

Conformally invariant spin- $\frac{3}{2}$ field equation in de Sitter space–time

N. Fatahi¹

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Abstract In the previous paper (Behroozi et al., Phys Rev D 74:124014, 2006; Dehghani et al., Phys Rev D 77:064028, 2008), conformal invariance for massless tensor fields (scalar, vector and spin-2 fields) was studied and the solutions of their wave equations and two-point functions were obtained. In the present paper, conformally invariant wave equation for massless spinor field in de Sitter space–time has been obtained. For this propose, we use Dirac’s six-cone formalism. The solutions of massless spin- $\frac{1}{2}$ and $-\frac{3}{2}$ equations, in the ambient space notation, have been calculated.

Introduction

Today, latest discoveries of the modern astrophysics data show that our universe in first approximation is in a de Sitter (dS) phase. Therefore, it is important to find a formulation of de Sitter quantum field theory with the same level of completeness and rigor as for its Minkowskian counterpart. The various physical stimulants for studying quantum field in dS space are:

The de Sitter space–time is a curved space–time manifold with maximum symmetry and it is a solution of the Einstein equation with nonzero cosmological constant. It is one of the simple curved space–time that one can find unique vacuum state [3] and one can replace the usual spectral condition by a certain geodesic spectral condition [4]. For large separated points, the graviton propagator on

dS space has a pathological behavior (infrared divergence) [5–7]. Therefore, one expectancy to discover inside the dynamics of the space–time is some natural mechanism to end inflation and to find a description for the cosmological constant [8–10]. Covariant quantum field theory in de Sitter space for the “massive” and “massless” conformally coupled scalar field [11, 12] and for the “massless” minimally coupled scalar field in de Sitter space [13] previously made. It has been shown that conformally coupled scalar field, vector field and spin-2 field in massive and massless cases correspond to the unitary irreducible representations (UIR) of de Sitter group [12, 14–18].

Conformal invariance was introduced for first time into physics by Cunningham and Bateman [19, 20] when they showed that Maxwell’s equations are covariant under the larger 15-parameter conformal group rather than 10-parameter Poincare group. Equations of motion of charged particles with nonzero mass are not conformal invariant. The massless field in Minkowski space–time propagates on the light cone. These fields are invariant under the conformal group $SO(2, 4)$. Fields with spin $s \geq 1$ are invariant under the gauge transformation as well. In dS space, for the set of observable transformations under the dS group $SO(1, 4)$, mass is not an invariant parameter. However, concept of light-cone propagator does exist and leads to the conformal invariance. “Massless” is used in reference to the conformal invariance (propagation on the de Sitter light cone).

One of the original objectives of the quantum field theory is the quantization of gravitational field [21]. The quanta of gravitational field are massless particle with spin-2 that call “graviton” so that the equation of this particle “Einstein equation” is not conformal invariant. Therefore, conformal invariance may solve the problem of quantum gravity. The fermionic partner of the gravitational field is a

✉ N. Fatahi
n83fatahi@yahoo.com

¹ Department of Physics, Science and Research Branch, Islamic Azad University, Tehran, Iran

spin- $\frac{3}{2}$ field that call “gravitino”. In the original formulation of supergravity, there was one real massless gravitino (accordingly $N = 1$) this means that one generator exists that change fermions to bosons $\delta B = \bar{\epsilon}F$. For unification between gravitational (spin-2) and nongravitational (spin-1) forces existence of this generator, that can change spin, is necessary. So, in this paper, we study conformal invariant equation spin- $\frac{3}{2}$ field in dS space–time.

In the previous papers [1, 2], the conformally invariant (CI) wave equations for scalar, vector and spin-2 fields in de Sitter space–time were obtained. We are interested in the conformal invariance properties of the massless spin- $\frac{1}{2}$ and spin- $\frac{3}{2}$ field in de sitter space–time. First, we introduce massless spinor field equations in de Sitter space. Afterwards explained the Dirac’s six-cone formalism. With using this formalism, we write the conformal invariant spinor wave equations. Then, in the following sections, de Sitter field solutions can be obtained.

Massless spinor field equations in de sitter space

de Sitter space–time is visualized as the hyperboloid embedded in a five-dimensional Minkowski space–time:

$$X_H = \{x \in \mathbb{R}^5; x^2 = \eta_{\alpha\beta}x^\alpha x^\beta = -H^{-2}\}, \quad \alpha, \beta = 0, 1, 2, 3, 4, \tag{2.1}$$

$$\eta_{\alpha\beta} = \text{diag}(1, -1, -1, -1, -1).$$

where H is Habel constant (for simplicity, we can choose $H = 1$). The de Sitter metric is:

$$ds^2 = \eta_{\alpha\beta}dx^\alpha dx^\beta \Big|_{x^2=-1} = g_{\mu\nu}^{ds}dX^\mu dX^\nu, \quad \mu, \nu = 0, 1, 2, 3, \tag{2.2}$$

where X^μ are the four space–time intrinsic coordinates in de Sitter hyperboloid and x^α are the five global coordinates in ambient space notation. Working in embedding space has two advantages: first it is close to the group theoretical language and second the equations are obtained in an easier way than they might be found in de Sitter intrinsic space.

