Gaillard *Journal of Theoretical and Applied Physics* 2013, **7**:45
http://www.jtaphys.com/content/7/1/45

Journal of Theoretical and Applied Physics
a SpringerOpen Journal

RESEARCH **Open Access**

Two-parameter determinant representation of seventh order rogue wave solutions of the NLS equation

Pierre Gaillard

Abstract

We present a new representation of solutions of focusing nonlinear Schrödinger equation (NLS) equation as a quotient of two determinants. We construct families of quasi-rational solutions of the NLS equation depending on two parameters, *a* and *b*. We construct, for the first time, analytical expressions of Peregrine breather of order 7 and multi-rogue waves by deformation of parameters. These expressions make possible to understand the behavior of the solutions. In the case of the Peregrine breather of order 7, it is shown for great values of parameters *a* or *b* the appearance of the Peregrine breather of order 5.

PACS: 35Q55; 37K10
Keywords: NLS equation; Wronskians determinants; Peregrine breather; Akhmediev's solutions; Rogue waves

Introduction

One of the most direct approaches to model the evolution of deep water waves is the use of nonlinear Schrödinger equation (NLS) [1,2]. This equation was first solved in 1972 by Zakharov and Shabat [3] using inverse scattering method. Its and Kotlyarov constructed in 1976 periodic and almost periodic solutions of the focusing NLS equation [4]. It is in 1979 that Ma found the first breather-type solution of the NLS equation [5]; this solution breathes temporally but is spatially localized. In 1983, Peregrine [6] gave the first example of quasi-rational solutions of the NLS equation which was localized both in space and time. Akhmediev, Eleonski, and Kulagin obtained the first higher order analog of the Peregrine breather [7] in 1986. Other families of higher order were constructed by Akhmediev et al. [8,9] using Darboux transformations; these solutions breathe spatially and are localized in time. The Peregrine solution appears to be a limiting case of a breather when the spatial period is taken to be infinite.

The notion of rogue waves first appeared in studies of ocean waves; then it moved to other domains of physics like optics [10], Bose-Einstein condensates [11], etc. A lot of experiments about solutions of NLS equation have been realized; in particular, the basic Peregrine soliton has been studied very recently in [12,13]; furthermore, the NLS equation accurately describes physical rogue waves of relatively high order according to the work [14].

Actually, there is growing interest in studying higher order rational solutions. In 2010, rational solutions of the NLS equation have been written as a quotient of two Wronskians [15]. In 2011, it has been constructed in [16] a new representation of the solutions of the NLS equation in terms of a ratio of two Wronskian determinants of even order $2N$ composed of elementary functions.

Recently, Guo, Ling, and Liu gave another representation of the solutions of the focusing NLS equation as a ratio of two determinants [17] using generalized Darboux transform. Ohta and Yang have realized a new approach in [18] which gives a determinant representation of solutions of the focusing NLS equation, obtained from Hirota bilinear method.

Here we present a new representation of quasi-rational solutions of NLS expressed as a quotient of two determinants which do not involve limits introduced in [19]. It is important to note that the formulations given in [16,19] depend only on two parameters as mentioned in the title; this remark was first made by VB Matveev in March 2012. With this method, we obtain very easily explicit analytical

Correspondence: Pierre.Gaillard@u-bourgogne.fr
Université de Bourgogne, 9 Avenue Alain Savary, BP 47870, Dijon CEDEX 21078, France

expressions of the solutions of the NLS equation as deformations of the Peregrine breather of order 7 depending on two parameters, a and b. The following orders will be the object of other publications.

These explicit solutions are important since it makes it possible to understand the behavior of rogue waves. In particular, it shows the appearance for great values of parameters a or b of the Peregrine breather of order 5. Precisely, according to the explicit formulation of the solution of order 7, the factor of a^2 or b^2 is exactly the analytical expression of the Peregrine breather of order 5. This fact was first pointed out for the orders 3 and 4 by VB Matveev in March 2012. It was also noticed and highlighted by Akhmediev et al. in the article [20].

