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A path-following feasible interior-point algorithm for mixed symmetric cone linear complementarity problems

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Abstract
In this paper, we propose a feasible interior-point algorithm for mixed symmetric cone linear complementarity problems which are a general class of complementarity problems. The symmetrization of the search directions used in this paper is based on Nesterov and Todd scaling scheme. By using Euclidean Jordan algebra, we prove the convergence analysis of the proposed algorithm and show that the complexity bound of the algorithm matches the currently best known iteration bound for feasible interior-point methods.

Keywords: Mixed symmetric cone linear complementarity problem; Feasible interior-point method; Convergence analysis; Polynomial complexity.

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1. Introduction

Let $(\mathcal{U},\circ)$ and $(\mathcal{V},\circ)$ be Euclidean Jordan algebras (EJAs) with dimensions m and n and ranks $r_1$and r_2 equipped with the standard inner product $\langle x, s\rangle = \mathbf{tr}\,(x \circ s)$ and $\mathcal{K}$ be the symmetric cone corresponding to $(\mathcal{V},\circ)$. Furthermore, suppose that $\mathcal{J} = \mathcal{U} \times \mathcal{V}$ is the Cartesian EJA with dimension $m + n$ and rank $r_1 + r_2$.

Let $M := \begin{bmatrix} M_{11} & M_{12} \\ M_{21} & M_{22} \end{bmatrix}$ be an $(m + n)$-matrix written as a 2×2 block matrix, where M_{11} and M_{22} are respectively $n \times n$ and $m \times m$ matrices (so, M_{12} and M_{21} are $n \times m$ and $m \times n$ matrices). The mixed symmetric cone linear complementarity problem (MSLCP) is the problem of computation a vector triple $(x, s, y) \in \mathcal{K} \times \mathcal{K} \times \mathcal{U}$ such that

$$\binom{s}{0} = M\binom{x}{y} + \binom{q_1}{q_2}, \qquad x \diamond s = 0, \qquad (1)$$

where $\binom{q_1}{q_2} \in \mathcal{J}$ and M is a Cartesian positive semidefinite matrix. That is, for all vectors $u \in \mathcal{J}$, $\langle u, Mu\rangle \geq 0$. Clearly, setting $m = 0$, the MSLCP reduced to standard symmetric cone linear complementarity problem (SLCP).

The MSLCP is a certain kind of mathematical problems, that has become quite important in recent history, due to the discoveries that many different kinds of problems may be formulated as an MSLCP, and to the development of stable and efficient numerical solution procedures. The MSLCP includes many various classes of mathematical problems. It includes symmetric cone linear optimization (SCLO), convex quadratic symmetric cone programming (CQSCP), semidefinite optimization (SDO), symmetric cone nonlinear complementarity problem (SCNCP) and SLCP. Considerable research effort was devoted by mathematicians and engineers to solve this problem. Among them, feasible and infeasible interior-point methods (IPMs) are one of the most efficient numerical approaches for solving this class of optimization problems.

A close look at the IPM literature tells us that the first IPM for linear complementarity problems (LCPs) was due to Kojima, Mizuno and Yoshise [11]. Potra [3] presented an infeasible IPM for LCPs with quadratic convergence and O(nL) complexity. The primal-dual full-Newton step infeasible IPM for linear optimization (LO) was first analyzed by Roos [1]. The Roos's algorithm was extended by Mansouri et al. [8,9] to SDO and LCP. Faybusovich [12] made the first attempt to generalize IPMs to SCLO and SCLCP using the EJAs and associated symmetric cones. Potra [4] proposed an infeasible corrector-predictor IPM for the monotone SCLCP. Gu et al. [8] and Zangiabadi et al. [18] extended the Roos's idea to SCLP and second-order cone optimization (SOCO) by using full Nesterov-Todd (NT) direction as search directions.

In all of mentioned works, various proximity measures have been used by different authors to measure closeness of generated point from the optimal solution of underling problem. A homogeneous model for solving monotone mixed complementarity problems over symmetric cones has been presented by Lin et al. [15]. Wang et al. [7] proposed a weighted-path-following interior-point algorithm based on Darvay's new search direction [17] for LO to monotone mixed LCP (MLCP). Recently, Zhang et al. [16] simplified the complexity analysis of full-Newton step infeasible IPM for SDO based on using a new proximity measure. The goal of this paper is to use the Zhang's proximity measure on SDO to present a feasible IPM for MSLCPs which are a more general class of complementarity problems.