Dirac [22] deduces wave equation for electron in de Sitter space–time. This equation, with using eigenvalue equation for second-order Casimir operator $Q^{(1)}$, previously is obtained [23]. There are two Casimir operators for dS group, these operators commute with all the action of the group generators and thus they are constant on each representation. Two Casimir operators for dS group are:

$$Q^{(1)} = -\frac{1}{2}L^{\alpha\beta}L_{\alpha\beta}, \tag{2.3}$$

$$Q^{(2)} = -W_\alpha W^\alpha, \tag{2.4}$$

where $W_\alpha = \frac{1}{8}\epsilon_{\alpha\beta\gamma\delta\eta}L^{\beta\gamma}L^{\delta\eta}$ and $\epsilon_{\alpha\beta\gamma\delta\eta}$ is the antisymmetric tensor in the ambient space notation with $\epsilon_{01234} = 1$. The generator of de Sitter group is

$$L_{\alpha\beta} = M_{\alpha\beta} + S_{\alpha\beta},$$

where the “orbital” part is:

$$M_{\alpha\beta} = -i(x_\alpha\partial_\beta - x_\beta\partial_\alpha) = -i(x_\alpha\bar{\partial}_\beta - x_\beta\bar{\partial}_\alpha), \tag{2.5}$$

and the “spinorial” part $S_{\alpha\beta}$ is [24]:

$$S_{\alpha\beta} = -\frac{i}{4}[\gamma_\alpha, \gamma_\beta], \tag{2.6}$$

the transverse derivative is $\bar{\partial}_\beta = \partial_\beta + x_\beta x \cdot \partial$.

In this case, five γ matrices needed instead of the four ones of the flat Dirac theory. These matrices are found within the Clifford algebra issued from the metric $\eta^{\alpha\beta}$:

$$\gamma^\alpha\gamma^\beta + \gamma^\beta\gamma^\alpha = 2\eta^{\alpha\beta}, \quad \gamma^{\alpha\dagger} = \gamma^0\gamma^\alpha\gamma^0, \tag{2.7}$$

an explicit representation is provided by:

$$\begin{aligned} \gamma^0 &= \begin{pmatrix} I1 & 0 \\ 0 & I1 \end{pmatrix}, \quad \gamma^4 = \begin{pmatrix} 0 & I1 \\ -I1 & 0 \end{pmatrix} \\ \gamma^1 &= \begin{pmatrix} 0 & i\sigma^1 \\ i\sigma^1 & 0 \end{pmatrix}, \quad \gamma^2 = \begin{pmatrix} 0 & -i\sigma^2 \\ -i\sigma^2 & 0 \end{pmatrix}, \quad \gamma^3 = \begin{pmatrix} 0 & i\sigma^3 \\ i\sigma^3 & 0 \end{pmatrix}, \end{aligned}$$

where $I1$ is unit 2×2 matrix and σ^i are Pauli matrices.

By the eigenvalue of this two Casimir operator, the unitary irreducible representation of de sitter group $SO(1,4)$ is characterized “the principal, complementary and discrete series” [23]. The UIR’s $U_{s,v}$ in the principal series correspond to the:

$$Q_s^{(1)} = \left(\frac{9}{4} + v^2 - s(s+1)\right), \tag{2.8}$$

$$Q_s^{(2)} = \left(\left(\frac{1}{4} + v^2\right)s(s+1)\right), \tag{2.9}$$

where s is spin and v is real positive parameter. The second-order field equations can be written as:

$$\langle Q_s^{(1)} - \langle Q_s^{(1)} \rangle \rangle \psi = 0. \tag{2.10}$$

For massive field with $s = \frac{1}{2}$, we have:

$$\left(Q_{\frac{1}{2}}^{(1)} - \frac{3}{2}\right)\psi(x) = v^2\psi(x), \tag{2.11}$$

where $\psi(x)$ is a 4-component spinor with arbitrary degree of homogeneity ($x \cdot \partial\psi = \sigma\psi$). The second-order Casimir operator for spin- $\frac{1}{2}$ is given by

$$Q_{\frac{1}{2}}^{(1)} = -\frac{1}{2}M_{\alpha\beta}M^{\alpha\beta} - \frac{1}{2}S_{\alpha\beta}S^{\alpha\beta} - S_{\alpha\beta}M^{\alpha\beta}, \tag{2.12}$$

where one can show

$$Q_0^{(1)} = -\frac{1}{2}M_{\alpha\beta}M^{\alpha\beta}, \quad \frac{1}{2}S_{\alpha\beta}S^{\alpha\beta} = \frac{5}{2},$$

$$S_{\alpha\beta}M^{\alpha\beta} = -\frac{i}{2}\gamma_\alpha\gamma_\beta M^{\alpha\beta} = -\not{x}\not{\partial}^T, \tag{2.13}$$

note that $\gamma_x x^\alpha \equiv \not{x}$. The massless elementary system with spin s is described by the discrete series $\Pi_{p,s}^\pm$ with $\Delta = (s, s)$ for which [23]:

$$Q_s^{(1)} = 2(1 - s^2), \quad Q_s^{(2)} = s^2(1 - s^2).$$

In the representation of parameters $\Delta = (\frac{1}{2}, \frac{1}{2})$, the eigenvalues of the Casimir operators become:

$$\langle Q_{\frac{1}{2}}^{(1)} \rangle = \frac{3}{2}, \quad \langle Q_{\frac{1}{2}}^{(2)} \rangle = \frac{3}{16}. \tag{2.14}$$

Then, Eq. (2.11) is substituted by

$$\left(Q_{\frac{1}{2}}^{(1)} - \frac{3}{2}\right)\psi(x) = 0, \tag{2.15}$$

after using the relations (2.13) this equation is written as:

$$(Q_0 + \not{x}\not{\partial}^T - 4)\psi = 0. \tag{2.16}$$

Now, consider the de Sitter–Dirac operator D that define by:

$$D \equiv -\frac{i}{2}\gamma_\alpha\gamma_\beta M^{\alpha\beta} + 2 = -\not{x}\not{\partial} + 2. \tag{2.17}$$