This result, in addition to being the first time that explicit expressions of deformations of order 7 of the solutions of the equation of NLS are given, makes it possible to understand the phenomenon of appearance of rogue waves and their asymptotic behavior.

The method given in this paper is a very powerful tool, better than that presented in [16,21], and leads to new results about NLS equation.

Expression of solutions of NLS equation in terms of a ratio of two determinants

We recall in this section the result exposed in [19]. We consider the following focusing NLS equation:

$$iv_t + v_{xx} + 2|v|^2 v = 0. \tag{1}$$

We use the following notations:

κ_ν, δ_ν, and γ_ν are functions of the parameters λ_ν and $\nu = 1, \ldots, 2N$, satisfying the relations

$$\begin{array}{l} 0 < \lambda_j < 1, \quad \lambda_{N+j} = -\lambda_j, \\ 1 \leq j \leq N. \end{array} \tag{2}$$

They are given by the following equations:

$$\begin{array}{l} \kappa_j = 2\sqrt{1-\lambda_j^2}, \quad \delta_j = \kappa_j \lambda_j, \\ \gamma_j = \sqrt{\frac{1-\lambda_j}{1+\lambda_j}}, \quad 1 \leq j \leq N, \end{array} \tag{3}$$

and

$$\begin{array}{l} \kappa_{N+j} = \kappa_j, \quad \delta_{N+j} = -\delta_j, \\ \gamma_{N+j} = 1/\gamma_j, \quad 1 \leq j \leq N, \end{array} \tag{4}$$

The terms $x_{r,\nu}$ $(r = 1, 3)$ are defined by

$$\begin{array}{l} x_{r,\nu} = (r-1) \ln \frac{\gamma_\nu - i}{\gamma_\nu + i}, \\ 1 \leq \nu \leq 2N. \end{array} \tag{5}$$

The parameters e_ν are defined by

$$\begin{array}{l} e_j = ia_j - b_j, \quad e_{N+j} = ia_j + b_j, \\ 1 \leq j \leq N, \end{array} \tag{6}$$

where a_j and b_j, for $1 \leq j \leq N$, are arbitrary real numbers.

We then define the following expressions:

$$\begin{array}{l} A_\nu = \kappa_\nu x/2 + i\delta_\nu t - ix_{3,\nu}/2 - ie_\nu/2, \\ B_\nu = \kappa_\nu x/2 + i\delta_\nu t - ix_{1,\nu}/2 - ie_\nu/2, \end{array}$$

for $1 \leq \nu \leq 2N$, with κ_ν, δ_ν, and $x_{r,\nu}$ defined previously in Equations 3, 4, and 5.

The parameters e_ν are defined in Equation 6. For the following reduction, we choose a_j and b_j in the form

$$\begin{array}{l} a_j = \tilde{a_1} j^{2N-1} \epsilon^{2N-1}, \\ b_j = \tilde{b_1} j^{2N-1} \epsilon^{2N-1}, \\ 1 \leq j \leq N. \end{array} \tag{7}$$

We consider the parameter λ_j written in the form

$$\lambda_j = 1 - 2j^2 \epsilon^2, \quad 1 \leq j \leq N. \tag{8}$$

Below we use the following functions for $1 \leq j \leq 2N$:

$$\begin{array}{l} f_{4j+1,k} = \gamma_k^{4j-1} \sin A_k, \\ f_{4j+2,k} = \gamma_k^{4j} \cos A_k, \\ f_{4j+3,k} = -\gamma_k^{4j+1} \sin A_k, \\ f_{4j+4,k} = -\gamma_k^{4j+2} \cos A_k, \end{array} \tag{9}$$

$1 \leq k \leq N$, and

$$\begin{array}{l} f_{4j+1,k} = \gamma_k^{2N-4j-2} \cos A_k, \\ f_{4j+2,k} = -\gamma_k^{2N-4j-3} \sin A_k, \\ f_{4j+3,k} = -\gamma_k^{2N-4j-4} \cos A_k, \\ f_{4j+4,k} = \gamma_k^{2N-4j-5} \sin A_k, \end{array} \tag{10}$$

for $N+1 \leq k \leq 2N$.