The paper is organized as follows. In Section 2, some concepts and results on EJAs and symmetric cones which are required in our analysis are listed. Section 3 describes the main idea of IPMs and presents a feasible IPM for solving MSLCPs. The convergence analysis of the proposed algorithm which is the main part of this paper is presented in Section 4. Subsection 4.2 is devoted to obtain the complexity bound of the proposed algorithm. Finally, some conclusions and remarks follow in Section 5.

2. Preliminaries

In this section, we briefly review and introduce Euclidean Jordan algebra (EJA) as well as some of its basic properties. The EJA $(\mathcal{A},\circ)$ is a finite dimensional vector space over $\mathbb{R}$ equipped with the bilinear map $\circ:(x,y)\rightarrow x\circ y\in\mathcal{A}$ and the standard inner product $\langle x,s\rangle=\mathbf{tr}\,(x\circ s)$ while the Cartesian EJA is a Cartesian product of a finite number (such as N) of classical EJAs with the canonical inner $\langle x,s\rangle=\sum_{i=1}^{N}\langle x^{(i)},s^{(i)}\rangle$. The related cone of squares corresponding with $(\mathcal{A},\circ)$ is called the classical symmetric cone $\mathcal{K}$. For each $x\in\mathcal{A}$, $L(x)y:=x\circ y$ and $P(x):=2L(x)^2-L(x^2)$, where $L(x)^2:=L(x)L(x)$, denote the linear and quadratic representation of $\mathcal{A}$, respectively.

A Jordan algebra has an identity element, if there exist a unique element $e\in\mathcal{A}$ such that $x\circ e=e\circ x=x$ for all $x\in\mathcal{A}$. an element $c\in\mathcal{A}$ is said to be idempotent if

$c^2 = c$. An idempotent c is primitive if it is nonzero and cannot be expressed by sum of two other nonzero idempotents. A set idempotents $\{c_1, c_2, \dots, c_k\}$ is called a Jordan frame if $c_i \circ c_j \neq 0$ for any $i \neq j$ and $\sum_{i=1}^{N} c_i = e$. For any $x \in \mathcal{A}$ let r the smallest positive integer such that $\{e, x, x^2, \dots, x^r\}$ is linearly dependent, r is called the degree of x and is denoted by $\deg(x)$. The rank of $\mathcal{A}$, denoted by $\mathrm{rank}(\mathcal{A})$, is defined as the maximum of $\deg(x)$ over all $x \in \mathcal{A}$.

The spectral decomposition theorem (Theorem III.1.2 of [2]) of an Euclidean Jordan algebra $\mathcal{A}$ states that for any $x \in \mathcal{A}$ there exists a Jordan frame $\{c_1, c_2, \dots, c_k\}$ and real numbers $\{\lambda_1, \lambda_2, \dots, \lambda_k\}$ (the eigenvalues of x) such that $x = \sum_{i=1}^{r} \lambda_i(x) c_i$. For any $x \in \mathcal{A}$, the norm induced by the standard inner product is named as the Frobenius norm, which is given by $\|x\|_F := \sqrt{\langle x, x\rangle}$. Some other norms related to absolute value of eigenvalues of x, namely norm 1 and norm infinity, defined as $\|x\|_1 := \sum_{i=1}^{r} |\lambda_i(x)|$ and $\|x\|_\infty := \max|\lambda_i(x)|$.

Here, we list some key lemmas which are required in our analysis.

Lemma 2.1 (Lemma 3.2 in [12]) Let $\mathrm{int}\mathcal{K}$ be the interior of $\mathcal{K}$. For $x, s \in \mathrm{int}\mathcal{K}$ there exists a unique $w \in \mathrm{int}\mathcal{K}$ such that $x = P(w)s$. Moreover,

$$w = P\left(x^{\frac{1}{2}}\right)\left(P\left(x^{\frac{1}{2}}\right)s\right)^{\frac{-1}{2}} = \left[P\left(s^{\frac{-1}{2}}\right)\left(P\left(s^{\frac{1}{2}}\right)x\right)^{\frac{1}{2}}\right].$$

The point w is called the scaling point of x and s.