With using this operator, Eq. (2.11) can be written as follow:

$$(iD - v)(iD + v)\psi(x) = 0, \tag{2.18}$$

this relation is similar to the $(i\not{\partial} + m)(i\not{\partial} - m)\psi(x) = 0$, in Minkowski space. Then, the first order of field equation for particle with spin- $\frac{1}{2}$ and nonzero mass in de Sitter space introduced as follow:

$$(iD + v)\psi(x) = 0, \tag{2.19}$$

where $v \in \mathbb{R}$ and $v \neq 0$. Solutions of the de Sitter dirac first-order equation are also the solutions of Eq. (2.18). For massless case we have:

$$iD\psi(x) = 0. \tag{2.20}$$

Therefore, with using the above equation, the de Sitter massless spin- $\frac{1}{2}$ field equation (2.15) will be:

$$(Q_0 - 2)\psi = 0. \tag{2.21}$$

Now let us consider de Sitter massless spin- $\frac{3}{2}$ field equation. As previously mentioned [25]. The spinor–vector unitary irreducible representations can be classified using the eigenvalues of $Q^{(1)}$ and the field equation can be written as:

$$\left(Q_{\frac{3}{2}}^{(1)} - \langle Q_{\frac{3}{2}}^{(1)} \rangle\right)\kappa_\alpha(x) = 0. \tag{2.22}$$

For spin- $\frac{3}{2}$ field, according to the possible values of p and q , two types of the unitary irreducible representations are distinguished for the de Sitter group $SO(1, 4)$ namely, the principal and discrete series.

1. The unitary irreducible representations $U_{\frac{3}{2},v}^\pm$ in the principal series where $p = s = \frac{3}{2}$ and $q = \frac{1}{2} + iv$ correspond to the Casimir spectral values:

$$\langle Q_{\frac{3}{2}}^{(1)} \rangle = v^2 - \frac{3}{2}, \quad v \in \mathbb{R} \quad v > \frac{3}{2},$$

Note that $U_{\frac{3}{2},v}^\pm$ and $U_{\frac{3}{2},-v}^\pm$ are equivalent.

2. The unitary irreducible representations $\Pi_{\frac{3}{2},q}^\pm$ of the discrete series, where $p = s = \frac{3}{2}$ correspond to:

$$\langle Q_{\frac{3}{2}}^{(1)} \rangle = -\frac{5}{2}, \quad q = \frac{3}{2}, \quad \Pi_{\frac{3}{2},\frac{3}{2}}^\pm, \tag{2.23}$$

$$\langle Q_{\frac{3}{2}}^{(1)} \rangle = -\frac{3}{2}, \quad q = \frac{1}{2}, \quad \Pi_{\frac{3}{2},\frac{1}{2}}^\pm, \tag{2.24}$$

where the sign \pm stands for the helicity. The “massless” spin- $\frac{3}{2}$ field in dS space corresponds to the discrete series $\Pi_{\frac{3}{2},\frac{3}{2}}^\pm$ and the field equation is:

$$\left(Q_{\frac{3}{2}}^{(1)} + \frac{5}{2}\right)\kappa_\alpha(x) = 0, \tag{2.25}$$

where

$$Q_{\frac{3}{2}}^{(1)}\kappa_\alpha(x) = \left(-\frac{1}{2}M_{\alpha\beta}M^{\alpha\beta} + \frac{i}{2}\gamma_\alpha\gamma_\beta M^{\alpha\beta} - 3 - \frac{5}{2}\right)\kappa_\alpha(x)$$

$$- 2\partial x.\kappa(x) + 2x\partial.\kappa(x) + \gamma\gamma.\kappa(x). \tag{2.26}$$

However, with the subsidiary condition $\partial.\kappa = \partial^\top.\kappa = \gamma.\kappa(x) = 0$, the “massless” spin- $\frac{3}{2}$ field is singular. This type of singularity is actually due to the divergencelessness condition needed to associate this field with a specific unitary irreducible representation of dS group. To solve this problem, the subsidiary condition must be dropped. Then, the field equation (2.25) replaced with the following equation

$$\left(Q_{\frac{3}{2}}^{(1)} + \frac{5}{2}\right)\kappa_\alpha(x) - D_{\frac{3}{2},x}\bar{\partial}.\kappa(x) = 0, \tag{2.27}$$

where

$$D_{\frac{3}{2},x} = -\bar{\partial}_x - \bar{\gamma}_x \not{x}.$$

One can show that this equation is invariant under the gauge transformation:

$$\kappa_\alpha(x) \rightarrow \kappa'_\alpha(x) = \kappa_\alpha(x) + D_{\frac{3}{2}\alpha}\zeta, \tag{2.28}$$

where ζ is an arbitrary spinor field. By introducing a gauge fixing parameter c , the wave equation now reads as:

$$\left(Q_{\frac{3}{2}}^{(1)} + \frac{5}{2} \right) \kappa(x) - c D_{\frac{3}{2}\alpha} \bar{\partial} \cdot \kappa(x) = 0, \tag{2.29}$$

The role of c is just to fix the gauge field ζ . The first-order field equation is now simply introduced as follows:

$$\not{x} \bar{\partial} \kappa_\alpha(x) - \kappa_\alpha(x) - x_\alpha \not{x} \not{x} + D_{\frac{3}{2}\alpha} \not{x} \not{x} = 0, \tag{2.30}$$

where this equation is invariant under the gauge transformation as follow:

$$\kappa_\alpha(x) \rightarrow \kappa'_\alpha(x) = \kappa_\alpha(x) + \partial_\alpha^\top \zeta. \tag{2.31}$$

The solutions of this equation obviously satisfied the field equation (2.29).

There exists another first-order field equation:

$$\not{x} \bar{\partial} \kappa_\alpha(x) - 3\kappa_\alpha(x) - x_\alpha \not{x} \not{x} - \partial_\alpha^\top \not{x} \not{x} = 0, \tag{2.32}$$

this equation is invariant under the gauge transformation $\kappa_\alpha(x) \rightarrow \kappa'_\alpha(x) = \kappa_\alpha(x) + D_{\frac{3}{2}\alpha}\zeta$.