We define the functions $g_{j,k}$ for $1 \leq j \leq 2N$, $1 \leq k \leq 2N$ in the same way; we replace only the term A_k by B_k:

$$\begin{array}{l} g_{4j+1,k} = \gamma_k^{4j-1} \sin B_k, \\ g_{4j+2,k} = \gamma_k^{4j} \cos B_k, \\ g_{4j+3,k} = -\gamma_k^{4j+1} \sin B_k, \\ g_{4j+4,k} = -\gamma_k^{4j+2} \cos B_k, \end{array} \tag{11}$$

for $1 \leq k \leq N$, and

$$\begin{array}{l} g_{4j+1,k} = \gamma_k^{2N-4j-2} \cos B_k, \\ g_{4j+2,k} = -\gamma_k^{2N-4j-3} \sin B_k, \\ g_{4j+3,k} = -\gamma_k^{2N-4j-4} \cos B_k, \\ g_{4j+4,k} = \gamma_k^{2N-4j-5} \sin B_k, \end{array} \tag{12}$$

for $N+1 \leq k \leq 2N$.

All functions $f_{j,k}$ and $g_{j,k}$ and their derivatives depend on ϵ and can all be prolonged by continuity when $\epsilon = 0$. Then they have the following result:

Theorem 1. *The function v defined by*

$$v(x,t) = \exp(2it - i\varphi) \times \frac{\det((n_{jk})_{j,k\in[1,2N]})}{\det((d_{jk})_{j,k\in[1,2N]})} \tag{13}$$

is a quasi-rational solution of the NLS Equation 1:

$$iv_t + v_{xx} + 2|v|^2 v = 0,$$

where

$$\begin{aligned}
n_{j1} &= f_{j,1}(x,t,0),\\
n_{jk} &= \frac{\partial^{2k-2} f_{j,1}}{\partial\epsilon^{2k-2}}(x,t,0),\\
n_{jN+1} &= f_{j,N+1}(x,t,0),\\
n_{jN+k} &= \frac{\partial^{2k-2} f_{j,N+1}}{\partial\epsilon^{2k-2}}(x,t,0),\\
d_{j1} &= g_{j,1}(x,t,0),\\
d_{jk} &= \frac{\partial^{2k-2} g_{j,1}}{\partial\epsilon^{2k-2}}(x,t,0),\\
d_{jN+1} &= g_{j,N+1}(x,t,0),\\
d_{jN+k} &= \frac{\partial^{2k-2} g_{j,N+1}}{\partial\epsilon^{2k-2}}(x,t,0),\\
& 2 \le k \le N,\ 1 \le j \le 2N.
\end{aligned}$$

The functions f and g are defined in Equations 9, 10, 11, and 12.

Proof. We do not have the space to reproduce the proof here. The reader can find it in [19].

In other words, the solutions of the NLS equation can be written as

$$v(x,t) = \exp(2it - i\varphi) \times Q(x,t)$$

with $Q(x,t)$ defined by

$$\frac{\begin{vmatrix} f_{1,1}[0] & \dots & f_{1,1}[N-1] & f_{1,N+1}[0] & \dots & f_{1,N+1}[N-1]\\ f_{2,1}[0] & \dots & f_{2,1}[N-1] & f_{2,N+1}[0] & \dots & f_{2,N+1}[N-1]\\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots\\ f_{2N,1}[0] & \dots & f_{2N,1}[N-1] & f_{2N,N+1}[0] & \dots & f_{2N,N+1}[N-1] \end{vmatrix}}{\begin{vmatrix} g_{1,1}[0] & \dots & g_{1,1}[N-1] & g_{1,N+1}[0] & \dots & g_{1,N+1}[N-1]\\ g_{2,1}[0] & \dots & g_{2,1}[N-1] & g_{2,N+1}[0] & \dots & g_{2,N+1}[N-1]\\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots\\ g_{2N,1}[0] & \dots & g_{2N,1}[N-1] & g_{2N,N+1}[0] & \dots & g_{2N,N+1}[N-1] \end{vmatrix}}$$

$\square$

Quasi-rational solutions of order 7

We have already constructed in [16,19,21] solutions of the NLS equation for the cases $N = 1$ until $N = 6$. Here, we give new solutions of the NLS equation in the case $N = 7$. We start to give some plots of these solutions in the (x, t) plan for various values of the parameters.