Lemma 2.2 (Lemma 28 in [14]) Let $u \in \mathrm{int}\mathcal{K}$. Then

$$x \circ s = \mu e \Leftrightarrow P(u)x \circ P(u)^{-1}s = \mu e.$$

Lemma 2.3 (Lemma 30 in [14]) Let $x, s \in \mathrm{int}\mathcal{K}$. Then

$$\left\|P\left(x^{\frac{1}{2}}\right)s - e\right\|_F \leq \|x \circ s - e\|_F.$$

Lemma 2.4 (Theorem 4 in [10]) Let $x, s \in \mathrm{int}\mathcal{K}$. Then

$$\boldsymbol{\lambda}_{\mathbf{min}}\left(P\left(x^{\frac{1}{2}}\right)s\right) \geq \boldsymbol{\lambda}_{\mathbf{min}}(x \circ s).$$

3. The optimality conditions and central path

In this section, we present a feasible interior-point algorithm for solving MSLCPs. More precisely, we first investigate the optimality conditions of MSLCP and then describe the idea behind of IPMs to solve this class of mathematical problems. Finding an approximate solution of MSLCP is equivalent to solve the following system:

$$\begin{aligned} &M_{11}x + M_{12}y - s = -q_1, \qquad x \in \mathcal{K} \\ &M_{21}x + M_{22}y = -q_2, \qquad s \in \mathcal{K} \qquad (2) \\ &x \circ s = 0. \end{aligned}$$

Similar to all of feasible interior-point algorithms, we assume that the MSLCP satisfies the interior point condition (IPC) ,i.e, there exist a vector triple $(x^0, s^0, y^0) \in \operatorname{int}\mathcal{K} \times \operatorname{int}\mathcal{K} \times \mathcal{U}$ such that

$$\begin{pmatrix} s^0 \\ 0 \end{pmatrix} = M \begin{pmatrix} x^0 \\ y^0 \end{pmatrix} + \begin{pmatrix} q_1 \\ q_2 \end{pmatrix} \qquad (3)$$

The vast majority of feasible interior-point methods approximates an ε-solution of the underlying problem by tracing the so-called central path. The central path $\big(x(\mu), y(\mu), s(\mu)\big)$ associated with MSLCP satisfies

$$\begin{aligned} &M_{11}x + M_{12}y - s = -q_1, \qquad x \in \operatorname{int}\mathcal{K} \\ &M_{21}x + M_{22}y = -q_2, \qquad s \in \operatorname{int}\mathcal{K} \qquad (4) \\ &x \circ s = \mu e \end{aligned}$$

Where the parameter μ is positive and e is the identity vector in $\mathcal{V}$. These equations may be interpreted as perturbed optimality conditions. In each iteration of the algorithm μ tends to zero, while the $\big(x(\mu), y(\mu), s(\mu)\big)$ converges to solution of MSLCP. In order to solve system (4), we apply Newton's method and find an approximate solution of the problem. After using Newton's method and neglecting the quadratic term $\Delta x \circ \Delta s$ from the third equation of (4) we obtain the following system

$$\begin{aligned} &M_{11}\Delta x + M_{12}\Delta y - \Delta s = 0 \\ &M_{21}\Delta x + M_{22}\Delta y = 0, \qquad (5) \\ &x \circ \Delta s + s \circ \Delta x = \mu e - x \circ s. \end{aligned}$$

Due to fact that x and s are not operator commute, i.e, $L(x)L(s) \neq L(s)L(x)$, system (5) doesn't always have a unique solution. This difficulty can be solved by applying Lemma 2.2 and replacing the third equation in (4) with $P(u)x \circ P(u)^{-1}s = \mu e$. After applying Newton's method, we obtain the following system

$$\begin{aligned} &M_{11}\Delta x + M_{12}\Delta y - \Delta s = 0 \\ &M_{21}\Delta x + M_{22}\Delta y = 0, \qquad (6) \\ &P(u)x \circ \Delta s + P(u)^{-1}s \circ \Delta x \\ &\qquad = \mu e - P(u)x \circ P(u)^{-1}s. \end{aligned}$$

Some best-known choices of the variable u, such as $u = x^{-\frac{1}{2}}$ and $u = s^{\frac{1}{2}}$ have been suggested by different authors. However, we choose u to be the scaling element introduced in Lemma 2.1 in which the resulting direction is called the Nesterov-Todd direction (NT-direction).

