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A Method for Designing Optimal Systems for the Centralized Structures in DEA

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Abstract

This study designs optimal systems for the centralized structures in data envelopment analysis (DEA). It assumes that a collection of decision making units (DMUs) with a master decision maker and a certain budget for them is available and introduces an optimal system for each DMU by maximizing their total revenue. A nmerical example is used to illustrate the proposed model.

Keywords: *DEA, revenue, centralized structures, budget, optimal system.*

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1. Introduction

Data envelopment analysis (DEA), developed by Charnes et al. [1], provides a nonparametric methodology for evaluating the efficiency of each of a set of comparable decision making units (DMUs), relative to one another [4].

In the original model of Charnes et. al. [1], efficiency is represented by the ratio of weighted outputs to weighted inputs.

In many real world applications, there are situations (such as bank branches, hospitals, university departments) in which all the DMUs are under a centralized Decision maker (DM) [5].

The centralized decision-maker wants to minimize the overall input consumption (or to maximize the overall output production) by all DMUS ([2,3]).

In recent years, Wei and Chang [7] developed optimal system design (OSD) network DEA models to optimally design a DMU's optimal network.

These models can help DMUs design their optimal input and output portfolios in terms of profit maximization, given their total available budget [6].

This paper develops optimal system design DEA models to centralized systems to design their optimal systems. It proposes models to optimize resource allocation in terms of revenue maximization given the DMU"s total available budget *B*.

The paper unfolds as follows. Section 2 provides a brief of preliminaries.

Section 3 proposes a model for designing of the optimal systems for the centralized structures.

Example and its results are shown in section 4 while Section 5 contains conclusions.

2. Preliminaries

Suppose there are *n* DMUs and each DMU $_j, j=1,\ldots,n$ produce s different outputs from m different inputs. Let the observed input and output vectors of DMU$_j$ be $X_j=(x_{1j},\ldots,x_{mj})$ and $Y_j=(y_{1j},\ldots,y_{sj})$, respectively, that all components of vectors X_j and Y_j for all DMUs are positive. Then, a general production possibility set (T) can be represented as follows:

$T=\{(x,y)\,|\,y$ can produce by $x\}$.

In addition, it is assumed that the costs and prices for data are available, and let $C=(c_1,\ldots,c_m)>0$ be the input prices, and $P=(p_1,\ldots,p_s)>0$ be the output prices. We consider the following postulates to construct a production possibility set to define OSD [7]:

1.Envelopment postulate: The observed $(x_j, y_j) \in T, \forall j = 1,\ldots,n$.

2.Convexity postulate: If $(x,y)\in T$, and $(\bar{x},\bar{y})\in T$, then $(x+(1-\alpha)\bar{x}, y+(1-\alpha)\bar{y})\in T, \forall \alpha\in[0,1]$.

3.Output inefficiency postulate: If $(x,y)\in T$, and $\bar{y}\le y$, then $(x,\bar{y})\in T$.

4.Minimum extrapolation postulate: T is the intersection set of all T satisfying Postulates 1-3.

The production possibility set that satisfies the above defined postulates can be represented as follows:

$$T=\left\{(x,y)\,|\,x=\sum_{j=1}^{n}\lambda_j x_j,\ y\le\sum_{j=1}^{n}\lambda_j y_j,\ \sum_{j=1}^{n}\lambda_j=1,\ \lambda_j\ge 0,\ j=,\ldots,n\right\}$$

The next section uses T to propose a method to determine optimal system.

3. Designing of the optimal systems for the centralized structures

In centralized scenario the DM wants to maximize the efficiency of individual units at the same time that total input consumption is minimized or total output production is maximized [7].

According to the definition of T, the radial centralized resource allocation in input oriented is as follows:

$$\begin{aligned}
\min\ & \theta \\
\text{s.t.}\ & \sum_{r=1}^{n}\sum_{j=1}^{n}\lambda_{jr}x_{ij}=\theta\sum_{j=1}^{n}x_{ij},\ i=1,\ldots,m \\
& \sum_{r=1}^{n}\sum_{j=1}^{n}\lambda_{jr}y_{kj}\ge\sum_{j=1}^{n}y_{kj},\ k=1,\ldots,s \\
& \sum_{r=1}^{n}\sum_{j=1}^{n}\lambda_{jr}=1,\ r=1,\ldots,n \\
& \lambda_{jr}\ge 0,\ r,j=1,\ldots,n.
\end{aligned}\qquad(1)$$

Using the defined T in section 2, the following linear programming is proposed to determine the corresponding optimal revenue-maximization system for the centralized structures:

$$\begin{aligned}
\min\ & \sum_{k=1}^{s}p_k\sum_{j=1}^{n}\hat{y}_{kj} \\
\text{s.t.}\ & \sum_{r=1}^{n}\sum_{j=1}^{n}\lambda_{jr}x_{ij}=\sum_{j=1}^{n}\hat{x}_{ij},\ i=1,\ldots,m \\
& \sum_{r=1}^{n}\sum_{j=1}^{n}\lambda_{jr}y_{kj}\ge\sum_{j=1}^{n}\hat{y}_{kj},\ k=1,\ldots,s \\
& \sum_{i=1}^{m}c_i\sum_{j=1}^{n}x_{ij}\le B \\
& \sum_{r=1}^{n}\sum_{j=1}^{n}\lambda_{jr}=1,\ r=1,\ldots,n \\
& \lambda_{jr}\ge 0,\ r,j=1,\ldots,n.
\end{aligned}\qquad(2)$$

