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Best proximity points in probabilistic metric spaces and applications to nonlinear Fredholm integral equations

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Abstract. This paper presents best proximity points in probabilistic metric spaces, deriving new fixed point results. We also include some examples that showcase our findings and their relevance to nonlinear Fredholm integral equations.

Keywords: Probabilistic metric space, best proximity point, nonlinear Fredholm integral equations.

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1. Introduction

Fixed point (FP) theory has emerged as one of the most significant branches of mathematics in the past decade. It has opened up numerous opportunities for those interested in the field by introducing various types of metric spaces, including e-metric spaces, z-metric spaces, and rectangular metric spaces. A notable addition to this area is the probability metric space, which was first introduced by Menger [22] as a metric space of possibilities. Following the work of Menger, Segal [29] significantly advanced the field by establishing several FP theorems. For comprehensive information regarding probabilistic metric space (PMS), readers are directed to [3–5, 11–13, 15–17] and the references therein. Subsequently, the theoretical framework of FPs in PMSs, applicable to both single-valued and multivalued mappings, has been extensively investigated by numerous mathematicians ([15, 17, 19, 23, 25]). In their analysis, Schweizer and Sklar [28] examined the properties of Menger PMSs. In 2010, Jachymski [19] improved upon the probabilistic

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interpretation of the classical Banach contraction principle, which was initially presented by Ćirić [5] for nonlinear contractions. An overview of this research is also available in [17, Hadzic and Pap]. Essential definitions such as distribution functions, t-norms, H-type t-norms, and Menger PMSs can be found in [2, 5, 6, 16, 17] alongside related references. In 2011, Raj [24] presented the concept of the best proximity point (BPP). Also, he formulated and proved several theorems that are relevant to the study of weakly contractive non-self mappings. These contributions have significantly advanced our understanding of the behavior of such mappings within the broader context of FP theory.

Definition 1.1 [17] A Menger PMS is defined as a triple (Ξ, F, Ψ), where Ξ is a nonempty set, Ψ is a continuous t-norm, and F is a mapping from $\Xi \times \Xi$ to D^+ (The set of all Menger distance distribution functions is denoted by D^+). This structure must satisfy the following properties:

(PM1) $F_{\zeta,\varsigma}(\ell) = 1$ for all $\ell > 0$ iff $\zeta = \varsigma$;
(PM2) $F_{\zeta,\varsigma}(\ell) = F_{\varsigma,\zeta}(\ell)$ for all $\zeta, \varsigma \in \Xi$ and $\ell > 0$;
(PM3) $F_{\zeta,z}(\ell + \wp) \geqslant \Psi(F_{\zeta,\varsigma}(\ell), F_{\varsigma,z}(\wp))$ for all $\zeta, \varsigma, z \in \Xi$ and $\ell, \wp \geqslant 0$.

Let (Ξ, η) denote a metric space and $\mathcal{E}$ and $\mathcal{F}$ represent nonempty subsets of Ξ. Consider a non-self mapping $T : \mathcal{E} \to \mathcal{F}$ and define $\eta(\mathcal{E}, \mathcal{F}) = inf\{\eta(\zeta, \varsigma) : \zeta \in \mathcal{E}, \varsigma \in \mathcal{F}\}$. An element $\zeta \in \mathcal{E}$ is termed a BPP for the mapping T if it satisfies the condition $\eta(\zeta, T\zeta) = \eta(\mathcal{E}, \mathcal{F})$. Define

$$\mathcal{E}_0 = \{\zeta \in \mathcal{E} : \eta(\zeta, \varsigma) = \eta(\mathcal{E}, \mathcal{F}) \text{ for some } \varsigma \in \mathcal{F}\},$$
$$\mathcal{F}_0 = \{\varsigma \in \mathcal{F} : \eta(\zeta, \varsigma) = \eta(\mathcal{E}, \mathcal{F}) \text{ for some } \zeta \in \mathcal{E}\}.$$

If ζ is a BPP for T, then $\zeta \in \mathcal{E}_0$ and $T\zeta \in \mathcal{F}_0$.

The objective of the BPP theory is to establish sufficient conditions for the existence of such points. Consequently, numerous researchers have investigated various contractions to ensure the existence and uniqueness of BPPs in different metric and partially ordered metric spaces (see [1, 7–10, 14, 18, 20, 21, 26, 27]). Let $\mathcal{E}$ and $\mathcal{F}$ denote nonempty subsets of the metric space (Ξ, η), with $\mathcal{E}_0 \neq \emptyset$. We say the pair $(\mathcal{E}, \mathcal{F})$ possesses the P-property [24] if $\eta(\zeta_1, \varsigma_1) = \eta(\mathcal{E}, \mathcal{F})$ and $\eta(\zeta_2, \varsigma_2) = \eta(\mathcal{E}, \mathcal{F})$ imply that $\eta(\zeta_1, \zeta_2) = \eta(\varsigma_1, \varsigma_2)$.

This paper introduces the concept of BPPs in PMSs. In Section 2, we present the latest results on BPPs for self-maps in PMS. Section 3 establishes a coupled BPP result in this context. Finally, Section 4 applies these findings to demonstrate the existence of solutions for a system of integral equations and nonlinear Fredholm integrals.

