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The Uniqueness of the Overall Assurance Interval for Epsilon in DEA Models by the Direction Method

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Abstract

The role of non-Archimedean ε in the DEA models has been clarified, so that the associated linear programs can be infeasible (for the multiplier side) and unbounded (for the envelopment side) with an unsuitable choice of ε. This paper shows that the overall assurance interval for ε in DEA models is unique by the concept of extreme directions. Also, it presents an assurance value for ε using only simple computations on inputs and outputs of DMUs.

Keywords: *Data Envelopment Analysis (DEA), Non-Archimedean Infinitesimal, Extreme Directions.*

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1. Introduction

DEA is a mathematical method for determining the relative efficiency of decision making units (DMUs). The data are input-output observations for a number of DMUs using varying amounts of the same inputs to produce varying amounts of the same outputs. Charnes et al. (1978) proposed a linear programming for determining the relative efficiency of DMUs.

In recent years, DEA has enjoyed both rapid growth and widespread acceptance.

A new bibliography in website www.deazone.com contains almost 9000 studies employing the methodology of DEA. In these studies, the two most frequently used models are the Charnes, Cooper and Rhodes (CCR) model and Banker, Charnes and Cooper (BCC) model, both of which involve the non-Archimedean ε.

Even though some researchers prefer to apply the Archimedean DEA models for their researches, the DEA literature shows that the non-Archimedean ε DEA models are still widely accepted and applied to a large number of practical problems. Mehrabian et al. (2000) defined the overall assurance interval of the non-Archimedean ε for all of DMUs in CCR and BCC models. They have shown that an assurance value for ε using a single LP is enough for finding non-Archimedean ε.

In this paper, the concept of extreme directions in mathematical programming is used to provide strong support for the validity and uniqueness of the overall assurance interval. Moreover, it is shown that an assurance value for ε can be determined using only simple computations on inputs and outputs of DMUs.

2-Non-Archimedean DEA Models and the Overall Assurance Interval for ε

Consider n DMUs, each consuming varying amounts of m inputs in the production of s outputs. The $m \times n$ matrix of inputs is denoted by X and the $s \times n$ matrix of outputs by Y. Furthermore, x_{ij} denotes the amount consumed of the ith input by the jth decision making unit, and y_{rj} denotes the amount production of its rth output. Finally, X_j and Y_j denote, respectively, the vector of inputs and outputs for the jth DMU.

The input-oriented linear programming problem formulation for the CCR and CC models (both the envelopment and the multiplier sides) is as follows:

$$
\begin{aligned}
&CCR_p : \text{Envelopment Side}\\
&\min \theta_k - \varepsilon(\mathbf{1}S^+ + \mathbf{1}S^-)\\
&\text{s.t. } Y\lambda - S^- = Y_k,\\
&\quad \theta_k X_k - X\lambda - S^- = 0,\\
&\quad \lambda \geq 0,\ S^+ \geq 0,\ S^- \geq 0.\\
&CCR_d : \text{ Multiplier Side}\\
&\max UY_k\\
&\text{s.t. } VX_k = 1,\\
&\quad UY - VX \leq 0,\\
&\quad U \geq \mathbf{1}\varepsilon,\ V \geq \mathbf{1}\varepsilon.
\end{aligned}
\tag{1}
$$

$$
\begin{aligned}
&BCC_p : \text{Envelopment Side}\\
&\min \theta_k - \varepsilon(\mathbf{1}S^+ + \mathbf{1}S^-)\\
&\text{s.t. } Y\lambda - S^- = Y_k,\\
&\quad \theta_k X_k - X\lambda - S^- = 0,\\
&\quad \mathbf{1}\lambda = 1,\\
&\quad \lambda \geq 0,\ S^+ \geq 0,\ S^- \geq 0.\\
&BCC_d : \text{ Multiplier Side}\\
&\max UY_k + u_0\\
&\text{s.t. } VX_k = 1,\\
&\quad UY - VX + \mathbf{1}u_0 \leq 0,\\
&\quad U \geq \mathbf{1}\varepsilon,\ V \geq \mathbf{1}\varepsilon,\\
&\quad u_0 \text{ free}
\end{aligned}
\tag{2}
$$

where $\mathbf{1}$ is a row vector of units.

