RESEARCH

Full structure of the thermal plasma including the ion isothermal drag

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Abstract

In order to study the ion temperature effect on the space-charge structure and the plasma variables on the wall, the ion thermal force has been added to the ion motion equation in the plasma fluidal model. In the eigenvalue problem of plasma, the plasma equations are numerically solved in a whole area from the plasma center to the wall and it is displayed that the ion temperature has significant effects on the plasma structure and floating variables. However, the fluidal theory of plasmas introduces a singular point among the space charge of plasma boundary layer if the static pressure and the inertial mass of the thermal ions are taken into account at the same time. Finding a full numerical solution for the thermal plasma equations needs to cross the singular point. The singular point and how crossing the point will be depicted too.

Keywords Isothermal drag · Ion temperature · Full solution · Singularity point

Introduction

The subject of plasma boundary layer is nearly as old as the subject of plasma body and is very important in many fields of plasma physics and plasma technology including fusion researches, plasma diagnostics and material plasma processing [1–4]. For example, in plasma immersion ion implantation (PIII) that is an effective technique for semiconductor fabrication and material processing, knowing the structure of plasma boundary layer plays the main role [5–7].

The mathematical examination of plasmas near walls and electrodes has attracted considerable interest and has been turned to the main issue of many explorations [8–19]. Among the several parameters that their effects have been studied on the plasma variables and structure, the ion temperature effects in thermal plasmas are significant and have been examined by many authors [20–35].

It is apparent that the plasma fluid equations introduce singularities among the quasi-neutral plasma core if the static pressure and the inertial mass of the thermal ions are taken into account at the same time [20, 21, 36]. Since the quasi-neutral one-fluid equations represent an acceptable smooth solution without any singularity, it follows that the singularity represented in the two-fluid equations of thermal plasmas should be obliterate [20, 36–38].

Friedman and Levi [36] have solved the two-fluid equations of thermal plasmas using a so-called neutral approach. Using a perturbation method, Franklin [20] has analyzed the same equations, and he got the ratio τ_i/τ_e as the perturbation parameter with τ_i and τ_e as the ion and electron absolute temperature, respectively. Valentini [21] has tried to remove the irregularities at the singular point and its near proximity, and has represented a systematic method to prevail the singularity problems via numerical calculations.

In the present paper, using the boundary conditions at the plasma center and on the wall, the two-fluid equations are solved throughout the whole plasma for some values of the ion temperature in order to obtain a precise description of the plasma boundary layer structure. In order to find a smooth solution, using an approximation method, the main equations are analyzed in an velocity interval around the singular point. The effects of ion temperature and ion generation rate are investigated on the boundary layer structure and its floating variables.

In "Model and equations" section of this inquiry, warm plasmas incorporating the ion isothermal drag are formulized. The asymptotic limits of plasma including the

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plasma asymptotic and sheath asymptotic limits are introduced and are analytically solved in the plasma and sheath scales, respectively, in "Asymptotic and full solutions" section. Also the plasma full equations in the plasma scale, including the smallness parameter α and their numerical solutions will be presented in this section. The *summary and results* section summarizes paper derivations.

Model and equations

A thermal plasma confined between two metal plane walls located at $z = \pm L$ is considered to investigate the thermal behavior of plasma variables. Whereas the electrodes are flat and there are no sidewalls, a one-dimensional study is considered. Indeed, the plasma vessel is unlimited laterally and a longitudinal plasma is formed just between the electrodes, so the plasma is wasted at the both ends on the electrodes and there is not any lateral instability for plasma. The warm ions are modeled using the fluidal approximation of plasma. Since they are in thermal equilibrium, the electrons are described using the Boltzmann's relation. Also the plasma is symmetrical around the central plane $z = 0$; therefore, it is studied in the half-space $0 \le z \le L$. Then, the two-fluid model of plasma is represented by [16, 21, 30, 32, 34, 36];

$$\frac{d(n_i v_z)}{dz} = \nu n_e, \tag{1}$$

$$m_i \frac{d(n_i v_z^2)}{dz} = e n_i E_z - \frac{dP_i}{dz}, \tag{2}$$

$$n_e = n_o \exp\left(\frac{e\varphi}{k_B \tau_e}\right), \tag{3}$$

$$\frac{d^2\varphi}{dz^2} = -\frac{e}{\varepsilon_o}\left(n_i - n_e\right). \tag{4}$$

