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A Note on Approximately Amenable Modulo an Ideal of Banach Algebras

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Abstract

In this paper, approximately amenable modulo an ideal of Banach algebras, approximately amenable modulo an ideal of second dual of Banach algebras are investigated. Also, using the obtained results, it is shown that $l^1(S)^{**}$ is approximately amenable modulo $I_\sigma{}^{**}$ if and only if S/σ is finite where I_σ is the induced ideal for the least group congruence σ on S.

Keywords: *Amenability modulo an ideal, Approximately amenable modulo an ideal, Semigroup algebra, group congruence*

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1. Introduction

The concept of amenability for Banach algebra was initiated by B. E. Johnson in 1972 [9]. The structure of approximately amenable (contractible) Banach algebras were considered in [5, 6, 7] through different ways.

It was shown that two concepts of approximately amenable and approximately contractible for Banach algebras are equivalent and for a locally compact group G, the group algebra $L^1(G)$ is approximately amenable (contractible) if and only if G is amenable.

The concept of approximately amenable (contractible) modulo an ideal of Banach algebras is introduced by the authors in [13]. Based on the given results, it is shown that the weighted semigroup algebra $l^1(S,\omega)$ is approximately amenable modulo an ideal if and only if S is amenable and ω_σ is diagonally bounded where ω_σ is the induced weight on S/σ for the least group congruence σ on S.

In this paper we shall focus on approximately amenable modulo an ideal of Banach algebra and we state some famous results of approximately amenable of Banach algebras for approximately amenable modulo an ideal of Banach algebras.

2. Preliminaries

For a Banach algebra A, let X be a Banach A-bimodule. A linear mapping D: A→ X is called a derivation if $D(ab) = a.D(b) + D(a).b \quad (a,b \in A)$ and D is called inner if there exists $x \in X$ such that $D(a) = a.x - x.a$.

The space of all continuous derivations from A to X is denoted by $Z_1(A,X)$, the space of inner derivations from A to X by $N_1(A,X)$, and the first (continuous) cohomology group of A with coefficients in X by the quotient space $H_1(A,X) = \frac{Z_1(A,X)}{N_1(A,X)}$.

A derivation D is called approximately inner if there exists $(x_\alpha)_\alpha \subseteq X$, such that $D(a) = lim_\alpha ad_{x_\alpha}(a)\ (a \in A)$. A Banach algebra A is called amenable (approximately amenable) if every continuous derivation $D: A \to X^\star$ is inner (approximately inner) for all Banach A-bimodules X.

We now begin by recalling some definitions and then we shall give some of the basic consequences of our definitions. Further results in this area are give in [1, 10, 11, 12, 14], which in particular contains interesting results on contractibility modulo an ideal, some hereditary properties of amenability modulo an ideal, together with several illuminating examples.

Definition 2.1.

Let A be a Banach algebra and I be a closed ideal of A. A Banach algebra A is called amenable (contractible) modulo I if every bounded derivation $D: A \to X^{\star}$ ($D: A \to X$) is inner on the set theoretic difference $A\backslash I := \{ a \in A : a \notin I\}$ for all Banach A-bimodule X such that $I.X = X.I = 0$.

All over this paper we fix A and I as above, unless they are otherwise specified.

Definition 2.2.

A Banach algebra A is called approximately amenable modulo I if every bounded derivation $D: A \to X^{\star}$ is approximately inner on the set theoretic difference $A\backslash I := \{a \in A: a \notin I\}$ for all Banach A-bimodule X such that $I.X = X.I = 0$.

It is shown that if a Banach algebra A is approximately amenable and $I^2 = I$, then A is approximately amenable modulo I and if A is approximately amenable modulo I, then $\frac{A}{I}$ is approximately amenable [13].

We recall that for a Banach algebra A, by the projective tensor product $A \widehat{\otimes} A$ we mean a Banach A-bimodule, where the module actions are given naturally by

$$c.a \otimes b = ca \otimes b \quad , \quad a \otimes b.c = a \otimes bc \qquad (a, b, c \in A).$$

and there is a continuous linear A-bimodule homomorphism $\pi: A \widehat{\otimes} A \to A$ by $\pi(a \otimes b) = ab$ $(a, b \in A)$.