In the following, for more simplicity, we note $\tilde{a}_1 = a$ and $\tilde{b}_1 = b$.

Figures

If we choose $a = 0$ and $b = 0$, we recognize the famous Peregrine breather of order 7 in Figure 1. If we choose $a = 0$ and $b = 10^8$, we obtain the following plot in Figure 2. If we choose $a = 10^{12}$ and $b = -10^8$, we obtain the following regular plot in Figure 3.

We saw in these plots that when parameters a and b increase, there is the appearance of circular configurations or more exactly annular configurations. This phenomenon was also highlighted in the article [22]. Indeed, the previous images given by Figures 2 and 3 are closely analogous to Figure 2 parameter b in that paper which gives the case of order 8 as an example. This fact is also highlighted in the works of the author [23]. In the work [22], it was pointed out that the shift (here corresponding to a and b nonzero) pulls out or 'emits' a ring of $2N - 1$ fundamental ($N = 1$) rogue elements, corresponding to 15 of them there and to 13 here. It leaves behind a rogue wave of order $N - 2$, i.e., 6 there (amplitude=13) and 5 (amplitude=11) in this paper. Of course, Figure 1 here is analogous to Figure 2 parameter a there (amplitudes 15 and 17, respectively).

Asymptotic behavior

In all these figures, for $N = 7$, we see that when the parameters a and b tend towards the infinite, there is the appearance of the Peregrine breather of order 5. This remark was made the first time by VB Matveev in March 2012 for orders 3 and 4 [24]. It was also reported in [20].

In fact, if one identifies the factors of a^2 or b^2 (even of $a^2 + b^2$) in the analytical expression of Peregrine of order 7 given in Additional file 1, one finds exactly that of the Peregrine breather of order 5. We do not have enough space in this text to detail these calculations. The reader who would like to find the expression of the Peregrine breather of order 5 will be able to refer to the article [21].

This explains why when one or the other of parameters a or b (even both) tend towards infinite in the case of the Peregrine breather of order $N = 7$, it appears in the center of the figures' Peregrine breather of order $N - 2 = 5$.

Analytic expressions

N takes the value 7; we make the following change of variables $X = 2x$ and $T = 4t$. The solution of the NLS equation can be written in the form

$$\begin{aligned}
v_N(x,t) &= \frac{n(x,t)}{d(x,t)}\exp(2it)\\
&= (1 - \alpha_N \frac{G_N(2x,4t) + iH_N(2x,4t)}{Q_N(2x,4t)})e^{2it}
\end{aligned}$$