These equations are the ion continuity and ion momentum conservation equations [Eqs. (1) and (2), respectively], the Boltzmann relation for the electron density [Eq. (3)], and the Poisson's equation [Eq. (4)] in which n_i and m_i are the ion number density and ion mass, respectively, P_i is the static pressure of ions, $E_z = -d\varphi/dz$ is the z-component of electric field, φ is the electric potential and v_z is the z-component of ion velocity. Also, n_e is the electron number density, $n_o = n_{eo} = n_{io}$ is the plasma density (ion or electron density) at the plasma center, k_B is the Boltzmann constant, ν is the ionization rate and ε_o stands for the vacuum permittivity.

Besides, the thermal drag of ions is $dP_i/dz = \gamma k_B \tau_i (n_i/n_o)^{(\gamma-1)}(dn_i/dz)$ in which γ is the ion polytrophic coefficient. In the case of ion isothermal flow ($\gamma = 1$),

the ion thermal drag is simplified to $dP_i/dz = k_B \tau_i (dn_i/dz)$ [21, 28, 30, 32].

It is efficient to represent some normalized variables and parameters as; $Z = (\nu/c_s)z$, $T = \tau_i/\tau_e$, $\alpha = (\nu/c_s)\lambda_D$, $\chi = (ec_s/k_B\tau_e\nu)E_z$, $\eta = -(e/k_B\tau_e)\varphi$, $V = v_z/c_s$, $I = n_i/n_o$ and $E = n_e/n_o$ in which $\chi = d\eta/dZ$ stands for the normalized electric field, $k_B\tau_e$ is the electron thermal energy, $c_s = \sqrt{k_B\tau_e/m_i}$ introduces the ion *sound* velocity and $\lambda_D = \sqrt{\varepsilon_o k_B\tau_e/n_o e^2}$ defines the Debye length at the plasma center. Therefore, Eqs. (1), (2) and (4) will be turned to;

$$\frac{d(IV)}{dZ} = E, \tag{5}$$

$$V\frac{dV}{dZ} = \chi - \frac{EV}{I} - \frac{T}{I}\frac{dI}{dZ}, \tag{6}$$

$$\alpha^2 \frac{d^2\eta}{dZ^2} = I - E, \tag{7}$$

along with $E = \exp(-\eta)$ for the normalized electron density. These equations must be integrated for proper boundary conditions at the plasma center and on the wall for some different values of the both parameters T and α. By integrating from the plasma center ($Z = 0$) to the position $Z = Z_f$ where the *floating* boundary condition $\eta = \eta_f$ is satisfied, one can simply solve the eigenvalue problem (Z_f and η_f are called the floating width and floating potential, respectively).

In the special case $\alpha \to 0$ the Poisson's equation (7) gives rise to $I = E = \exp(-\eta)$ which is named the quasi-neutrality with no space charge. In other words, a neutral plasma and a very thin sheath with positive space charge are formed in front of the electrodes. In order to form a thick sheath layer attached to each electrode in which the Poisson's equation plays a critical role, it is essential to exchange the space coordinate Z to $\zeta = (Z - Z_f)/\alpha$ with ζ as the new high resolution coordinate and Z_f as the origin of the new coordinate. In the new coordinate space, set Eqs. (5)–(7) will be exchanged to

$$\frac{d(IV)}{d\zeta} = \alpha E, \tag{8}$$

$$V\frac{dV}{d\zeta} = \chi - \alpha\frac{EV}{I} - \frac{T}{I}\frac{dI}{d\zeta}, \tag{9}$$

$$\frac{d^2\varphi}{d\zeta^2} = I - E, \tag{10}$$

where $E = \exp(-\varphi)$. There are apparent differences between set Eqs. (5)–(7) and (8)–(10) in the special case $\alpha \to 0$; however, there is not any difference between them in the case $\alpha > 0$ except for the space coordinate.