UIRs of the conformal group

In the Minkowski space, the massless field equations are conformally invariant. For every massless representation of Poincare group, there exists only one corresponding representation in the conformal group [26, 27]. The massless field with spin- $\frac{1}{2}$ associated with discrete series $\prod_{p,s}^\pm$ and their unitary irreducible representation are $\prod_{\frac{3}{2},\frac{3}{2}}^\pm$, where $p = q = s = \frac{1}{2}$ correspond to $\langle Q_{\frac{1}{2}}^{(1)} \rangle = \frac{3}{2}$ where these two representations have a Minkowskian interpretation. For spin- $\frac{3}{2}$ field, the two unitary irreducible representations $\prod_{\frac{3}{2},\frac{3}{2}}^\pm$ have a Minkowskian interpretation. The direct sum of two UIR's $C(j+1, j, 0)$ and $C(-(j+1), j, 0)$ of the conformal group $SO(2,4)$, with positive and negative energy and $j = \frac{1}{2}$ for spinor field $j = \frac{3}{2}$ for spin- $\frac{3}{2}$ field, is a unique extension of representation $\prod_{j,j}^\pm$. The massless Poincare UIR's $P^>(0, j)$ and $P^<(0, j)$ with positive and negative energies, respectively, and positive helicity. The following diagrams illustrate these relations:

$$\begin{array}{ccccc} & C(j+1, j, 0) & & C(j+1, j, 0) & \leftrightarrow & \mathcal{P}^>(0, j) \\ \prod_{j,j}^+ \leftrightarrow & \oplus & \xrightarrow{H=0} & \oplus & & \oplus \\ & C(-(j+1), j, 0) & & C(-(j+1), j, 0) & \leftrightarrow & \mathcal{P}^<(0, j), \end{array} \tag{3.1}$$

$$\begin{array}{ccccc} & C(j+1, 0, j) & & C(j+1, 0, j) & \leftrightarrow & \mathcal{P}^>(0, -j) \\ \prod_{j,j}^- \leftrightarrow & \oplus & \xrightarrow{H=0} & \oplus & & \oplus \\ & C(-(j+1), 0, j) & & C(-(j+1), 0, j) & \leftrightarrow & \mathcal{P}^<(0, -j), \end{array} \tag{3.2}$$

where the arrows \leftrightarrow designate unique extension. It is important to note that the representations $\prod_{\frac{3}{2},\frac{3}{2}}^\pm$ do not have corresponding flat limit.

Diracs six-cone and the projective of the six-cone

Diracs six-cone is a five-dimensional super-surface in \mathbb{R}^6 that

define by:

$$u^2 = \eta_{ab} u^a u^b = u_0^2 - \vec{u}^2 + u_5^2 = 0, \tag{4.1}$$

where

$$\eta_{ab} = \text{diag}(1, -1, -1, -1, -1, 1), \quad a, b = 0, 1, 2, 3, 4, 5$$

$$\vec{u} \equiv (u_1, u_2, u_3, u_4).$$

If operator \hat{A} that act on the field ϕ in \mathbb{R}^6 satisfy the relation

$$\hat{A} u^2 \phi = u^2 \hat{A} \phi,$$

for any ϕ , it is to be intrinsic. Wave equations, subsidiary conditions, etc., must be expressed in terms of operators that are defined intrinsically on the cone. One of the intrinsic operators is powers of d'Alembertian $(\partial_a \partial^a)^n$ which acts intrinsically on fields of conformal degree $(n-2)$. For tensor or spinor fields of degree $-1, 0, 1, \dots$, the intrinsic wave operators are $\partial^2, (\partial^2)^2, (\partial^2)^3, \dots$ respectively [28]. Thus, the following CI system of equations, on the cone, has been used [1]:

$$\begin{cases} (\partial_a \partial^a)^n \Psi = 0, \\ \hat{N}_5 \Psi = (n-2) \Psi, \end{cases} \tag{4.2}$$

where Ψ is tensor or spinor field of a definite rank and symmetry. Other examples of the intrinsic operators are:

1. Fifteen generators of the conformal group $SO_0(2, 4)$

$$M_{ab} = i(u_a \partial_b - u_b \partial_a),$$
2. The conformal-degree operator N_5

$$N_5 \equiv u^a \partial_a,$$
3. The intrinsic gradient

$$\text{Grad}_a \equiv u_a \partial_b \partial^b - (2N_5 + 4) \partial_a,$$
4. The powers of d'Alembertian

$$(\partial_a \bar{\partial}^a)^n,$$

which acts intrinsically on fields of conformal degree (n-2).

With CI conditions that added to the above system, the space of solutions restricted. The following conditions are introduced by:

1. transversality

$$u_a \Psi^{ab\dots} = 0,$$

2. divergencelessness

$$\text{Grad}_a \Psi^{ab\dots} = 0,$$

3. tracelessness

$$\Psi^a_{ab\dots} = 0.$$

Now, with regard to the following relations:

$$\begin{cases} x^\alpha = (u^5)^{-1} u^\alpha, \\ x^5 = u^5, \end{cases} \quad (4.3)$$

the coordinates on the cone $u^2 = 0$ projected to the 4 + 1 de sitter space. Note that in the projective coordinate, x^5 becomes superfluous. So that intrinsic operators that introduced read as:

1. The ten $SO_0(1, 4)$ generators

$$M_{\alpha\beta} = i(x_\alpha \partial_\beta - x_\beta \partial_\alpha), \quad (4.4)$$

2. The conformal-degree operator N_5

$$N_5 = x_5 \frac{\partial}{\partial x_5}, \quad (4.5)$$

3. The conformal gradient [29]

$$\text{Grad}_\alpha = -x_5^{-1} x_\alpha [Q^0 - N_5(N_5 - 1)] + 2\bar{\partial}_\alpha (N_5 + 1), \quad (4.6)$$

4. The powers of d'Alembertian $(\partial_a \bar{\partial}^a)^n$ which acts intrinsically on field of conformal degree (n-2)

$$(\partial_a \bar{\partial}^a)^n = -x_5^{-2n} \prod_{j=1}^n [Q_0 + (j + 1)(j - 2)]. \quad (4.7)$$

In the next section, we use this formalism to write conformal invariant wave equation of massless field.