To simplify matters, we define

$$\begin{aligned} &v := \frac{P(w)^{\frac{1}{2}}s}{\sqrt{\mu}} = \left[\frac{P(w)^{-\frac{1}{2}}x}{\sqrt{\mu}}\right], \\ &d_x := \frac{P(w)^{-\frac{1}{2}}\Delta x}{\sqrt{\mu}}, \qquad (7) \\ &d_s := \frac{P(w)^{\frac{1}{2}}\Delta s}{\sqrt{\mu}}. \end{aligned}$$

Using (7), after some elementary calculations, we obtain

$$\bar{M}_{11}d_x + \bar{M}_{12}\frac{\Delta y}{\sqrt{\mu}} - d_s = 0,$$
$$\bar{M}_{21}d_x + M_{22}\frac{\Delta y}{\sqrt{\mu}} = 0, \qquad (8)$$
$$d_x + d_s = v^{-1} - v.$$

where

$$\bar{M}_{11} = P(w)^{\frac{1}{2}}M_{11}P(w)^{\frac{1}{2}},$$
$$\bar{M}_{12} = P(w)^{\frac{1}{2}}M_{12}, \qquad (9)$$
$$\bar{M}_{21} = M_{21}P(w)^{\frac{1}{2}}.$$

System (8) uniquely defines the search direction (d_x, d_s) so that $(\Delta x, \Delta s)$ are computed via (6). If $(x, y, s) \neq \big(x(\mu), y(\mu), s(\mu)\big)$, then $(\Delta x, \Delta s, \Delta y)$ is nonzero. The new iterate is obtained by taking a full-Newton step as follows:

$$X^+ = x + \Delta x,$$
$$y^+ = y + \Delta y, \qquad (10)$$
$$s^+ = s + \Delta s$$

From the third equation of system (8), we have

$$d_x + d_s = v^{-1} - v \iff v \circ (d_x + d_s) = e - v^2. \qquad (11)$$

Due to basic idea of IPMs, we need a parameter to measure the closeness of iterates to the central path. Similar to Zhang et al. [16], we use the following proximity measure in our analysis

$$\delta(x, s; \mu) = \delta(v) := \|e - v^2\|_F. \qquad (12)$$

3.1. Generic feasible algorithm for MSLCP

We can now describe the feasible algorithm in a more formal way. At the start of the algorithm, we choose a strictly feasible triple vector (x^0, s^0, y^0) with $\mu^0 = \frac{\mathbf{tr}(x^0 \circ s^0)}{\mu^0}$ such that $\delta(x^0, s^0, y^0) \leq \tau$ with $\tau \in (0,1)$. By using Newton's method, we find a new iterate close to the central path. Then, μ is reduced by the factor $1 - \theta$ with $0 < \theta < 1$. This process is repeated until μ reduces to a small enough value. Now, at this stage we have found an ε-approximate solution of MSLCP. The generic full-Newton step feasible interior-point algorithm for MSLCP is depicted in **Fig.1**.

Fig 1: Primal-Dual Feasible IPM for MSLCP

Step 0 (Initialization): Choose an accuracy parameter $\varepsilon > 0$, a barrier parameter $0 < \theta < 1$ as $\theta = \frac{1}{6\sqrt{r_2}}$ and an initial feasible point (x^0, y^0, s^0) with $\mu^0 = \frac{\mathbf{tr}(x^0 \circ s^0)}{r_2}$ and $\delta(x^0, s^0, y^0) \leq \tau = \frac{1}{2}$.

Step1 (Test convergence): If $\mu r_2 \leq \varepsilon$, declare convergence and stop. Otherwise, proceed to the next step.

Step2 (Computation): Update the parameter μ as $\mu = (1 - \theta)\mu$ and compute the scaled search direction $(d_x, \Delta y, d_s)$ by solving system (8).

Step3 (Update iterate): Generate new iterate (x^+, y^+, s^+) as (10), set $(x, y, s) = (x^+, y^+, s^+)$, and go to **step 1**.

4 Convergence analysis

The aim of this section is to investigate the feasibility and quadratic convergence of the generated points by the algorithm in **Fig. 1**. To simplify analysis, let

$$p_v = d_x + d_s, \quad q_v = d_x - d_s \tag{13}$$

It follows

$$d_x = \frac{p_v + q_v}{2},$$
$$d_s = \frac{p_v - q_v}{2}, \tag{14}$$
$$d_x \circ d_s = \frac{{p_v}^2 - {q_v}^2}{4}.$$

In following lemmas, we get some bounds for the eigenvalues of the variance vector v and standard inner product $\langle d_x, d_s \rangle$.