Asymptotic and full solutions

Applying the limit condition $\alpha = 0$ in the set Eqs. (5)–(7) or (8)–(10) creates the asymptotic equations. In this case, the plasma asymptotic equations created from Eqs. (5)–(7) are

$$\frac{d(I_o V_o)}{dZ} = E_o, \tag{11}$$

$$V_0 \frac{dV_o}{dZ} = \frac{d\eta_o}{dZ} - V_o - \frac{T}{I_o}\frac{dI_o}{dZ}, \tag{12}$$

$$I_o = E_o = \exp(-\eta_o), \tag{13}$$

that can be combined and simplified to

$$\frac{d\eta_o}{dZ} = \frac{2V_o}{1 + T - V_o^2}, \tag{14}$$

$$\frac{dV_o}{dZ} = \frac{1 + T + V_o^2}{1 + T - V_o^2}. \tag{15}$$

Subscription "o", representing the *outer* asymptotic solution, has been added to the variables V, I and η. These equations are just held in the quasi-neutral plasma and their analytical solutions are

$$\eta_o = \ln\left(1 + \frac{V_o^2}{1 + T}\right), \tag{16}$$

$$Z_o = 2\sqrt{1 + T}\arctan\left(\frac{V_o}{\sqrt{1 + T}}\right) - V_o, \tag{17}$$

$$I_o = E_o = \exp(-\eta_o) = \frac{1 + T}{1 + T + V_o^2}, \tag{18}$$

These functions are authentic in the interval $0 \leq V_o \leq \sqrt{1 + T}$. A singular point, located in $V_s = \sqrt{1 + T} = V_B$, is introduced by Eqs. (14) and (15) and is called the general Bohm criterion. The singular point V_s represents the sheath edge where the breakdown of the plasma quasi-neutrality is commenced. Using the general Bohm criterion in Eqs. (16)–(18), the other variables at the sheath edge $\eta_s = \ln 2 = 0.6931$, $Z_s = \sqrt{1 + T}(\pi/2 - 1)$ and $I_s = \exp(-\eta_s) = 1/2$ are attained in which the subscription "s" refers to the *sheath* edge. One can easily find the well-known amounts $V_s = 1$ and $Z_s = \pi/2 - 1 = 0.5708$ in the cold plasmas [18].

On the other hand, applying the condition $\alpha = 0$ in the set Eqs. (8)–(10) generates the sheath asymptotic equations

$$\frac{d(I_i V_i)}{d\zeta} = 0, \tag{19}$$

$$V_i \frac{dV_i}{d\zeta} = \chi_i - \frac{T}{I_i}\frac{dI_i}{d\zeta}, \tag{20}$$

$$\frac{d\chi_i}{d\zeta} = I_i - E_i, \tag{21}$$

$$\frac{d\eta_i}{d\zeta} = \chi_i. \tag{22}$$

The additional subscription "i" has been added to the variables V, I, η and χ to introduce the *inner* asymptotic solution. These equations are just held in the positive space-charge sheath. These equations are not included the ionization term any more. Using Eqs. (19) and (22) in Eq. (20) and integrating of the obtained equation from the sheath edge to a general point in the sheath region results to;

$$\left(\frac{V_i}{V_s}\right)^2 - \frac{2T}{V_s^2}\ln\left(\frac{V_i}{V_s}\right) = 2\frac{\eta_i - \eta_s}{V_s^2} + 1, \tag{23}$$

$$\left(\frac{I_s}{I_i}\right)^2 - \frac{2T}{V_s^2}\ln\left(\frac{I_s}{I_i}\right) = 2\frac{\eta_i - \eta_s}{V_s^2} + 1, \tag{24}$$

for the ion velocity and ion density as functions of the electric potential η_i. These equations are lessened to the specified relations;

$$V_i = V_s\left(2\frac{\eta_i - \eta_s}{V_s^2} + 1\right)^{1/2}, \tag{25}$$

$$I_i = I_s\left(2\frac{\eta_i - \eta_s}{V_s^2} + 1\right)^{-1/2}, \tag{26}$$

in a nonthermal plasma [16].