3. Approximately amenable modulo an ideal of Banach algebras.

In this section we first recall some properties of approximately amenable modulo I; see [13] for the proofs of the statements.

Proposition 3.1 .[13]

The following conditions are equivalent:

(a) A is approximately amenable modulo I;

(b) There is a net $(\mathrm{M}_\nu) \subset (\frac{\mathrm{A}^{\neq}}{\mathrm{I}} \widehat{\otimes} \frac{\mathrm{A}^{\neq}}{\mathrm{I}})^{\star\star}$ such that $\bar{a}\,.M_\nu - M_\nu.\bar{a} \to 0$ and $\pi^{\star\star}(M_\nu) \to \bar{e}$ $(\forall \bar{a} \in \frac{A^{\neq}}{I})$;

(c) There is a net $(M'_\nu) \subset (\frac{A^{\neq}}{I} \widehat{\otimes} \frac{A^{\neq}}{I})^{\star\star}$ such that $\bar{a}\,.M'_\nu - M'_\nu.\bar{a} \to 0$ and $\pi^{\star\star}(\mathrm{M}'_\nu) \to \bar{e}$ *for every* ν $(\forall \bar{a} \in \frac{A^{\neq}}{I})$;

Corollary 3.2. [13]

A is approximately amenable modulo I if and only if there are nets $(\mathrm{M}''_\nu) \subseteq (\frac{\mathrm{A}}{\mathrm{I}} \widehat{\otimes} \frac{\mathrm{A}}{\mathrm{I}})^{\star\star}$, (F_ν) , $(\mathrm{G}_\nu) \subseteq (\frac{\mathrm{A}}{\mathrm{I}})^{\star\star}$, such that for each $\bar{\mathrm{a}} \in \frac{\mathrm{A}^{\neq}}{\mathrm{I}}$;

(i) $\bar{a}.\mathrm{M}''_\nu - \mathrm{M}''_\nu.\bar{a} + \mathrm{F}_\nu \otimes \bar{a} - \bar{a} \otimes \mathrm{G}_\nu \to 0$

(ii) $\bar{a}\,.F_\nu \to \bar{a}\,, G_\nu.\bar{a} \to \bar{a}\,; and$

(iii) $\pi^{\star\star}\,(M''_\nu)\,.\bar{a} - F_\nu\,.\bar{a} - G_\nu.\bar{a} \to 0.$

Theorem 3.3.

A Banach algebra A is approximately amenable modulo I if and only if, for each $\varepsilon > 0$ and each finite subset S of $\frac{A}{I}$, there exist $F \in \frac{A}{I} \otimes \frac{A}{I}$ and $u, v \in \frac{A}{I}$ such that $\pi(F) = u + v$ and, for each $\bar{a} \in S$:

(i) $\|\bar{a}.F - F.\bar{a} + u \otimes \bar{a} - \bar{a} \otimes v\| < \varepsilon;$

(ii) $\|\bar{a} - \bar{a}u\| < \varepsilon \;\; and \;\; \|\bar{a} - v\bar{a}\| < \varepsilon.$

Proof**.** Suppose that A is approximately amenable. Then by Corollary 3.2, there are nets $(M''_\nu) \subseteq (\frac{A}{I} \widehat{\otimes} \frac{A}{I})^{\star\star}$, $(F_\nu)\,, (G_\nu) \subseteq (\frac{A}{I})^{\star\star}$, such that for each $\bar{a} \in A/I$:

(i) $\bar{a}.M''_\nu - M''_\nu\,.\bar{a} + F_\nu \otimes \bar{a} - \bar{a} \otimes G_\nu \to 0;$

(ii) $\bar{a} - \bar{a}.F_\nu \to 0 \;\; and \;\; \bar{a} - G_\nu.\bar{a} \to 0;$

$(iii)\, \pi^{\star\star}(M''_\nu) - F_\nu - G_\nu \to 0.$

In each case convergence is in the $\|.\|$-topology. Let Y denote the Banach space $\left(\frac{A}{I} \widehat{\otimes} \frac{A}{I}\right) \oplus \frac{A}{I} \oplus \frac{A}{I} \oplus \frac{A}{I}$. For each $\bar{a} \in \frac{A}{I}$, define a convex set in Y by setting