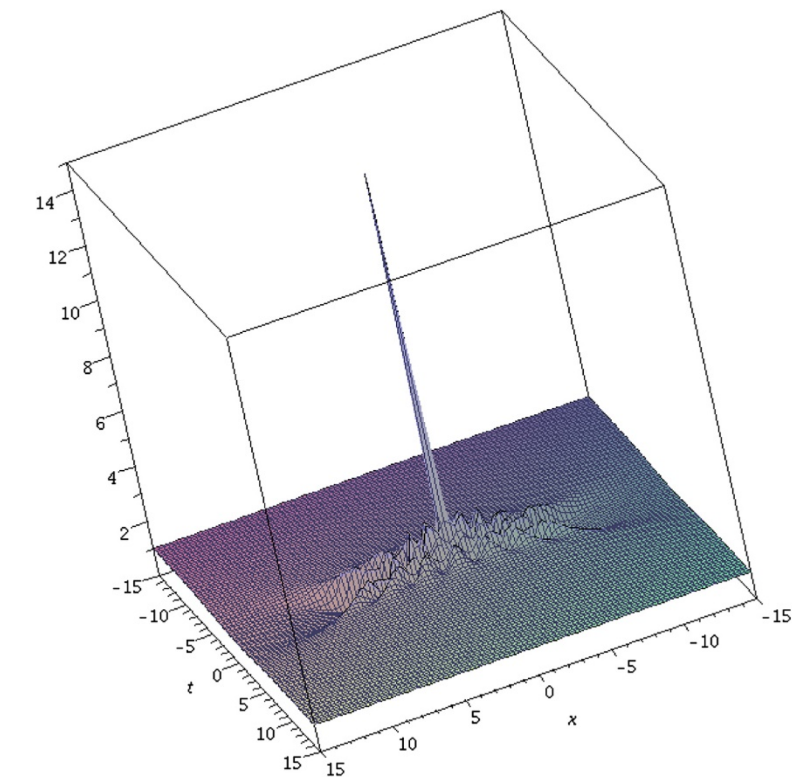

Figure 1 Solution of NLS, $N = 7, a = 0, b = 0$.

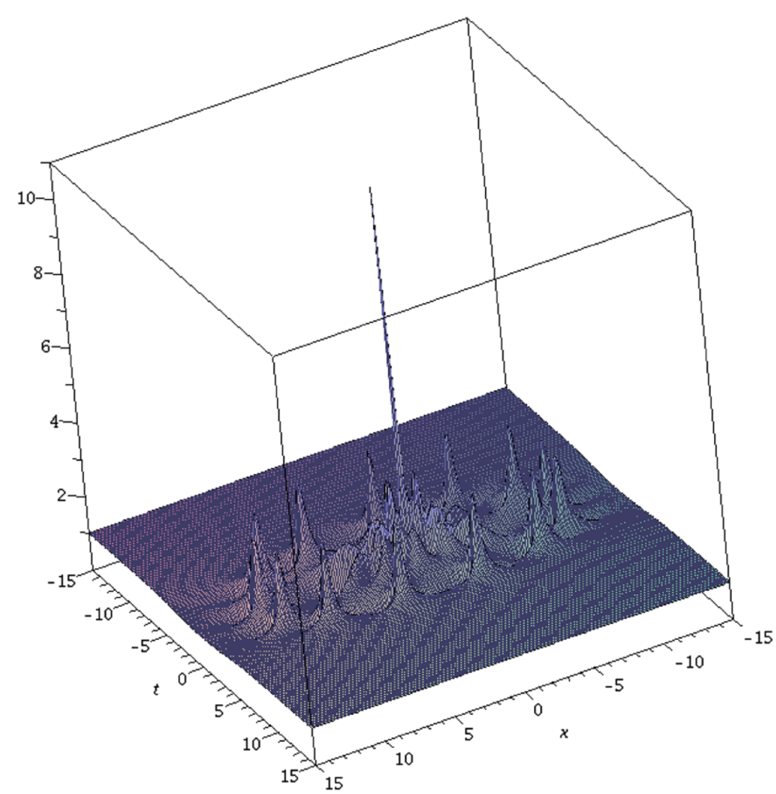

Figure 2 Solution of NLS, $N = 7, a = 0, b = 10^8$.