Following some separate discussions on the both plasma asymptotic limit (introduced by the quasi-neutrality condition) and sheath asymptotic limit (with no ionization), it is the time to study the behavior of the whole plasma in the body and near the wall simultaneously. It means that the plasma Eqs. (5)–(7) should be solved for some small finite amounts of α. It is easy to verify that set Eqs. (5)–(7) give rise to the first-order differential equations

$$\frac{dV}{dZ} = \frac{1}{V^2 - T}\left(V\chi - \frac{E}{I}V^2 - T\frac{E}{I}\right), \tag{27}$$

$$\frac{dI}{dZ} = \frac{2VE - I\chi}{V^2 - T},$$ (28)

$$\frac{d\chi}{dZ} = \frac{I - E}{\alpha^2},$$ (29)

$$\frac{d\eta}{dZ} = \chi.$$ (30)

Numerical solution of these equations, from the plasma center to the wall, is the main goal of the rest of this paper. The required boundary conditions at the plasma center are computed using power series expansions for the variables I, V and η at that point. Evidently, the plasma variables I and η are even functions of the space coordinate Z, whereas ion velocity V is an odd function. Therefore, they might be introduced by

$$I \approx i_1 + i_2 Z^2,$$

$$\eta \approx (f_1 + f_2 Z^2) Z^2,$$

$$V \approx (v_1 + v_2 Z^2) Z,$$

and consequently

$$\chi = \frac{d\eta}{dZ} \approx (2f_1 + 4f_2 Z^2) Z,$$

$$E = \exp(-\eta) \approx 1 - f_1 Z^2 + \left(\frac{f_1^2}{2} - f_2\right) Z^4,$$

in which there are six unknown coefficients i_1, i_2, f_1, f_2, v_1 and v_2 that must be determined. Applying these approximate relations in Eqs. (27)–(30) and comparing the coefficients of the two lowest order in Z in the both sides of the acquired relations result in,

$$i_1 = 1 + \frac{2(i_1^3 + 12\alpha^2)\alpha^2}{[4i_1^3 + (T - 3)i_1^2 + 2(4T + 3)\alpha^2]i_1^2},$$ (31)

and

$$i_2 = \frac{f_1 i_1 - v_1}{T}, \quad v_1 = \frac{1}{i_1}, \quad f_1 = \frac{i_1 - 1}{2\alpha^2},$$

$$f_2 = \frac{i_2 + f_1}{12\alpha^2}, \quad v_2 = -\frac{f_1 + 3i_2 v_1}{3i_1}.$$

Starting with $i_1 = 1$ as the initial amount and utilizing relation (31) iteratively, one can find the principle factor i_1; the other coefficients are determined using i_1.

By finding the boundary conditions in a near vicinity of the plasma center, say $Z = 0.01$, one can proceed to solve

Eqs. (27)–(30). But, it is apparent that there is a singularity in these equations. It is essential to transmit the singularity point smoothly when solving the equations. To cross this singular point correctly and smoothly, an approximate technique are utilized. In this method, a velocity interval is symmetrically chosen around the singular point $V = \sqrt{T}$ in which the both following equations are worked out instead of Eqs. (27) and (28)

$$\frac{dV}{dZ} = \frac{\chi}{V} - \frac{E}{I} - \frac{T}{VI}\frac{dI}{dZ},$$ (32)

$$\frac{dI}{dZ} = \frac{E}{V} - \frac{I}{V}\frac{dV}{dZ}.$$ (33)

Equations (32) and (33) are the direct consequences of Eqs. (5) and (6). In this method, the gradient of the ion density I and the ion velocity V have been specified just before the interval; therefore, they can be used in Eqs. (32) and (33) to find I and V and their gradients in the next step and this sequences stay on to the end of the interval. Again, we will come back to the singular Eqs. (27) and (28) at the end of the interval. It is important to mention that there is no need to replace Eqs. (29) and (30) since they are not singular.

In order to find the full solution of Eqs. (27)–(30), it is essential to end the computations at a suitable location. *Floating* point, in which the ion directional current is equal to the electron random current, is selected as the right location. The floating point is specified by; $n_{if}v_{zf} = n_{ef}c_e/4$ in which n_{if} is the ion density, v_{zf} is the ion velocity and n_{ef} is the electron density at that point, and $c_e = \sqrt{8k_B T/\pi m_e}$ is the electron thermal velocity [16]. The recent floating relation can be rewrite in the normalized form as follows

$$I_f V_f = E_f \sqrt{\frac{m_i}{2\pi m_e}} = \sqrt{\frac{m_i}{2\pi m_e}} \exp(-\eta_f),$$

leading to

$$\eta_f = \ln\left(\frac{\beta}{I_f V_f}\right),$$ (34)

in which, $\beta = \sqrt{m_i/2\pi m_e}$. We have done the computations with $\beta = 108.0832$ calculated for the electropositive gas Argon. Relation (34) is known as the floating condition and η_f is called the floating potential.