$K_{\bar{a}} \coloneqq \left\{(\bar{a}.m - m.\bar{a} + u \otimes \bar{a} - \bar{a} \otimes v, \bar{a} - \bar{a}u, \bar{a} - v\bar{a}, \pi(m) - u - v) \;: m \in \frac{A}{I} \widehat{\otimes} \frac{A}{I}, u, v \in \frac{A}{I}\right\}.$

For the specified finite subset S of $\frac{A}{I}$, $K \coloneqq \prod\{K_{\bar{a}} : \bar{a} \in S\}$ is a convex set in the Banach space Y^S. The conditions above show that the weak closure of K in Y^S contains the zero element 0 of Y^S . By $Mazur's$ theorem, it follows that 0 belongs to the $\|.\|$-closure of K in Y^S . Thus with $\varepsilon > 0$ as specified, there exist $F \in \frac{A}{I} \widehat{\otimes} \frac{A}{I}$ $and\; u, v \in \frac{A}{I}$ such that clauses (i) and (ii) of the Theorem are satisfied and further, such that $\|\pi(F) - u - v\| < \varepsilon$. By modifying F and u slightly, we may suppose further, that $F \in \frac{A}{I} \otimes \frac{A}{I}$ and that $\pi(F) = u + v.$

Conversely, suppose that there are $F \in \frac{A}{I} \otimes \frac{A}{I}$ and $u, v \in \frac{A}{I}$ such that $\pi(F) = u + v,$ and for each $\bar{a} \in S \subset \frac{A}{I}$,

i)$\|\bar{a}.F - F.\bar{a} + u \otimes \bar{a} - \bar{a} \otimes v\| < \varepsilon$

ii)$\|\bar{a} - v\bar{a}\| < \varepsilon, \|\bar{a} - \bar{a}u\| < \varepsilon.$

Consider the set $D \coloneqq (0,1) \times F(A)$, where $F(A)$ is the family of finite subsets of $/I$, and order D by setting

$$(\varepsilon_1, S_1) \ll (\varepsilon_2, S_2) \quad whenever \quad \varepsilon_1 \geq \varepsilon_2 \;\; and \;\; S_1 \subset S_2.$$

Then $(D, \ll)$ is a directed set. The conditions show that there exist nets $(F_\alpha) \subset \frac{A}{I} \widehat{\otimes} \frac{A}{I}$, and $(u_\alpha), (v_\alpha) \subset \frac{A}{I}$ such that $\pi(F_\alpha) = u_\alpha + v_\alpha$ and such that for each $\bar{a} \in \frac{A}{I}$, we have:

$\bar{a}.F_\alpha - F_\alpha.\bar{a} + u_\alpha \otimes \bar{a} - \bar{a} \otimes v_\alpha \to 0;$

$\bar{a} - \bar{a}u_\alpha \to 0, \bar{a} - v_\alpha\bar{a} \to 0.$

Then by (corollary 3.2), A is approximately amenable modulo I.

Corollary 3.4. Let I be a closed ideal of A and A be a Banach algebra with identity e. Then A is approximately amenable modulo I if and only if for each $\varepsilon > 0$ and each finite subset S of $\frac{A}{I}$, there exists $G \in \frac{A}{I} \otimes \frac{A}{I}$ with $\pi(G) = \bar{e}$ and $\|\bar{a}.G - G.\bar{a}\| < \varepsilon \quad (\bar{a} \in S)$.