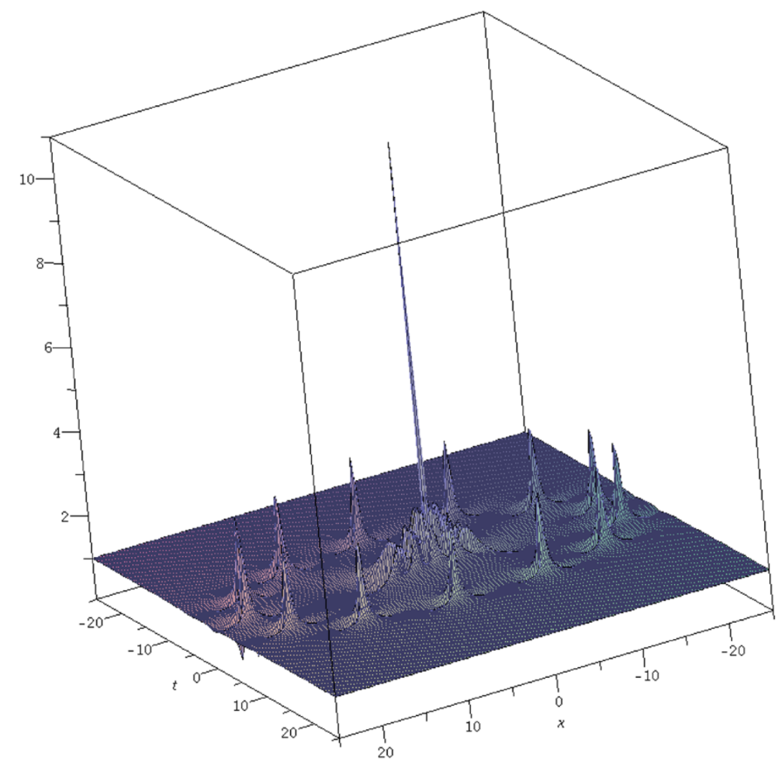

Figure 3 Solution of NLS, $N = 7, a = 10^{12}, b = -10^{8}$.

with

$$G_N(X,T) = \sum_{k=0}^{N(N+1)} \mathbf{g}_k(T)X^k$$

$$H_N(X,T) = \sum_{k=0}^{N(N+1)} \mathbf{h}_k(T)X^k$$

$$Q_N(X,T) = \sum_{k=0}^{N(N+1)} \mathbf{q}_k(T)X^k$$

The analytic expression of the Peregrine breather of order 7 is obtained as a particular case when $a = 0$ and $b = 0$. Because of the length of the coefficients of the polynomials G_N, H_N, and Q_N, we present them in Additional file 1.

Conclusions

We presented here for the first time to my knowledge the analytical expressions of the Peregrine breather of order 7 and its associated deformations. The conjecture about the form of the breather of order N in coordinates (x, t) is checked, also that the maximum of amplitude equal to $2N + 1$; the degree of the polynomials in x and t is quite equal to $N(N + 1)$ as that presented in [8]. For a and b nonzero, the maximum is less than that, as discussed above and seen in Figures 2 and 3. This new formulation gives a powerful novel method to obtain the nonsingular solutions of the NLS equation. We will present Peregrine breathers of a higher order in other publications. We can also imagine to obtain deformations with $2N - 2$ parameters for Peregrine breathers of order N. Some advances were made in this direction recently [24-26].

Also, we explained why, in the case of order $N = 7$, when one or the other of the parameters a or b (even both) tend towards infinite in the case of Peregrine breather of order N, it appears in the center of the figures' Peregrine breather of order $N - 2$. This phenomenon was checked for the orders of $N = 3$ until $N = 7$. It would be important to prove this conjecture in the general case.

Additional file

Additional file 1: Expressions of the coefficients of the polynomials *G*, *H*, and *Q*.

Competing interests
The author declares that he has no competing interests.

Acknowledgements
I would like to thank Vladimir B. Matveev for the long discussions that we had and Christian Klein for the financial aid by ANR via the program ANR-09-BLANC-0117-041.