As noted at the outset, in the plasma asymptotic limit (with $\alpha = 0$), the sheath region is infinitely thin. Then, in the full solution approach and for $\alpha = 0$, all of the variables behave such as the step function. Therefore, in this limiting case, using the sheath width $Z_s = \sqrt{1 + T(\pi/2 - 1)}$ as the floating width Z_f is reasonable [16]. Moreover, relation (24) results in

$$I_{\mathrm{f}} = I_{\mathrm{s}} \left[1 + 2\frac{\eta_{\mathrm{f}} - \eta_{\mathrm{s}}}{V_{\mathrm{s}}^2} + \frac{2T}{V_{\mathrm{s}}^2} \ln\left(\frac{I_{\mathrm{s}}}{I_{\mathrm{f}}}\right) \right]^{-1/2}. \qquad (35)$$

By using relation (35), one can determine the floating ion density I_{f} by iteration and starting with $I_{\mathrm{f}} = I_{\mathrm{s}}[1 + 2(\eta_{\mathrm{f}} - \eta_{\mathrm{s}})/V_{\mathrm{s}}^2]^{-1/2}$ as the initial value. Finally, using the relation $V_{\mathrm{f}}I_{\mathrm{f}} = V_{\mathrm{s}}I_{\mathrm{s}}$ it is easy to find the floating ion velocity V_{f}.

The spatial variations of the plasma variables in the full solution approach are shown in Fig. 1. It is clear from this figure that the spatial distribution of the ion density I is broken at the beginning of the ion velocity interval selected symmetrically around the singular point $V = \sqrt{T}$; similarly, the spatial distribution of the ion velocity V suffers a small rippling at that point. At the end point of the interval, however, the both distributions display complete smoothness. Also there is not any unevenness or breaking at the spatial distributions of the both electric potential η and electron density E.

Figures 2, 3, 4, 5, 6, 7 and 8 exhibit some of the plasma variables including; the floating ion density I_{f}, floating electron density E_{f}, central ion density I_c, floating electric potential η_{f}, spatial distribution of electric potential η, floating ion velocity V_{f} and floating width X_{f} (extended from the plasma center to the floating point) as functions of the ion temperature T and smallness parameter α, respectively. The equations have been solved for $\alpha = 0$ to 0.2 and $T = 0$ to 1..
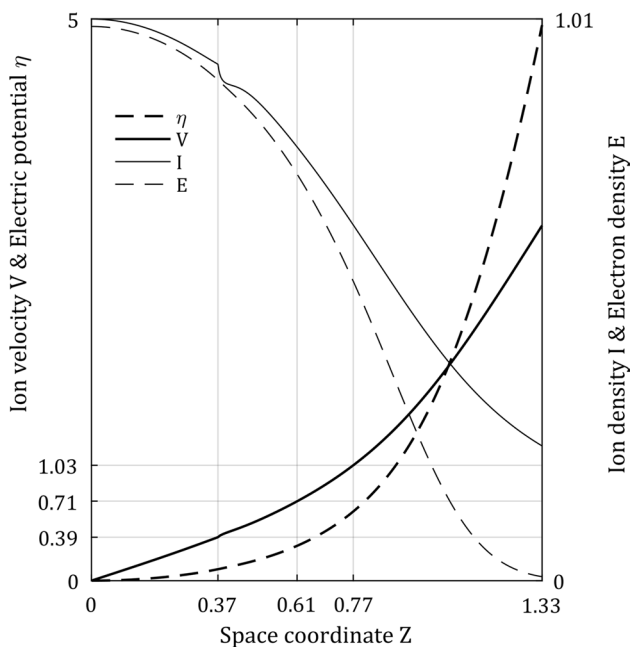