Proof. Suppose that such G exists, and set $u = v = \bar{e}$ and $F = G + \bar{e} \otimes \bar{e}$. Then $\pi(F) = \pi(G) + \pi(\bar{e} \otimes \bar{e}) = \pi(G) + \bar{e} = \bar{e} + \bar{e} = u + v$ and

$\|\bar{a}.F - F.\bar{a} + u \otimes \bar{a} - \bar{a} \otimes v\| =$

$= \|\bar{a}.(G + \bar{e} \otimes \bar{e}) - (G + \bar{e} \otimes \bar{e}).\bar{a} + \bar{e} \otimes \bar{a} - \bar{a} \otimes \bar{e}\|$

$= \|\bar{a}.G + \bar{a}.(\bar{e} \otimes \bar{e}) - G.\bar{a} - (\bar{e} \otimes \bar{e}).\bar{a} + \bar{e} \otimes \bar{a} - \bar{a} \otimes \bar{e}\|$

$= \|\bar{a}.G - G.\bar{a}\| < \varepsilon.$

Moreover

$\|\bar{a} - v\bar{a}\| = \|\bar{a} - \bar{e}\bar{a}\| = 0 < \varepsilon$ and $\|\bar{a} - \bar{a}u\| = \|\bar{a} - \bar{a}\bar{e}\| = 0 < \varepsilon.$

Then satisfy the conditions of theorem 3.3.

Conversely, let A be approximately amenable modulo of I and F, u, v satisfy the above condition for a finite subset S and with $\frac{\varepsilon}{3\|e\|}$, replacing ε, and set $G = F - u \otimes \bar{e} - \bar{e} \otimes v + \bar{e} \otimes \bar{e}$. Then $\pi(G) = \pi(F) - \pi(u \otimes \bar{e}) - \pi(\bar{e} \otimes v) + \pi(\bar{e} \otimes \bar{e}) = u + v - u - v + \bar{e} = \bar{e}$ and

$\|\bar{a}.G - G.\bar{a}\| =$

$= \|\bar{a}.F - \bar{a}.(u \otimes \bar{e}) - \bar{a}.(\bar{e} \otimes v) + \bar{a}.(\bar{e} \otimes \bar{e}) - F.\bar{a} + (u \otimes \bar{e}).\bar{a} + (\bar{e} \otimes v).\bar{a} - (\bar{e} \otimes \bar{e}).\bar{a}\|$

$\leq \|\bar{a}.F - F.\bar{a} + u \otimes \bar{a} - \bar{a} \otimes v\| + \|\bar{a} - \bar{a}u\| + \|\bar{a} - v\bar{a}\| < \varepsilon$.

Remark 3.5.

For each Banach algebra A, there is an isometry $\iota : \frac{A}{I} \widehat{\otimes} \frac{A}{I} \to \frac{A}{I} \widehat{\otimes} \frac{A}{I}$ such that $\iota(\bar{a} \otimes \bar{b}) = \bar{b} \otimes \bar{a} \qquad (\bar{a}, \bar{b} \in A/I)$.

Theorem 3.6.

Let I be a closed ideal of A where A is a commutative Banach algebra. Then A is approximately amenable modulo of I if and only if for each $\varepsilon > 0$ and each finite subset S of $\frac{A}{I}$ there exist $F \in \frac{A}{I} \otimes \frac{A}{I}$ with $\iota(F) = F$ and $u \in \frac{A}{I}$ such that $\pi(F) = 2u$, and for each $\bar{a} \in S$:

i) $\|\bar{a}.F - F.\bar{a} + u \otimes \bar{a} - \bar{a} \otimes v\| < \varepsilon$;

ii) $\|\bar{a} - \bar{a}u\| < \varepsilon$.

Proof. Since A is commutative, $\iota(\bar{a}.F) = \iota(F).\bar{a} \ \left(\bar{a} \in \frac{A}{I}, F \in \frac{A}{I} \otimes \frac{A}{I}\right)$. Because for $F \in \frac{A}{I} \otimes \frac{A}{I}$ set $= \sum \bar{b}_i \otimes \bar{c}_i \ \left(\bar{b}_i, \bar{c}_i \in \frac{A}{I}\right)$, Then

$$\iota(\bar{a}.F) = \iota(\bar{a}.(\sum \bar{b}_i \otimes \bar{c}_i))$$

$$= \iota(\sum \bar{a}\bar{b}_i \otimes \bar{c}_i)$$

$= \sum \bar{c}_i \otimes \bar{a}\bar{b}_i = \sum \bar{c}_i \otimes \bar{b}_i\bar{a} =$
$(\sum \bar{c}_i \otimes \bar{b}_i).\bar{a} = \iota(F).\bar{a}.$