Lemma 4.5 Let $\delta := \delta(x, s; \mu)$. Then

$$q(\delta) = \sqrt{1-\delta} \le \lambda_i(v) \le \sqrt{1+\delta} = p(\delta),$$
$$i = 1,2, \dots, n. \tag{15}$$

Proof According to the definition of $\delta := \delta(x, s; \mu)$ in (12), we have

$$|\lambda_i(e - v^2)| = |1 - \lambda_i(v)^2| \le$$
$$\|e - v^2\|_F = \delta, \quad i = 1,2, \dots, n$$

This follows the proof. ■

Lemma 4.6 Let $\delta := \delta(x, s; \mu)$. Then

$$0 \le \langle d_x, d_s \rangle \le \frac{\delta^2}{4(1-\delta)}. \tag{16}$$

Proof Since M is a Cartesian positive semidefinite matrix, by using the two first equations of the search direction system (6) and applying the notions in (7) we conclude that

$$\langle d_x, d_s \rangle = \frac{1}{\mu} \langle \Delta x, \Delta s \rangle \ge 0.$$

On the other hand, from (13) we obtain

$$\|p_v\|_F^2 = \|d_x + d_s\|_F^2 =$$
$$\|d_x - d_s\|_F^2 + 4\langle d_x, d_s \rangle \ge 4\langle d_x, d_s \rangle.$$

Using (11) and Lemma 4.5 in the later inequality, we obtain

$$\langle d_x, d_s \rangle \le \frac{1}{4} \|d_x + d_s\|_F^2 =$$
$$\frac{1}{4} \|v^{-1} \circ (e - v^2)\|_F^2$$
$$\le \frac{1}{4\lambda_{\min}^2(v)} \|e - v^2\|_F^2 \le \frac{\delta^2}{4(1-\delta)}.$$

This completes the proof. ■

The above lemma is crucial in finding the bounds for the eigenvalues of the product $d_x \circ d_s$.

Lemma 4.7 Let $x, s \in \mathcal{V}$. Then

$$|\lambda_j(d_x \circ d_s)| \le \frac{\delta^2}{4(1-\delta)}. \tag{17}$$

Proof Since $d_x \circ d_s = \frac{{p_v}^2 - {q_v}^2}{4}$, the elementary relations of norms and this fact that $\|p_v\|_F^2 \le \|q_v\|_F^2$ imply that

$$\|d_x \circ d_s\|_\infty \le \frac{1}{4} \max\{\|q_v\|_\infty^2, \|p_v\|_\infty^2\}$$
$$\le \frac{1}{4} \max\{\|q_v\|_F^2, \|p_v\|_F^2\}$$
$$\le \frac{1}{4} \|p_v\|_F^2$$
$$\le \frac{1}{4} \frac{\delta^2}{(1-\delta)}$$

This follows the Lemma. ■

Now, our aim is to find some conditions that guarantee the feasibility of the iterates after the full NT-step. As before, let $x, s \in \mathrm{int}\mathcal{K}$, $\mu > 0$ and w be the scaling point of x and s. Then, using (7), we obtain

$$x^+ = x + \Delta x = \sqrt{\mu} P(w)^{\frac{1}{2}}(v + d_x),$$

$$s^+ = s + \Delta s = \sqrt{\mu} P(w)^{-\frac{1}{2}}(v + d_s) \qquad (18)$$

Since $P(w)^{\frac{1}{2}}$ and $P(w)^{-\frac{1}{2}}$ are automorphisms of $\mathrm{int}\mathcal{K}$ due to Lemma III.2.2 in [4]

$x^+, s^+ \in \mathrm{int}\mathcal{K}$ if and only if $v + d_x$ and $v + d_s$ belong to $\mathrm{int}\mathcal{K}$, respectively.

Now, we investigate under what conditions the full-NT step is strictly feasible. Defining $x(\alpha) := x + \alpha\Delta x$ and $s(\alpha) := s + \alpha\Delta s$, for $\alpha \in [0,1]$ the following lemma gives a sufficient condition that guarantees the iterates $x(\bar{\alpha})$ and $s(\bar{\alpha})$ are strictly feasible when $\bar{\alpha} > 0$. For proof and more details see Lemma 4.1 in [7].