Conformal invariant spinor field equation in de sitter space

For spinor field, the simplest conformally invariant system is obtained from (4.2) with $n = 1$:

$$\begin{cases} (\partial_a \bar{\partial}^a) \Psi = 0, \\ N_5 \Psi = -\Psi. \end{cases} \quad (5.1)$$

We introduce $\psi = x_5 \Psi$ where Ψ is spinor field on the cone and ψ is spinor field on the de Sitter space. With using (4.7) the conformally invariant equation derived as:

$$(Q_0 - 2)\psi = 0, \quad (5.2)$$

which is a massless conformally coupled spinor field in de Sitter space. After making use of (2.16), the first-order field equation in this case becomes as follows:

$$(\not{x} \not{\partial}^\top - 2)\psi(x) = 0, \quad (5.3)$$

therefore, the field $\psi(x)$ associates with the UIR of dS group $\Pi_{\frac{1}{2}, \frac{1}{2}}^\pm$ and propagates on the dS light cone.

Now, we apply this method to the spin- $\frac{3}{2}$ field. In this case, six degrees of freedom of spinor field on the cone are classified as:

$$\kappa_\alpha = x_5(\Psi_\alpha + x_\alpha x \cdot \Psi), \quad \psi_1 = x_5 \Psi_5, \quad \psi_2 = x_5 x \cdot \Psi, \quad (5.4)$$

where ψ_1, ψ_2 are spinor fields on dS space. The condition $x \cdot \kappa = 0$ satisfied by the above definition, therefore, κ_α is a spin- $\frac{3}{2}$ field that lives on de Sitter hyperboloid. With using $n = 1$ in (4.2) conformal invariant wave equation for Ψ_α obtained as follows (Appendix 2):

$$(Q_0 - 2)\Psi_\alpha = 0, \quad (5.5)$$

where Ψ_α is a spin- $\frac{3}{2}$ field on the cone.

After doing some calculations, following CI system of field equations is obtained (see Appendix 2):

$$(Q_0 - 2)\kappa_\alpha + 2x_\alpha \bar{\partial} \cdot \kappa + \bar{\partial}_\alpha \bar{\partial} \cdot \kappa = 0, \quad (5.6)$$

$$(Q_0 - 2)\psi_1 = 0, \quad (Q_0 - 2)\psi_2 = 0, \quad (5.7)$$

that indicates ψ_1 and ψ_2 are both conformal invariant massless spinor fields. Using the transversality condition on the cone we obtain:

$$\psi_2 = \frac{1}{2} \bar{\partial} \cdot \kappa,$$

$$(Q_0 - 2)\bar{\partial} \cdot \kappa = 0. \quad (5.8)$$

However, one can use (2.26), to write the CI equation (5.6) as:

$$\left(Q_{\frac{3}{2}}^{(1)} - \not{x} \not{\partial} + \frac{7}{2} \right) \kappa_\alpha + \bar{\partial}_\alpha \bar{\partial} \cdot \kappa - \gamma_\alpha \gamma \cdot \kappa = 0, \quad (5.9)$$

with using Eqs. (2.25) and (2.30) the above field equation can be rewritten as:

$$\left(Q_{\frac{3}{2}}^{(1)} - \frac{5}{2}\right)\kappa_x - (\not{x} \not{\partial}^\top \kappa_x - \kappa_x - x_\alpha \not{x} \not{\partial}^\top \kappa - \bar{\partial}_x \bar{\partial} \cdot \kappa + \gamma_\alpha^\top \not{x}) = 0, \tag{5.10}$$

the fields should be projected to the de Sitter space, the transverse projection implies the transversality of fields, $x \cdot \kappa = 0$, so that from the homogeneity condition, one obtains $x^\alpha x \cdot \partial \kappa_\alpha = 0$. These two conditions impose the following constrains on the projected fields: $x \cdot \Psi = 0 = \Psi^\alpha x_\alpha$, and consequently $\kappa_x = \Psi_x$. In Appendix 2, it is shown that the CI divergencelessness condition on the cone, namely $\nabla_a \Psi^a = 0$, results in $\partial^T_\alpha \kappa^\alpha = 0 = \partial^T_\alpha \Psi^\alpha$, which indicates the divergenceless fields are only mapped from the cone on dS hyperboloid..

For simplicity and irreducibility of vector–spinor field representation, the CI condition $\gamma^a \Psi_a = 0$ on the cone is imposed, this leads to $\gamma^\beta \kappa_\beta = \not{x} = 0$, which is the conformally invariant condition on the de Sitter hyperboloid. Imposing this condition and irreducibility (see (2.25)), one receives the following first- and second-order CI field equations

$$(\not{x} \not{\partial}^T - 1)\kappa_x(x) = 0, \quad \text{and} \quad \left(Q_{\frac{3}{2}}^{(1)} + \frac{5}{2}\right)\kappa_x = 0. \tag{5.11}$$

In this case, κ_x associates with the UIR of dS group, namely $\Pi_{\frac{3}{2}, \frac{3}{2}}^+$, and note that it propagates on the dS light cone. In the following sections, we find the solution of this vector–spinor field.