Suppose that A is approximately amenable modulo I and take $\varepsilon > 0$ and a finite subset S of $\frac{A}{I}$. By Theorem 3.3, there exist F, u and v satisfying conditions (i) and (ii) of that result. For each $\bar{a} \in S$, we have;
$\|\iota(F).\bar{a} - \bar{a}.\iota(F) + \bar{a} \otimes u - v \otimes \bar{a}\| < \varepsilon$

Set $G = \frac{(F+\iota(F))}{2}$ and $W = \frac{(u+v)}{2}$. Then $\iota(G) = G$ and $\pi(G) = 2W$.

Further
$\|\bar{a}.G - G.\bar{a} + W \otimes \bar{a} - \bar{a} \otimes W\| < \varepsilon$
and $\|\bar{a} - \bar{a}W\| < \varepsilon$.

Thus the specified conditions are satisfied (with W for u).

The convers is clear.

4. Approximately amenable modulo an ideal of second dual of Banach algebra.

Let A be a Banach algebra and A^{**} be the second dual of A. It is shown that the Banach algebra A inherits amenability from A^{**} [8, 7]. Approximatly amenability of second dual of group algebra $L^1(G)^{**}$ is characterized in [8],
$L^1(G)^{**}$ is approximately amenable if and only if G is finite.

Theorem 4.1.

Let I be a closed ideal of A with $I^2 = I$ then;

(i) if A^{**} is amenable modulo I^{**} then A is amenable modulo I;

(ii) if A^{**} is approximately amenable modulo I^{**} then A is approximately amenable modulo I.

Proof.

(i) Suppose A^{**} is amenable modulo I^{**}. Then by [[1], Theorem 1], $\frac{A^{**}}{I^{**}} \simeq (\frac{A}{I})^{**}$ is amenable. Thus $\frac{A}{I}$ is amenable and since $I^2 = I$, A is amenable modulo I

(ii) Suppose A^{**} is approximately amenable modulo I^{**}, then by Theorem 3.2 [13], $\frac{A^{**}}{I^{**}} \simeq (\frac{A}{I})^{**}$ is approximately amenable, so $\frac{A}{I}$ is approximately amenable. Hence A is approximately amenable modulo I.

We now give some interesting result for semigroup algebras by using the previous results. In especial case, some results are given in [2, 3, 4].

Let S be a (discrete) semigroup and $E = E(S)$ is the set of idempotents of S.

A congruence ρ on a semigroup S is called a group congruence if $\frac{S}{\rho}$ is a group. We denote the least group congruence on S by σ. A semigroup S is called an E-semigroup if $E(S)$ forms a subsemigroup of S, and E-inversive if for all $x \in S$, there exists $y \in S$ such that $xy \in E(S)$.

Lemma 4.2. [1]

Let S be an E-inversive E-semigroup with commuting idempotents and σ be the least group congruence on S, then $l^1(s/\sigma) \simeq \frac{l^1(S)}{I_\sigma}$ where I_σ is a closed ideal of $l^1(S)$ and $I_\sigma^2 = I_\sigma$.

Theorem 4.3.

Let S be an E-inversive E-semigroup with commuting idempotents and σ be the least group congruence on S, Then $l^1(S)^{\star\star}$ is approximately amenable modulo ${I_\sigma}^{\star\star}$ if and only if S/σ is finite.

Proof. It is clear that $\left(l^1\left(\frac{s}{\sigma}\right)\right)^{**} \simeq \left(\frac{l^1(S)}{I_\sigma}\right)^{**} \simeq \frac{l^1(S)^{**}}{I_\sigma^{**}}$ and ${I_\sigma}^2 = I_\sigma$ (by [1]). Using Theorem 3.2, [13], $l^1(S)^{\star\star}$ is approximately amenable modulo ${I_\sigma}^{\star\star}$ if and only if $\left(l^1\left(\frac{s}{\sigma}\right)\right)^{**}$ is approximately amenable,. If and only if S/σ is finite (because S/σ is a group).

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