Received: 9 March 2013 Accepted: 2 September 2013
Published: 10 September 2013

References

1. Osborne, AR: Nonlinear Ocean Waves and the Inverse Scattering Transform. Elsevier, New York (2010)
2. Yuen, HC, Lake, BM: Nonlinear deep water waves: theory and experiment. Phys. Fluids **18**(8), 956–960 (1975)
3. Zakharov, VE, Shabat, AB: Exact theory of two dimensional self focusing and one dimensional self modulation of waves in nonlinear media. Sov. Phys. JETP **34**, 62–69 (1972)
4. Its, AR, Kotlyarov, VP: Explicit expressions for the solutions of nonlinear Schrödinger equation. Dockl. Akad. Nauk. SSSR, S. A. **965**(11), 965–968 (1976)
5. Ma, YC: The perturbed plane-wave solutions of the cubic nonlinear Schrödinger equation. Stud. Appl. Math. **60**, 43–58 (1979)
6. Peregrine, D: Water waves, nonlinear Schrödinger equations and their solutions. J. Austral. Math. Soc. Ser. B. **25**, 16–43 (1983)
7. Eleonski, V, Krichever, I, Kulagin, N: Rational multisoliton solutions to the NLS equation. Soviet Doklady Sect. Math. Phys. **287**, 606–610 (1986)
8. Akhmediev, N, Ankiewicz, A, Soto-Crespo, JM: Rogue waves and rational solutions of nonlinear Schrödinger equation. Phys. Rev. E. **80**, 026601-1-8 (2009)
9. Ankiewicz, A, Clarkson, PA, Akhmediev, N: Rogue waves, rational solutions, the patterns of their zeros and integral relations. J. Phys. A : Math. Theor. **43**, 122002 (2010)
10. Erkintalo, M, Genty, G, Dudley, JM: Rogue-wave-like characteristics in femtosecond supercontinuum generation. Opt. Lett. **34**, 2468–2470 (2009)
11. Bludov, YV, Konotop, VV, Akhmediev, N: Matter rogue waves. Phys. Rev. A. **80**, 033610, 1–5 (2009)
12. Chabchoub, A, Hoffmann, N, Akhmediev, N: Rogue wave observation in a water wave tank. Phys. Rev. Let. **106**, 204502, 1–4 (2011)
13. Kibler, B, Fatome, J, Finot, C, Millot, G, Dias, F, Genty, G, Akhmediev, N, Dudley, JM: The Peregrine soliton in nonlinear fibre optics. Nat, Phys. **6**, 790–795 (2010)
14. Chabchoub, A, Hoffmann, H, Onorato, M, Akhmediev, N: Super rogue waves: observation of a higher-order breather in water waves. Phys. Review X. **2**, 1–6 (2012)
15. Dubard, P, Gaillard, P, Klein, C, Matveev, VB: On multi-rogue waves solutions of the NLS equation and positon solutions of the KdV equation. Eur. Phys. J. Spec. Topics **185**, 247–258 (2010)
16. Gaillard, P: Families of quasi-rational solutions of the NLS equation and multi-rogue waves. J. Phys. A : Meth. Theor. **44**, 1–15 (2011)
17. Guo, B, Ling, L, Liu, QP: Nonlinear Schrödinger equation: generalized Darboux transformation and rogue wave solutions. Phys. Rev. E. **85**, 026607 (2012)
18. Ohta, Y, Yang, J: General high-order rogue waves and their dynamics in the nonlinear Schrödinger equation. Proc. R. Soc. A. **468**(2142), 1716–1740 (2012)
19. Gaillard, P: Degenerate determinant representation of solution of the NLS equation, higher Peregrine breathers and multi-rogue waves. J. Math. Phys. **54**, 013504–1-32 (2013)
20. Kedziora, DJ, Ankiewicz, A, Akhmediev, N: Triangular rogue wave cascades. Phys. Rev. E. **86**, 056602, 1–9 (2012)
21. Gaillard, P: Wronskian representation of solutions of the NLS equation and higher Peregrine breathers. J. Math. Sci.: Adv. Appl. **13**(2), 71–153 (2012)
22. Kedziora, DJ, Ankiewicz, A, Akhmediev, N: Circular rogue wave clusters. Phys. Rev. E. **84**, 056611, 1–7 (2011)
23. Gaillard, P: Eighth Peregrine breather solution of the NLS equation and multi-rogue waves. hal-00664052 (2012)
24. Dubard, P, Matveev, VB: Multi-rogue waves solutions: from NLS equation to KP-I equation. RIMS 1777. 1–38, Kyoto University (2013)
25. Gaillard, P: Deformations of third order Peregrine breather solutions of the NLS equation with four parameters. hal-00783882 (2013)
26. Gaillard, P: Six-parameters deformations of fourth order Peregrine breather solutions of the NLS equation. J. Math. Phys. **54**, 073519-1-21 (2013)

doi:10.1186/2251-7235-7-45
Cite this article as: Gaillard: Two-parameter determinant representation of seventh order rogue wave solutions of the NLS equation. *Journal of Theoretical and Applied Physics* 2013 **7**:45.