Mehrabian et al. (2000) introduced the overall assurance interval for ε as $[0,\ \varepsilon^*]$ where $\varepsilon^* = \min\{\varepsilon_1^*, \ldots, \varepsilon_n^*\}$, such that ε_k^* is the optimal value of the following problem:

$$
\begin{aligned}
&\max\ \varepsilon\\
&\text{s.t. } VX_k = 1,\\
&\quad UY - VX \leq 0,\\
&\quad \mathbf{1}\varepsilon - U \leq 0,\\
&\quad \mathbf{1}\varepsilon - V \leq 0.
\end{aligned}
\tag{3}
$$

Each element of the overall assurance interval $[0,\ \varepsilon^*]$ is defined as an assurance value of the non-Archimedean ε for feasibility/boundedness of the multiplier/envelopment side in the CCR model for all DMUs.

3. The Directions Method

The concept of extreme directions plays an important role in the theory of mathematical programming (see Bazaraa et al. (2006) and Murty (1993)). This concept is used to develop a new method which we call the *Directions Method*, for calculating an overall assurance interval of $[0,\ \varepsilon_d^*]$.

Definition 1:

Let S be a nonempty convex set in $\mathbb{R}^n$. A nonzero vector d in $\mathbb{R}^n$ is called a *direction* of S if for each $x \in S$, $x + \mu d \in S$ for all $\mu \geq 0$. Two directions d_1 and d_2 of S are called *distinct* if $d_1 \neq \alpha d_2$ for any $\alpha > 0$. A direction d of S is called an *extreme direction* if it can not be written as a positive combination of two distinct directions, that is, if

$d = \mu_1 d_1 + \mu_2 d_2$ for $\mu_1, \mu_2 \geq 0$ then $d_1 = \alpha d_2$ for some $\alpha > 0$.

Lemma1: Given that $S = \{x : Ax = b, x \geq 0\}$ is a nonempty set where A is an $m \times n$. Then, d is a direction of S if and only if $d \geq 0$, $d \neq 0$ and $Ad = 0$.

Proof: See Bazaraa et al. (2006).

Theorem 1:
Suppose that the set of $S = \{x : Ax = b, x \geq 0\}$ is not empty and let $d_1, \ldots, d_\ell$ be the extreme directions of the set S. Then, there is a finite optimal solution to LP of $\min\{cx : x \in S\}$ if and only $cd_j \geq 0$ for $j = 1, \ldots, \ell$.

Proof: See Bazaraa et al. (2006).

4. The Uniqueness of the Overall Assurance Interval

Let CCR_j be the CCR_p model for the evaluating DMU_j. Without losing generality, we suppose $\theta \geq 0$ in (1). Therefore, the matrix form of CCR_j is as follows:

$$\begin{aligned} \min\ & c(\varepsilon)x \\ \text{s.t.}\ & Ax = b, \\ & x \geq 0, \end{aligned} \qquad (4)$$

where $A = \begin{bmatrix} 0 & Y & -I_s & 0 \\ X_j & -X & 0 & -I_m \end{bmatrix}$, $x = \begin{bmatrix} \theta \\ 0 \\ S^+ \\ S^- \end{bmatrix}$,

$$c(\varepsilon) = \begin{bmatrix} 1 \\ 0 \\ -\varepsilon 1 \\ -\varepsilon 1 \end{bmatrix}, b = \begin{bmatrix} Y_j \\ 0 \end{bmatrix}$$

Now, suppose that $d_1, \ldots, d_{\ell_j}$ are extreme directions for CCR_j ($j = 1, \ldots, n$), where $d_k = (d_k^\theta, d_k^\lambda, d_k^+, d_k^-)^T$, $k = 1, \ldots, \ell_j$. In order to guarantee the boundedness of CCR_j, we show that there is a positive ε such that $c(\varepsilon)d_k \geq 0$ for $k = 1, \ldots, \ell_j$ which is equivalent to the following inequality:

$$\varepsilon\left[\mathbf{1}d_k^+ + \mathbf{1}d_k^-\right] \leq d_k^\theta,\ k = 1, \ldots, \ell_j \qquad (5)$$

Lemma 2:

The set $H^j = \left\{k : \frac{d_k^\theta}{\mathbf{1}d_k^+ + \mathbf{1}d_k^-} > 0,\ k = 1, \ldots, \ell_j\right\}$ is nonempty, for $j = 1, \ldots, n$.