Fig. 1 The spatial distributions of ion density I, electron density E, ion velocity V and electric potential η as functions of the space coordinate Z for $T = 0.5$ and $\alpha = 0.1$. The quantities on the V axis are the singular point $V = \sqrt{T} \approx 0.71$ and the symmetrical interval around it (and their corresponding locations on the Z-axis)
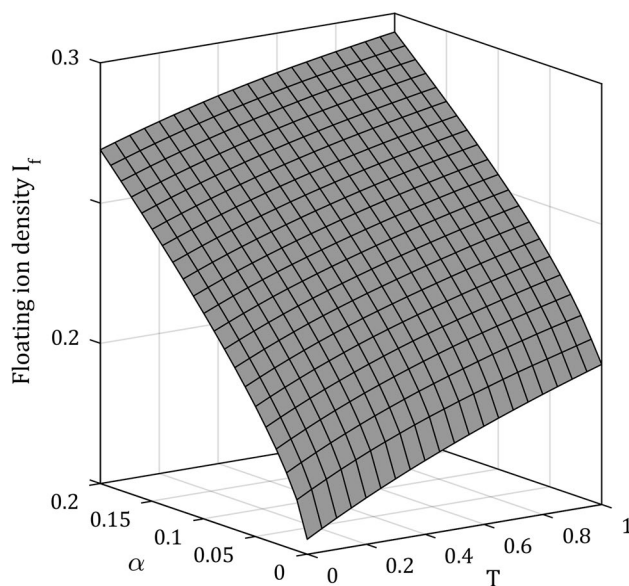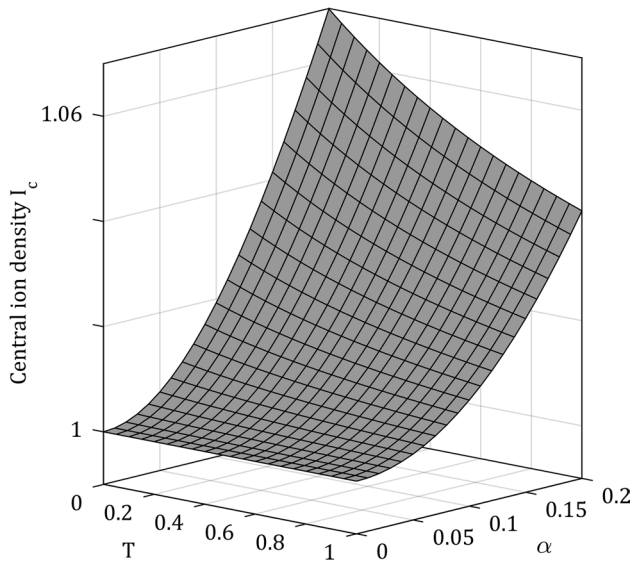

Fig. 2 Floating ion density I_{f} as a function of the ion temperature T and smallness parameter α