The solutions of conformally invariant field equation

Using the de Sitter plane waves [30], the de Sitter–Dirac plane wave for spinor field calculated [23]. In this section, the solution of the conformally invariant wave equation in terms of the de Sitter–Dirac plan wave calculated. A general solution of Eq. (5.11) can be written in terms of spinor fields ψ_1, ψ_2, ψ_3 as follows:

$$\kappa_x(x) = Z_\alpha^\top \psi_1 + D_{\frac{3}{2}\alpha} \psi_2 + \gamma_\alpha^\top \psi_3, \tag{6.1}$$

where Z is an arbitrary five-component constant vector field

$$Z_\alpha^\top = \theta_{\alpha\beta} Z^\beta = Z_\alpha + H^2 x_\alpha x \cdot Z, \quad x \cdot Z^\top = 0.$$

If we want the field κ_x to obey simultaneously the second- and first-order field equation (5.11) we find that, the spinor fields ψ_1, ψ_2 and ψ_3 must obey the following equations:

$$(Q_0 + \not{x} \not{\partial}^\top - 3)\psi_1 = 0, \tag{6.2}$$

$$2(x \cdot Z)\psi_1 + (Q_0 + \not{x} \not{\partial}^\top)\psi_2 = 0, \tag{6.3}$$

$$[\not{x}(Z \cdot x) + \not{Z}]\psi_1 + (Q_0 + \not{x} \not{\partial}^\top)\psi_3 = 0, \tag{6.4}$$

and

$$(\not{x} \not{\partial}^\top - 1)\psi_1 = 0, \tag{6.5}$$

$$(\not{x} - 2)\psi_2 - 2 \not{x} \psi_3 = 0, \tag{6.6}$$

$$\not{x} \not{\partial}^\top \psi_3 - (4 \not{x} + 1)\psi_2 - \not{x} \cdot Z \psi_1 = 0, \tag{6.7}$$

with using (6.2) and (6.5) we find that:

$$(Q_0 - 2)\psi_1 = 0, \tag{6.8}$$

that indicates ψ_1 is massless conformally coupled spinor field with homogeneity degree of $\sigma = -1$ and -2 [23].

In Eq. (6.1), all sentences must have the same degree of homogeneity. The homogeneity degree of Z_α^\top and γ_α^\top is zero. Therefore, the homogeneity degree of ψ_3 must be equal with ψ_1 . So, the homogeneity degree of ψ_2 must be $\sigma = 0, -3$.

Eq. (6.6) results in

$$\psi_2 = \frac{2}{5}(1 - 2 \not{x})\psi_3, \quad \text{or} \quad \psi_3 = \frac{1}{2}(1 + 2 \not{x})\psi_2. \tag{6.9}$$

Now let us multiply $\not{\partial}^T$ from the left on Eqs. (6.6) and (6.7), after doing some calculations, this yields

$$Q_0 \psi_2 = \frac{2}{5}(1 - 2 \not{x})\psi_2 + 2[3x \cdot Z - \not{x} \not{Z}^\top]\psi_1, \tag{6.10}$$

$$Q_0 \psi_3 = -\frac{2}{5}(2 \not{x} + 7)\psi_3 + [3x \cdot Z - \not{x} \not{Z}^\top]\psi_1. \tag{6.11}$$

Inserting these results in Eqs. (6.2), (6.3), (6.4) and after making use of (6.9), one can write ψ_2 and ψ_3 in term of ψ_1 as follows

$$\psi_2 = \frac{1}{2}(1 + 3 \not{x})[\not{Z} + (1 + 3 \not{x})x \cdot Z]\psi_1, \tag{6.12}$$

$$\psi_3 = -\frac{5}{4}(1 - \not{x})[\not{Z} + (1 + 3 \not{x})x \cdot Z]\psi_1. \tag{6.13}$$

Using the divergencelessness condition and the above relations, one obtains

$$(Z \cdot \partial^T + 3x \cdot Z)\psi_1 + \frac{1}{2}(\not{x} - 3)[\not{Z} + (1 + 3 \not{x})x \cdot Z]\psi_1 = 0, \tag{6.14}$$

and making use of $\not{x} = 0$ leads one to write

$$\not{Z}^\top \psi_1 - (4 \not{x} + \not{\partial}^\top)\psi_2 + 4\psi_3 = 0,$$

and in terms of ψ_1 , this equation becomes

$$\not{Z}^\top \psi_1 + 2(\not{x} + 5)[\not{Z} + (1 + 3 \not{x})x \cdot Z]\psi_1 - \frac{2}{5}(1 + 2 \not{x})x \cdot Z \psi_1 = 0. \tag{6.15}$$

Actually what we have obtained is that by dealing with ψ_1 , the other two fields will be established as well. Consequently by gathering all the results, κ can be written as follows

$$\kappa_\alpha(x) = \mathbf{D}_\alpha(\mathbf{x}, \partial^T, \mathbf{Z})\psi_1, \tag{6.16}$$

where we have defined

$$\mathbf{D}_\alpha(\mathbf{x}, \partial^T, \mathbf{Z}) \equiv Z_\alpha^T + \left[\frac{1}{2} \mathcal{D}_\alpha^{(\frac{3}{2})}(1 + 3 \not{x}) - \frac{5}{4} \gamma_\alpha^T(1 - \not{x}) \right] [\mathcal{Z} + (1 + 3 \not{x})x \cdot \mathcal{Z}].$$

In this formalism, a given spin- $\frac{3}{2}$ field could be constructed from the multiplication of the polarization vector \mathbf{D}_α or Z_α (5 degrees of freedom) with the spinor field ψ_1 (2 degrees of freedom), which appears naturally ten polarization states. After making use of (6.14) and (6.15), the degrees of freedom are indeed reduced to the usual four polarization states $m_{\frac{3}{2}} = \frac{3}{2}, \frac{1}{2}, -\frac{1}{2}$ and $-\frac{3}{2}$, where two of them are the physical states $\pm \frac{3}{2}$ [31].

Now, ψ_1 should be identified. Making use of the relation between the $s = 1/2$ and spin-zero Casimir operators, Eq. (6.8) can be written as

$$\left(Q_{\frac{1}{2}}^{(1)} - \frac{1}{2} \right) \psi_1(x) = 0. \tag{6.17}$$

This means that ψ_1 and its related two-point functions can in fact be extracted from a massive spinor field in the principal series representation given by (2.11) by setting $v = -i$.