Lemma 4.8 Let $x, s \in \mathcal{K}$, and $x(\alpha) \circ s(\alpha) \in \mathrm{int}\mathcal{K}$ for $\alpha \in [0, \bar{\alpha}]$. Then

$$x(\bar{\alpha}) \in \mathrm{int}\mathcal{K}, \qquad s(\bar{\alpha}) \in \mathrm{int}\mathcal{K}.$$

Lemma 4.9 Let $\delta \leq 2\sqrt{2} - 2$. Then $e + d_x \circ d_s \in \mathcal{K}$.

Proof Due to Lemma 4.7 the absolute value of eigenvalues of $d_x \circ d_s$ do not exceed $\frac{\delta^2}{4(1-\delta)}$. This implies that $1 - \frac{\delta^2}{4(1-\delta)}$ is a lower bound for the eigenvalues of $e + d_x \circ d_s$. Applying some simple calculations, we conclude that if $\delta \leq 2\sqrt{2} - 2$ then $e + d_x \circ d_s \in \mathcal{K}$. This completes the proof. ■

Lemma 4.10 Let $\delta < 2\sqrt{2} - 2$. Then the iterates x^+, s^+ with full NT-step are strictly feasible.

Proof Defining a step length $\alpha \in [0,1]$, $v_x(\alpha) = v + \alpha d_x$, and $v_s(\alpha) = v + \alpha d_s$ and using the third equation in (8), we have

$$v_x(\alpha) \circ v_s(\alpha) = (v + \alpha d_x) \circ (v + \alpha d_s)$$
$$= v^2 + \alpha v \circ (d_x + d_s) + \alpha^2 (d_x \circ d_s)$$
$$= v^2 + \alpha v \circ (v^{-1} - v) + \alpha^2 =$$
$$(1-\alpha) v^2 + \alpha e + \alpha^2 (d_x \circ d_s). \qquad (19)$$

Since, $\delta < 2\sqrt{2} - 2$ Lemma 4.9 implies that $d_x \circ d_s \succ -e$ Substituting in (19), we get

$$v_x(\alpha) \circ v_s(\alpha) \succ (1-\alpha) v^2 + \alpha e - \alpha^2 e$$
$$= (1-\alpha)(v^2 + \alpha e).$$

If $\alpha \in [0,1]$, the vector $v^2 + \alpha e$ belongs to $\mathcal{K}$, hence we have $v_x(\alpha) \circ v_s(\alpha) \succ 0$. Lemma

4.8 implies that, for $\alpha = 1$, we have $v_x(1) = v + d_x \in \mathrm{int}\mathcal{K}$ and $v_s(1) = v + d_s \in \mathrm{int}\mathcal{K}$.

Due to automorphism property of quadratic mapping P, we conclude that x^+, s^+ belong to int$\mathcal{K}$ and this completes the proof. ■

We are ready to prove the property of quadratic convergence of the iterates. Let w^+ and $v^+ = \frac{P(w^+)^{\frac{1}{2}}s^+}{\sqrt{\mu}}\left[= \frac{P(w^+)^{-\frac{1}{2}}x^+}{\sqrt{\mu}}\right]$, respectively be the scaling point and variance vector related to the new iterates x^+ and s^+. One of the most important lemma in our analysis is as follows.

Lemma 4.11 (Proposition 5.9.3 in [13]) One has

$$(v^+)^2 \sim P(v + d_x)^{\frac{1}{2}}(v + d_s), \qquad (20)$$

In which the notation $x \sim s$ means x and s share the same set of the eigenvalues.

In the next lemma, we proceed to prove the local quadratic convergence of full NT-step.

Lemma 4.12 Let $\delta < 2\sqrt{2} - 2$. Then after a full NT-step one has

$$\delta(x^+, s^+; \mu) < \frac{\delta^2}{2(1-\delta)}.$$

Proof Using Lemmas 2.3, 4.6 and 4.11, and applying (19) for $\alpha = 1$, we have

$$\delta(x^+, s^+; \mu) = \|e - v^+ \circ v^+\|_F = \left\|e - P(v + d_x)^{\frac{1}{2}}(v + d_s)\right\|_F$$

$$\leq \|e - (v + d_x) \circ (v + d_s)\|_F = \|-d_x \circ d_s\|_F$$

$$\leq \frac{1}{2}(\|d_x\|_F^2 + \|d_s\|_F^2) = \frac{1}{2}(\|d_x + d_s\|_F^2 - 2\langle d_x, d_s\rangle)$$

$$\frac{1}{2}\left(\frac{1}{\lambda_{\min}^2(v)}\|e - v^2\|_F^2\right) \leq \frac{\delta^2}{2(1-\delta)},$$

Where the last inequality is obtained because of Lemma 4.5. This concludes the desired result. ■

Corollary 1

If $\delta \leq \frac{1}{2}$, then $\delta(x^+, s^+; \mu) \leq \delta^2$ which shows the quadratic convergence of the algorithm.