According to Figs. 2 and 3, floating ion and electron densities are increasing functions of the both ion temperature and smallness parameter. Since smallness parameter α is the only factor for plasma generation in the boundary layer, so it is reasonable that the plasma density become an increasing function of this parameter throughout the boundary layer [21].
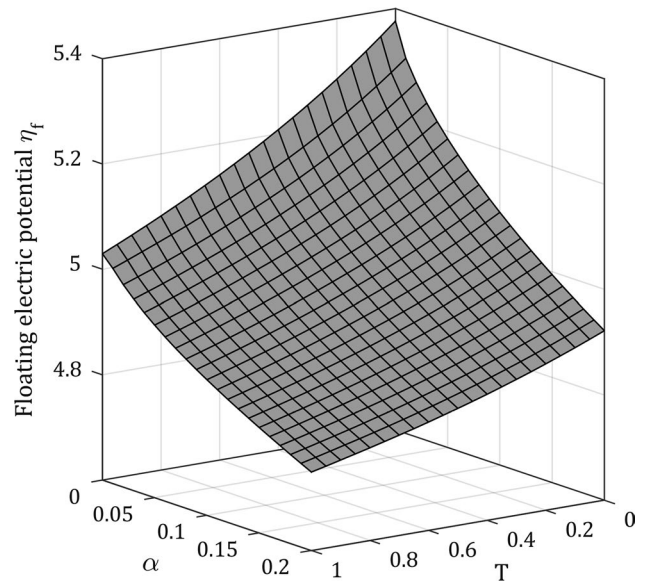
On the other hand, Fig. 4 shows that the ion density at the plasma center decreases by rising the ion temperature, while
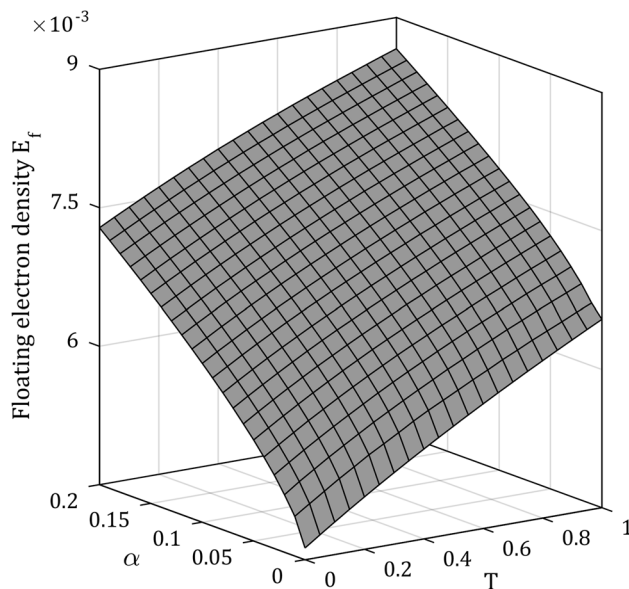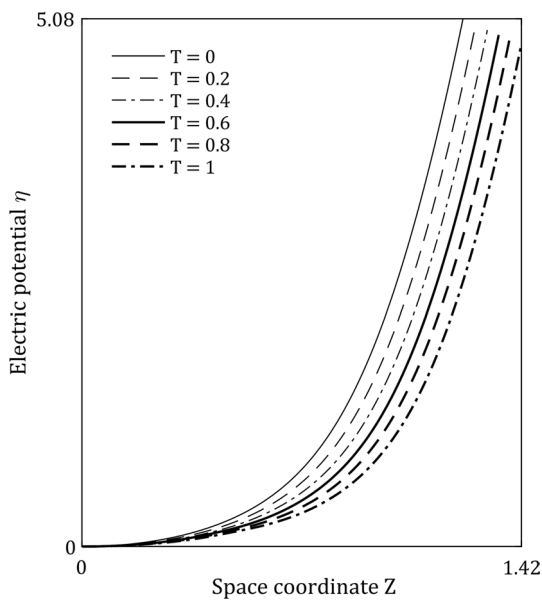

Fig. 3 Floating electron density E_{f} as a function of the ion temperature T and smallness parameter α
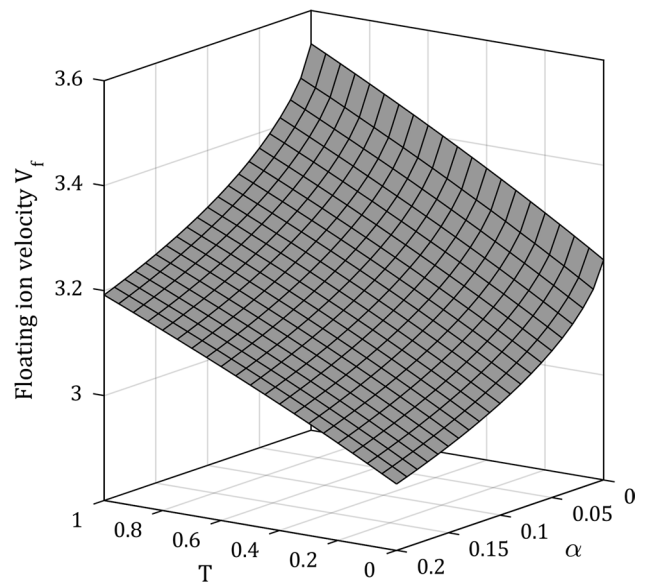
Fig. 4 The ion density at the plasma center I_c as a function of the ion temperature T and smallness parameter α

Fig. 6 Floating electric potential η_f as a function of the ion temperature T and smallness parameter α