Proof: For a given j, we only need to show that there is a $k_0 \in \{1, \ldots, \ell_j\}$ such that $\mathbf{1}d_k^+ + \mathbf{1}d_k^- > 0$ and $d_{k_0}^\theta > 0$. By contradiction, suppose that $\mathbf{1}d_k^+ + \mathbf{1}d_k^- = 0$ for $k = 1, \ldots, t$ and $d_{k_0}^\theta > 0$ for $k = t+1, \ldots, \ell_j$. Therefore, $c(\varepsilon)d_k \geq 0$ for $k = 1, \ldots, t$ and $c(\varepsilon)d_k = 0$ for $k = t+1, \ldots, \ell_j$ imply that CCR_j is bounded for all $\varepsilon > 0$, which is a contradiction.

Theorem 2:

The problems $\text{CCR}_j \ (j=1,\ldots,n)$ are bounded for $0<\varepsilon \le \varepsilon_d^*$ where

$$\varepsilon^j = \min\left\{\frac{d_k^{\theta}}{\mathbf{1}d_k^+ + \mathbf{1}d_k^-} : k \in H^j\right\}, j=1,.. \quad (6)$$

$$\varepsilon_d^* = \min\left\{\varepsilon^j : j=1,\ldots,n\right\}. \quad (7)$$

Proof: The proof is obvious from Lemma 2.

By the following theorem, we are now able to show that the overall assurance interval introduced in Definition 1 is equal to the one obtained by the directions method.

Theorem 3: $\varepsilon^* = \varepsilon_d^*$

Proof:.Suppose $\varepsilon^* \neq \varepsilon_d^*$. We consider two outcomes:

- **Case 1.** $\varepsilon^* < \varepsilon_d^*$**.** Let $\varepsilon = \varepsilon_d^*$. Since $[0,\ \varepsilon^*]$ is the largest overall assurance interval (see Mehrabian et al. (2006)), there is at least one j for which the CCR_j problem is unbounded and this is impossible by Theorem 2.

- **Case 2.** $\varepsilon_d^* < \varepsilon^*$ Let $\varepsilon = \varepsilon^*$. Hence, there is at least one j such that the CCR_j problem is unbounded. From (7), there is a $j_0 \ (1 \le j_0 \le n)$ such that $\varepsilon_d^* = \varepsilon^{j_0}$.Now, (6) implies that there exists a $k_0 \in H^{j_0}$ that $\varepsilon^{j_0} = \frac{d_{k_0}^{\theta}}{\mathbf{1}d_k^+ + \mathbf{1}d_k^-}$.Therefore, $\frac{d_{k_0}^{\theta}}{\mathbf{1}d_k^+ + \mathbf{1}d_k^-} < \varepsilon^*$ implies that $c(\varepsilon^*)d_{k_0} < 0$. This means that the CCR_{j_0} problem is unbounded, which contradicts Theorem 2.

From the above two cases, we can deduce that $\varepsilon^* = \varepsilon_d^*$.

5. An Example

In the following example, we will obtain a unique upper bound for non-Archimedean ε in the CCR model. Consider the following data domain consisting of three DMUs each consuming one input to produce one output (Table1).

	DMU_1	DMU_2	DMU_3
Input	1	$\frac{1}{2}$	2
Output	1	1	2

Table 1: Three DMUs with one input and one output

For calculating ε_d^*, we need to obtain all the extreme directions $d^j = (d_j^{\theta},\ d_j^1,\ d_j^2,\ d_j^3\ d_j^+,\ d_j^-)^T$ for each CCR_j problem $(j=1,2,3)$ as reported in Table 2.

CCR_1	CCR_2	CCR_3
$d_1^1=\left(\frac{1}{3},\frac{1}{3},0,0,\frac{1}{3},0\right)^T$	$d_2^1=\left(\frac{52}{91},0,0,\frac{13}{91},\frac{26}{91},0\right)^T$	$d_3^1=\left(\frac{1}{4},0,0,\frac{1}{4},\frac{1}{2},0\right)^T$
$d_1^2=\left(\frac{2}{5},0,0,\frac{1}{5},\frac{2}{5},0\right)^T$	$d_2^2=\left(\frac{1}{2},\frac{1}{4},0,0,\frac{1}{4},0\right)^T$	$d_3^2=\left(\frac{1}{5},\frac{2}{5},0,0,\frac{2}{5},0\right)^T$
$d_1^3=\left(\frac{1}{5},0,\frac{2}{5},0,\frac{2}{5},0\right)^T$	$d_2^3=\left(\frac{1}{3},0,\frac{1}{3},0,\frac{1}{3},0\right)^T$	$d_3^3=\left(\frac{1}{9},0,\frac{4}{9},0,\frac{4}{9},0\right)^T$
$d_1^4=\left(\frac{1}{2},0,0,0,0,\frac{1}{2}\right)^T$	$d_2^4=\left(\frac{2}{3},0,0,0,0,\frac{1}{3}\right)^T$	$d_3^1=\left(\frac{1}{3},0,0,0,0,\frac{2}{3}\right)^T$