The following lemma investigates the effect on the proximity measure of a full NT-step followed by an update of the parameter μ.

Lemma 4.13 Let $\delta < 2\sqrt{2} - 2$ and $r_2$be the rank of EJA $(\mathcal{V},\circ)$ and $\mu^+ = (1 - \theta)\mu$, where $\theta \in [0,1]$. Then

$$\delta(x^+, s^+; \mu^+) \leq \frac{1}{1-\theta}\left(\theta\sqrt{r_2} + \frac{\delta^2}{2(1-\delta)}\right).$$

Proof Due to the definition of $\delta(v)$ and Lemma 2.3, we have

$$\delta(x^+, s^+; \mu^+) = \left\|e - \frac{v^+}{\sqrt{1-\theta}} \circ \frac{v^+}{\sqrt{1-\theta}}\right\|_F = \frac{1}{1-\theta}\|(1 - \theta)e - v^+ \circ v^+\|_F$$

$$= \frac{1}{1-\theta}\left\|(1 - \theta)e - P(v + d_x)^{\frac{1}{2}}(v + d_s)\right\|_F$$

$$\leq \frac{1}{1-\theta}\|(1 - \theta)e - (v + d_x) \circ (v + d_s)\|_F$$

$= \frac{1}{1-\theta} \|-\theta e - d_x \circ d_s\|_F$

$\leq \frac{1}{1-\theta}\left(\theta\sqrt{r_2} + \frac{\delta^2}{2(1-\delta)}\right).$

This completes the proof. ■

Lemma 4.14 After a full NT-step,

$\mu r_2 \leq \langle x^+, s^+\rangle \leq \mu\left(r_2 + \frac{\delta^2}{4(1-\delta)}\right).$

Proof From (18) and the third equation in (8), we have

$\langle x^+, s^+\rangle = \mu\langle v + d_x, v + d_s\rangle =$

$\mu(\langle v, v\rangle + \langle v, d_x + d_s\rangle + \langle d_x, d_s\rangle)$

$= \mu(\langle v, v\rangle + \langle v, v^{-1} - v\rangle + \langle d_x, d_s\rangle) =$

$\mu(r_2 + \langle d_x, d_s\rangle).$

Lemma 4.6 implies the desired results. ■

4.2. Complexity analysis

We conclude this subsection with a theorem that gives the complexity of the algorithm in **Fig. 1**. The algorithm starts with the iterate (x, s, y) and a $\mu > 0$ such that $\delta(x, s; \mu) \leq \frac{1}{2}$ where $(x, s) \in \mathrm{int}\mathcal{K} \times \mathrm{int}\mathcal{K}$ and $y \in \mathcal{U}$. Then the barrier parameter μ updated to $\mu^+ = (1-\theta)\mu$ with $\theta = \frac{1}{6\sqrt{r_2}}$. Assuming $r_2 \geq 4$, after a full NT-step, Lemma 4.13 implies $\delta(x^+, s^+; \mu^+) \leq \frac{1}{2}$. Hence, after each iteration of the algorithm we have

$\delta(x, x; \mu) \leq \frac{1}{2}, \qquad \langle x, s\rangle \leq \mu(r_2 + \delta^2),$

Therefore the proposed algorithm is well-defined. The following lemma states the main result of this section.

Lemma 4.15 If $\theta = \frac{1}{6\sqrt{r_2}}$, the number of iterations of feasible algorithm with full NT-step doesn't exceed

$$\sqrt{r_2}\log\frac{\mathrm{tr}(x^0 \circ s^0)}{\varepsilon}.$$

5. Concluding and remarks

In this paper, we proposed a feasible interior-point algorithm based on using the NT-search direction for MSLCPs. At each iteration of the algorithm, the duality gap μ is reduced by the rate $1 - O\left(\frac{1}{\sqrt{r_2}}\right)$ and the complexity of the algorithm is $O(\sqrt{r_2}\log\varepsilon^{-1})$. This complexity coincides with the currently best known complexity bound obtained so far for this class of mathematical problems.

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