By $\bar{y}_k=\sum_{j=1}^{n}\hat{y}_{kj},(k=1,\ldots,s)$ and $\bar{x}_i=\sum_{j=1}^{n}\hat{x}_{ij},(i=1,\ldots,m)$ model (3) is converted the following model:

$$\begin{aligned}
\min \quad & \sum_{k=1}^{p} d_k \bar{y}_k \\
\text{s.t.} \quad & \sum_{r=1}^{n}\sum_{j=1}^{n} \lambda_{jr} x_{ij} = \bar{x}_i, i = 1,\ldots,m \\
& \sum_{r=1}^{n}\sum_{j=1}^{n} \lambda_{jr} y_{kj} \geq \bar{y}_k \geq \sum_{j=1}^{n} y_{kj}, k = 1,\ldots,s \\
& \sum_{i=1}^{m} c_i \bar{x}_i \leq B \\
& \sum_{r=1}^{n}\sum_{j=1}^{n} \lambda_{jr} = 1, r = 1,\ldots,n \\
& \lambda_{jr} \geq 0, r, j = 1,\ldots,n.
\end{aligned} \tag{3}$$

Model (3) identifies the optimal resource allocation with revenue maximization when the total available budget for all DMU's is B We add the constraints $\bar{y}_{kj} \geq \sum_{j=1}^{n} y_{kj}, k = 1,\ldots,s$ to model (3) to preserve the total value of k^{th} $k \in \{1,\ldots,s\}$ output of all DMUs.

Definition 1: Using the optimal solution of model (3) the optimal system for DMU $_r, (r = 1,\ldots,n)$ is defined as follows:

$$\begin{cases} x_{ir}^* = \sum_{j=1}^{n} \lambda_{rj}^* x_{ij}, \ i = 1,\ldots,m \\ y_{kr}^* = \sum_{j=1}^{n} \lambda_{rj}^* y_{kj}, \ k = 1,\ldots,p. \end{cases} \tag{4}$$

Theorem 1: The constraints $\sum_{r=1}^{n}\sum_{j=1}^{n} \lambda_{jr} y_{kj} \geq y_k, k = 1,\ldots,p$ are binding at the optimal solution of model (3).

Proof: By contradiction, let $\exists t \in \{1,\ldots,n\}, \sum_{r=1}^{n}\sum_{j=1}^{n} \lambda_{rj}^* y_{tj} > y_t$ and set $s_t = \sum_{r=1}^{n}\sum_{j=1}^{n} \lambda_{rj}^* y_{tj} - y_t > 0$. Then, we have $\tilde{y}_t = y_t^* + s_t > y_t^*$ and it increases the optimal value, which is a contradiction.

Using Theorem 1 and the constraints $\sum_{r=1}^{n}\sum_{j=1}^{n} \lambda_{jr} x_{ij} = x_i, i = 1,\ldots,m$ model (3) can be modified as follows:

$$\begin{aligned}
\min \quad & \sum_{k=1}^{p} d_k \sum_{r=1}^{n}\sum_{j=1}^{n} \lambda_{jr} y_{kj} \\
\text{s.t.} \quad & \sum_{i=1}^{m} c_i \sum_{r=1}^{n}\sum_{j=1}^{n} \lambda_{jr} x_{ij} \leq B \\
& \sum_{r=1}^{n}\sum_{j=1}^{n} \lambda_{jr} = 1, r = 1,\ldots,n \\
& \sum_{j=1}^{n} \lambda_{jr} y_{kj} \geq \sum_{j=1}^{n} y_{kj}, k = 1,\ldots,s \\
& \lambda_{jr} \geq 0, r, j = 1,\ldots,n.
\end{aligned} \tag{5}$$

Since models (3) and (5) are equivalent, therefore model (5) can be used to determine the optimal system, too, according to (4).

4. Numerical Example

To illustrate the capability of the proposed model, consider 12 hospital branches (DMUs) with two inputs (x_1, x_2) and two outputs (y_1, y_2). The data set has been shown in Table 1.

Suppose that *B*=4000 available as a total budget for 12 DMUs. According to the above discussion, by model (5) the optimal systems for these DMUs are obtained. The results are shown in Table 2. The 2nd and 3th columns of Table 2 show the optimal value of inputs for the optimal systems and the columns 4th and 5th show the optimal value of outputs.

Table1. The inputs and outputs of DMUs

DMU	x_1	x_2	y_1	y_2
1	20	151	100	500
2	19	131	80	350
3	25	160	90	450
4	27	168	120	600
5	22	158	70	300
6	55	255	80	450
7	33	235	100	500
8	31	206	85	450
9	30	244	76	380
10	50	268	75	410
11	53	306	80	440
12	38	284	70	400

Table2. The targets of DMUs according to available budget

DMU	x_1^*	x_2^*	y_1^*	y_2^*
1	20	151	100	500
2	20	151	100	500
3	19	131	80	350
4	19	131	80	350
5	19.867	148.333	97.333	480
6	19	131	80	350
7	19	131	80	350
8	20	151	100	500
9	20	151	100	500
10	20	151	100	500
11	20	151	100	500
12	19	131	80	350

5- Conclusion

This paper proposed a model to determine optimal systems for the centralized structures in DEA by assuming that a collection of DMUs with a master decision maker and a certain budget for them is available. Numerical example used to illustrate the capability of the proposed model.

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