Therefore, the solutions of (6.8) are found to be [23, 32]:

$$\psi_{1\xi, \mathcal{V}}(x) = \mathcal{V}(x, \xi)(Hx \cdot \xi)^{-1}, \tag{6.18}$$

$$\psi'_{1\xi, \mathcal{U}}(x) = \mathcal{U}(\xi)(Hx \cdot \xi)^{-3}, \tag{6.19}$$

where $\mathcal{V}(x, \xi) = \frac{x \cdot \xi}{x \cdot \xi} \mathcal{V}(\xi)$ and

$$\xi \in \mathcal{C}^+ = \{ \xi; \eta_{\alpha\beta} \xi^\alpha \xi^\beta = (\xi^0)^2 - \vec{\xi} \cdot \vec{\xi} - (\xi^4)^2 = 0, \xi^0 > 0 \}.$$

The two spinors $\mathcal{V}(\xi)$ and $\mathcal{U}(\xi)$ are

$$\mathcal{U}^a(\xi) = \frac{\xi^0 - \vec{\xi} \cdot \vec{\gamma} \gamma^0 + 1}{\sqrt{2(\xi^0 + 1)}} \mathcal{U}^a(\xi_+^o), \tag{6.20}$$

$$\mathcal{V}^a(\xi) = \frac{1}{\sqrt{2(\xi^0 + 1)}} \mathcal{U}^a(\xi_-^o), \quad a = 1, 2,$$

where

$$\mathcal{U}_1(\xi_+^o) = \frac{1}{\sqrt{2}} \begin{pmatrix} \alpha \\ \alpha \end{pmatrix}, \quad \mathcal{U}_2(\xi_+^o) = \frac{1}{\sqrt{2}} \begin{pmatrix} \beta \\ \beta \end{pmatrix}, \tag{6.21}$$

$$\mathcal{U}_1(\xi_-^o) = \frac{1}{\sqrt{2}} \begin{pmatrix} \alpha \\ \text{alpha} \end{pmatrix}, \quad \mathcal{U}_2(\xi_-^o) = \frac{1}{\sqrt{2}} \begin{pmatrix} \beta \\ -\beta \end{pmatrix}, \tag{6.22}$$

with $\alpha = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$, $\beta = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$ and $\xi = \frac{o}{\pm} \equiv (1, \vec{0}, \pm 1)$.

Eventually, two solutions for $\kappa_\alpha(x)$ are

$$\begin{aligned} \kappa_{1\alpha}(x) &= \left[Z_\alpha^T + \left[\frac{1}{2} \mathcal{D}_\alpha^{(\frac{3}{2})}(1 + 3 \not{x}) - \frac{5}{4} \gamma_\alpha^T(1 - \not{x}) \right] \right. \\ &\quad \left. [\mathcal{Z} + (1 + 3 \not{x})x \cdot \mathcal{Z}] \right] \mathcal{V}(\xi)(Hx \cdot \xi)^{-1} \\ &\equiv \mathcal{V}_\alpha(x, \xi, \mathcal{Z})(Hx \cdot \xi)^{-1}, \end{aligned} \tag{6.23}$$

and

$$\begin{aligned} \kappa_{2\alpha}(x) &= \left[Z_\alpha^T + \left[\frac{1}{2} \mathcal{D}_\alpha^{(\frac{3}{2})}(1 + 3 \not{x}) - \frac{5}{4} \gamma_\alpha^T(1 - \not{x}) \right] \right. \\ &\quad \left. [\mathcal{Z} + (1 + 3 \not{x})x \cdot \mathcal{Z}] \right] \mathcal{U}(\xi)(Hx \cdot \xi)^{-3} \\ &\equiv \mathcal{U}_\alpha(x, \xi, \mathcal{Z})(Hx \cdot \xi)^{-3}. \end{aligned} \tag{6.24}$$

By taking the derivative of plan wave $(x \cdot \xi)^\sigma$, the explicit forms of \mathcal{U}_α and \mathcal{V}_α are obtained in terms of ξ as follows:

$$\begin{aligned} \mathcal{V}_\alpha(x, \xi, \mathcal{Z}) &= \left[Z_\alpha^T + \left[\frac{1}{2} \mathcal{D}_\alpha^{(\frac{3}{2})}(1 + 3 \not{x}) - \frac{5}{4} \gamma_\alpha^T(1 - \not{x}) \right] \right. \\ &\quad \left. [\mathcal{Z} + (1 + 3 \not{x})x \cdot \mathcal{Z}] \right] \frac{\not{x} \cdot \not{\xi}}{x \cdot \xi} \mathcal{V}(\xi), \end{aligned} \tag{6.25}$$

and

$$\begin{aligned} \mathcal{U}_\alpha(x, \xi, \mathcal{Z}) &= \left[Z_\alpha^T + \left[\frac{1}{2} \mathcal{D}_\alpha^{(\frac{3}{2})}(1 + 3 \not{x}) - \frac{5}{4} \gamma_\alpha^T(1 - \not{x}) \right] \right. \\ &\quad \left. [\mathcal{Z} + (1 + 3 \not{x})x \cdot \mathcal{Z}] \right] \mathcal{U}(\xi). \end{aligned} \tag{6.26}$$

Conclusion

Conformal transformations and conformal techniques have been used in general relativity for a long time [33]. As one knows, the quantum theory of gravity based on Einstein equation is not renormalizable [34]. Also, it is proved that the conformal theories of gravity are better to renormalize [35, 36]. The gravitational field is long range and propagates with the speed of light, thus in the linear approximation, expected the equations governing its dynamic must be conformally invariant, whereas, the Einstein equation is not conformally invariant equation.

In the linear approximation, gravitational field resembles as a massless particle with spin-2 that propagates on the background space–time. In this paper, we study fermionic partner of gravitational field, massless spin- $\frac{3}{2}$ field, in de Sitter space. We used Dirac’s six-cone formalism to obtain CI massless spin- $\frac{3}{2}$ wave equation in dS space which corresponds to UIRs of the dS group. We obtain the solutions in ambient space notation.

