/loss realized in the interval $[t, t + \tau]$ and with θ the level of confidence, then the VaR is the percentile at the $(1-\theta)$ of the profit/loss distribution in the interval $[t, t + \tau]$:

$$VaR_{\theta,t+\tau}(W_{t+\tau} - W_t) = \inf\{q \mid \Pr(W_{t+\tau} - W \leq q) > 1-\theta\}$$

On the other hand the CVaR measures the expected value of profit/loss given that the VaR has not been exceeded:

$$CVaR_{\theta,t+\tau}(W_{t+\tau} - W_t) = \frac{1}{1-\theta}\int_0^{1-\theta} VaR_{q,t+\tau}(W_{t+\tau} - W_t)dq$$

We can think to use the Markovian tree to compute the possible losses (VaR, CVaR) at a given future time T. Suppose we build a homogeneous Markov chain of 50 states. Thus, for our choice of the states, we can make a Markovian tree that growths linearly with the time because it recombines every period.

Then, after $T = 60$ days, we have $(N - 1)T + 1 = 49 * 60 + 1 = 2941$ nodes in the markovian tree. Starting to count from the lowest node, let $p(j)$ be the probability of being at the j-th node where the portfolio return is given by $z(j)\ T$ ($j = 1, \ldots, (N-1)T + 1$). Considering a confidence level θ, we can compute VaR and CVaR with the Markovian hypothesis:

$$VaR_\theta = \{ z_T^{(s)} | \sum_{i=1}^{s-1} p(i) < (1-\theta); \sum_{i=1}^{s} p(i) \geq (1-\theta) \}$$

$$CVaR_\theta = \frac{1}{1-\theta} \sum_{i: z_T^{(i)} \leq VaR} p(i) z_T^{(i)}$$

An ex-post analysis on 60 days portfolio return distributions shows that the markovian tree better approximates the heavy tails than the Riskmetrics Gaussian model (B&S).

Figure 3 compares the ex post empirical return distribution (of an arbitrary portfolio) with the forecasted 60 days Markovian (mkv) and Riskmetrics (B&S) ones. This graphical comparison is confirmed by some simple statistical tests (Kolmogorv Smirnov and Anderson Darling) valuated on some US indexes (see TEDIX, TEDPIX) .

$$KS = \sup_x | F_{emp}(\mathrm{x}) - \mathrm{F}_{theo}(x) |$$

and Anderson-Darling test :

$$\mathrm{AD} = \sup_x \frac{| F_{emp}(\mathrm{x}) - \mathrm{F}_{theo}(x) |}{\sqrt{F_{theo}(\mathrm{x})(1 - F_{theo}(\mathrm{x}))}}$$

As we can see from Table 1 the Markovian approach presents the best performance in approximating the 60 days return distributions. These results are confirms that the proposed Markovian approach takes into account much better the aggregated 60 days risk as compared to classical Riskmetrics model.

Table 1. Kolmogorov-Smirnoff and Anderson-Darling tests for the indexes: Nasdaq, S&P500, and Dow Jones Industrials

		RiskMetircs	Markovian
TEDPIX	KS	0.0555	0.0629
	AD	37.021	32.021
50 Comapanies	KS	0.0424	0.0401
	AD	41.027	36.012

Concluding Remarks

This paper proposes alternative models for the portfolio selection and the VaR and CVaR calculation. In the first part we describe a portfolio selection model that uses a Markov chain to capture the behavior and the evolution of portfolio returns. In the second part we present some alternative markovian VaR and CVaR models. It is important to underline that the numerical procedure to compute the percentiles and the expected loss with the markovian approach is quite complex. As far as large portfolios or on-line VaR and CVaR calculation are concerned, the implementation of the above mentioned models should be evaluated on the basis of the tradeoff between costs and benefits. On the other hand, we believe that further very interesting markovian and semi-markovian approaches to value the expected risk exposure of portfolios can be easily expressed using some recently studied methodologies: either based on the approximation of more or less complex diffusion processes and capturing their markovianity with a Markov chain (see [5,6]) or using semi-markovian approaches (see [10,2,4]).

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