Table 2: The extreme directions for the problems CCR_j $(j=1,2,3)$

We have $\varepsilon^1=\frac{1}{2}$, $\varepsilon^2=1$ and $\varepsilon^3=\frac{1}{4}$. Therefore, $\varepsilon_d^*=\min\left\{\varepsilon^1,\varepsilon^2,\varepsilon^3\right\}=\frac{1}{4}$. So, Theorem 3 implies that CCR_j problems $(j=1,2,3)$ are bounded for each $\varepsilon\le\frac{1}{4}$. Also, it is obtained that $\varepsilon^*=\frac{1}{4}$. Thus, $\varepsilon^*=\varepsilon_d^*$

6. An approach for obtaining an assurance value

It is proved that for determining an assurance value of the non-Archimedean ε . in the CCR model, solving the following LP is enough (see [4]).

$$
\begin{aligned}
P:&\max\ \varepsilon \\
\text{s.t.}\quad & VX_j=1, \qquad j=1,\dots n \\
& UY_j-VX_j\le 0,\ j=1,\dots n \qquad (8) \\
& \mathbf{1}\varepsilon-U\le 0, \\
& \mathbf{1}\varepsilon-V\le 0.
\end{aligned}
$$

The following theorem presents an assurance value for ε using only simple computations on inputs and outputs of DMUs.

Theorem 4: $\bar{\varepsilon}=\min\{p,\ pq\}$ Is an assurance value of the non-Archimedean ε in the CCR model where, $p=1/\max_j\left\{\mathbf{1}X_j\right\}$ and $q=\min_j\left\{\mathbf{1}X_j/\mathbf{1}Y_j\right\}$

Proof: It is sufficient that, prove $\left(\bar{\varepsilon},\bar{V},\bar{U}\right)$ is belongs to the feasible region of the problem P, where $\bar{V}=p\mathbf{1}$ and $\bar{U}=pq\mathbf{1}$. For this means, we have

$\bar{V}X_i=p1X_i\ \le,\ \bar{U}Y_i-\bar{V}X_i=pq1Y_i-\ p1X_i=p1Y_i\,(q-1X_i/Y_i)\le 0, j=1,\dots n$

Also, $\mathbf{1}\bar{\varepsilon}\le pq\mathbf{1}=\bar{U}$ and $\mathbf{1}\bar{\varepsilon}\le p\mathbf{1}=\bar{V}$.

In the given example, $p=1/\max_j\{\mathbf{1}X_j\}=1/\max\{1,\frac{1}{2},2\}=\frac{1}{2}$

and

$$q = \min_j \{\mathbf{1}X_j / \mathbf{1}Y_j\} = \min\{\tfrac{1}{1}, \tfrac{1/2}{1}, \tfrac{2}{2}\} = \tfrac{1}{2}$$

Therfore, $\overline{\varepsilon} = \min\{p,\ pq\} = \frac{1}{4}$.

7. Conclusion

In this paper, it is shown that the overall assurance interval for ε in DEA models is unique by the concept of extreme directions. Also, it is provided an assurance value for non-archimedean epsilon, using only arithmetic operations on the inputs and outputs of DMUs.

References

[1] Bazaraa, M. S., John J. Jarvis, and H. D. Sherali (2006), *Linear Programming and Network Flows*, John Wiley and Sons, Third Edition, New York.

[2] Banker, R. D., A. Charenes, and W. W. Cooper (1984), *Some Models for Estimating Technical and Scale Inefficiencies in Data Envelopment Analysis*, Management Science, Vol. 30, No. 9, pp.1078-1092.

[8] Charnes, A., W. W. Cooper and E. L. Rhodes (1978), *Measuring the Efficiency of Decision Making Units*, European Journal of Operational Research, Vol. 2, No. 6, pp. 429-444.

[4] Mehrabian, Saeid, Gholam R. Jahanshahloo, Mohammad R. Alirezaee and Gholam R. Amin, (2000), *An Assurance Interval for the Non-Archimedean Epsilon in DEA models*, Operations Research, 48(2), pp.344-347.

[5] Murty, K. G. (1985), *Linear Programming*, John Wiley and Sons, New York.