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Appendix 1: Some useful relations

In this appendix, some useful relations are given.

$$\bar{\partial}_\alpha = \partial_\alpha + x_\alpha x \cdot \partial = \partial_\alpha - x_\alpha + x \cdot \partial x_\alpha \quad (7.1)$$

$$Q_0 \bar{\partial}_\alpha = \bar{\partial}_\alpha Q_0 + 2\bar{\partial}_\alpha + 2x_\alpha Q_0 \quad (7.2)$$

$$Q_0 x_\alpha = x_\alpha Q_0 - 4x_\alpha - 2\bar{\partial}_\alpha \quad (7.3)$$

$$\bar{\gamma}_\alpha = \Theta_\alpha^\beta \gamma_\beta = \gamma_\alpha + x_\alpha x \cdot \gamma \quad (7.4)$$

$$D_{\frac{3}{2}\alpha} = -\bar{\partial}_\alpha - \bar{\gamma}_\alpha \not{x} \quad (7.5)$$

$$Q_{\frac{1}{2}}^{(1)} D_{\frac{3}{2}} = D_{\frac{3}{2}} Q_{\frac{1}{2}}^{(1)} \quad (7.6)$$

$$Q_0 = -\bar{\partial}^2 \quad (7.7)$$

$$\gamma_\alpha^\top \not{x} = -\not{x} \gamma_\alpha^\top \quad (7.8)$$

$$Q_{\frac{3}{2}} \partial_\alpha^\top = \partial_\alpha^\top (Q_0 + \not{x} \not{\partial}^\top) \quad (7.9)$$

$$\not{\partial}^\top D_{\frac{3}{2}\alpha} = D_{\frac{3}{2}\alpha} \not{\partial}^\top - \not{x} \partial_\alpha^\top - x_\alpha \not{\partial}^\top - 4x_\alpha \not{x} + 3\gamma_\alpha^\top \quad (7.10)$$

$$Q_0 D_{\frac{3}{2}\alpha} = D_{\frac{3}{2}\alpha} Q_0 - 4\partial_\alpha^\top + 4\gamma_\alpha^\top \not{x} - 2\gamma_\alpha^\top \not{\partial}^\top - 2x_\alpha \not{x} \not{\partial}^\top \quad (7.11)$$

$$\not{x} \not{\partial}^\top x_\alpha = x_\alpha \not{x} \not{\partial}^\top + \not{x} \gamma_\alpha^\top \quad (7.12)$$

$$Q_0 \not{x} \not{\partial}^\top = \not{x} \not{\partial}^\top Q_0 \quad (7.13)$$

$$Z_\alpha^\top = Z_\alpha + x_\alpha x \cdot Z \quad (7.14)$$

$$Q_0 Z_\alpha^\top = Z_\alpha^\top Q_0 - 2x_\alpha Z \cdot \partial^\top - 4x_\alpha x \cdot Z \quad (7.15)$$

$$\partial^\top \cdot Z^\top = Z \cdot \partial^\top + 4x \cdot Z \quad (7.16)$$

$$\not{Z}^\top = \not{Z} + \not{x} x \cdot Z \quad (7.17)$$

$$\not{\partial}^\top \not{Z}^\top = \not{Z}^\top \not{\partial}^\top + \not{x} \not{Z}^\top + 4x \cdot Z \quad (7.18)$$

$$Q_0 \not{x} (x \cdot Z) = \not{x} x \cdot Z Q_0 - 8 \not{x} x \cdot Z - 2 \not{x} (Z \cdot \partial^\top) - 2 \not{\partial}^\top (x \cdot Z) \quad (7.19)$$

$$Q_0 \not{x} (Z \cdot \partial^\top) = \not{x} (Z \cdot \partial^\top) Q_0 - 2 \not{x} (Z \cdot \partial^\top) + 2 \not{x} (x \cdot Z) Q_0 - 2 \not{\partial}^\top (Z \cdot \partial^\top) \quad (7.20)$$

$$\not{\partial}^\top (Z \cdot \partial^\top) = -2 \not{x} (Z \cdot \partial^\top) + \not{x} (x \cdot Z) \quad (7.21)$$

$$\not{\partial}^\top (x \cdot Z) = -\not{x} (x \cdot Z) + \not{Z}^\top \quad (7.22)$$

Appendix 2: CI wave equation

By considering (4.7) and (5.1) we can write:

$$(Q_0 - 2)\Psi_\alpha = 0 \quad (8.1)$$

then using transversality condition we have:

$$(Q_0 - 2)x \cdot \Psi = 0 \quad (8.2)$$

If we multiply the relation (2.1) this appendix in x_α from left we obtain:

$$Q_0 x_\alpha \Psi^\alpha + 2x_\alpha \Psi^\alpha + 2\bar{\partial}_\alpha \Psi^\alpha = 0$$

then

$$\bar{\partial} \cdot \Psi = -2x \cdot \Psi \quad (8.3)$$

Now we get divergence from κ_α in (5.4) as follows:

$$\bar{\partial} \cdot \kappa = 2x_\alpha x \cdot \Psi \quad (8.4)$$

where the relation (2.3) in this appendix has been used to obtain the above relation. Also, it is known that

$$(Q_0 - 2)\bar{\partial} \cdot \kappa = 0$$

Now the operator $(Q_0 - 2)$ can work on relationship κ_α in (5.4):

$$(Q_0 - 2)\kappa_\alpha = x_5 [-8x_\alpha x \cdot \Psi - 2x_\alpha \bar{\partial} \cdot \Psi - 2\bar{\partial}_\alpha (x \cdot \Psi)]$$

then

$$(Q_0 - 2)\kappa_\alpha = x_5 [-4x_\alpha x \cdot \Psi - 2\bar{\partial}_\alpha (x \cdot \Psi)]$$

Finally, we obtain CI spin- $\frac{3}{2}$ equation as follows:

$$(Q_0 - 2)\kappa_\alpha + 2x_\alpha \bar{\partial} \cdot \kappa + \bar{\partial}_\alpha \bar{\partial} \cdot \kappa = 0 \quad (8.5)$$

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