Fig. 5 Spatial distribution of electric potential η for $\alpha = 0.1$ and some different values of ion temperature T
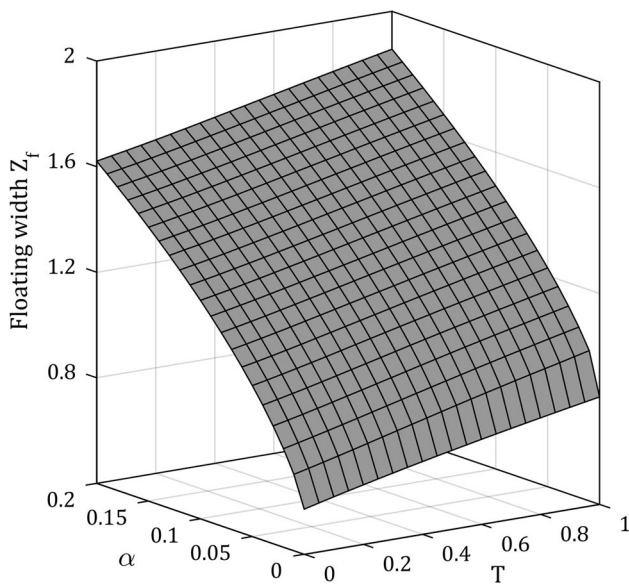

Fig. 7 Floating ion velocity V_f as a function of the ion temperature T and smallness parameter α

Fig. 2 displays that the ion temperature raises the floating ion density. It means that the ion temperature expands the plasma and extends it from the center to the outside of the plasma. Moreover, these figures show that smallness parameter α intensifies the ion temperature effect [21].

Figure 5 exhibits the spatial distribution of electric potential η from the plasma center to the floating point for $\alpha = 0.1$ and some values of ion temperature. It is apparent that the ion temperature decreases the electric potential distribution throughout the boundary layer. Also Fig. 6 shows that the

floating electric potential η_f is a descending function of the both ion temperature and smallness parameter. It means that the more plasma heating and the more plasma generating, the less floating potential.

Figure 7 displays the dependency of the floating ion velocity V_f to the ion temperature T and smallness parameter α. According to this figure, ion temperature increases and smallness parameter decreases the ion velocity and ion kinetic energy on the floating wall. Indeed, growing the ion

Fig. 8 Floating width Z_f as a function of the ion temperature T and smallness parameter α

temperature amplifies the thermal force on the ions and accelerates them outward to the wall. As a result of this process, the ion density on the floating wall increases and causes to pull out more electrons from the plasma center to the wall according to the electrostatic absorption. Moreover, the increased electron density on the floating wall reduces the floating potential according to the Boltzmann relation.

Finally, it can be seen from Fig. 8 that the ion temperature T and smallness parameter α increase the floating width Z_f. This is a result of growing the ion and electron density by increasing the ion temperature and the ion generation rate (or smallness parameter α) in the boundary layer [21].

It is important to note that the width of the ion velocity interval adjusted symmetrically around the singular point $\sqrt{T}$ has no effect on the floating variables and the general aspects of the boundary layer structure. Indeed, the velocity interval has been specified at the least possible amount to ensure the least unevenness in the spatial distribution of variables.

Summary and results

A fluidal treatment has been used to investigate how the ion temperature and the ion production rate affect on the plasma boundary layer structure as well as on the floating variables. First, the thermal plasma equations have been stated in two plasma and sheath scales and in the special case $\alpha = 0$ (no ion generation in the boundary layer) have been analytically solved. Furthermore, in the more common case $\alpha \gtrsim 0$ the equations have been numerically

solved from the plasma center to the wall in the plasma scale which is named the full solution approach.

In the full solution approach, the warm plasma equations including the smallness parameter α and ion temperature T which are singular at $V = \sqrt{T}$ have been analyzed in the plasma scale. In order to pass the singular point, an approximate method has been introduced. On the basis of this procedure, the full solution of the plasma equations introduces a breaking point near the singular point in the ion density distribution. This fracture is appeared much more faintly in the ion velocity distribution and is disappeared in the electric potential distribution.

The computations show that the ion temperature increases the floating width and all of the other variables at the floating point except the electric potential which is decreased by temperature. In addition, floating ion and electron densities and floating width are in direct relation with the ion generation rate (or smallness parameter) but the ion velocity and electric potential at the floating point are descending functions of this parameter.

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