2. New BPPs for self maps in PMSs

Let $\Upsilon = [0, 1]$, $\mho = [0, +\infty)$, $\hbar = C(\Upsilon, \mathbb{R})$ and $\top = (0, 1)$. This section delineates the latest advancements in BPPs for self-maps within PMSs. We now introduce the concept of a BPP for self-maps in PMSs.

Definition 2.1 Assume (Ξ, F, Ψ) is a PMS with nonempty subsets $\mathbb{A}$ and $\mathbb{B}$ of Ξ. Let $T : \mathbb{A} \to \mathbb{B}$ represents a non-self mapping, and $F_{\mathbb{A},\mathbb{B}}(\ell) = \sup\{F_{a,b}(\ell) : a \in \mathbb{A}, b \in \mathbb{B}\}$. An element $\zeta \in \Xi$ is termed a BPP for T if $F_{\zeta,T\zeta}(\ell) = F_{\mathbb{A},\mathbb{B}}(\ell)$. Let

$$\mathbb{A}_0 = \{a \in \mathbb{A} : F_{a,b}(\ell) = F_{\mathbb{A},\mathbb{B}}(\ell) \ for\ some\ b \in \mathbb{B}\},$$
$$\mathbb{B}_0 = \{b \in \mathbb{B} : F_{a,b}(\ell) = F_{\mathbb{A},\mathbb{B}}(\ell) \ for\ some\ a \in \mathbb{A}\}.$$

It is noteworthy that if ζ is a BPP for G, then $\zeta \in \mathbb{A}_0$ and $\mathcal{G}\zeta \in \mathbb{B}_0$.

Definition 2.2 Let (Ξ, F, Ψ) be a PMS, and $\mathbb{A}$ and $\mathbb{B}$ be two nonempty subsets of Ξ. We assert that the pair $(\mathbb{A}, \mathbb{B})$ possesses the PPM-property if $F_{\zeta_1,\varsigma_1}(\ell) = F_{\mathbb{A},\mathbb{B}}(\ell)$ and $F_{\zeta_2,\varsigma_2}(\ell) = F_{\mathbb{A},\mathbb{B}}(\ell)$ imply that $F_{\zeta_1,\zeta_2}(\ell) = F_{\varsigma_1,\varsigma_2}(\ell)$.

Theorem 2.3 Assume (Ξ, F, Ψ) is a PMS with nonempty subsets $\mathbb{A}$ and $\mathbb{B}$ of Ξ. Moreover, let $T : \mathbb{A} \to \mathbb{B}$ be a mapping such that $T(\mathbb{A}_0) \subset \mathbb{B}_0$, T be continuous and there exist $\zeta_0, \zeta_1 \in \mathbb{A}_0$ provided that $F_{\zeta_1,T\zeta_0}(\ell) = F_{\mathbb{A},\mathbb{B}}(\ell)$. Suppose that $(\mathbb{A}, \mathbb{B})$ have the PPM-property, Φ is all functions $\varphi : \mho \longrightarrow \mho$ which satisfy $\varphi(r) < r$ and $\lim\limits_{n\to\infty} \varphi^n(r) = 0$ for all $r > 0$ and

$$F_{T\zeta,T\varsigma}(\varphi(\ell)) \geqslant \min\{F_{\zeta,\varsigma}(\ell), F_{\zeta,T\zeta}(\ell), F_{\varsigma,T\varsigma}(\ell)\} \tag{1}$$

for all $\zeta, \varsigma \in \mathbb{A}$. Then T has a BPP in $\mathbb{A}$.

Proof. Given the assumption that there exist $\zeta_0, \zeta_1 \in \mathbb{A}_0$ such that $F_{\zeta_1,T\zeta_0}(\ell) = F_{\mathbb{A},\mathbb{B}}(\ell)$. From $\zeta_1 \in \mathbb{A}_0$ and $T(\mathbb{A}_0) \subset \mathbb{B}_0$, there exists $\zeta_2 \in \mathbb{A}$ such that $F_{\zeta_2,T\zeta_1}(\ell) = F_{\mathbb{A},\mathbb{B}}(\ell)$. In particular, $\zeta_2 \in \mathbb{A}_0$. By continuing this process, we construct a sequence $\{\zeta_n\}$ in $\mathbb{A}_0$ for all $n = 0, 1, ...$ so that $F_{\zeta_{n+1},T\zeta_n}(\ell) = F_{\mathbb{A},\mathbb{B}}(\ell)$. Since $(\mathbb{A}, \mathbb{B})$ has the PPM-property, we obtain $F_{\zeta_n,T\zeta_{n-1}}(\ell) = F_{\mathbb{A},\mathbb{B}}(\ell)$ and $F_{\zeta_{n+1},T\zeta_n}(\ell) = F_{\mathbb{A},\mathbb{B}}(\ell)$. Consequently, $F_{\zeta_n,\zeta_{n+1}}(\ell) = F_{T\zeta_{n-1},T\zeta_n}(\ell)$ for all $n \in \mathbb{N}$. Thus, the contractive condition leads us to the conclusion.

$$F_{\zeta_n,\zeta_{n+1}}(\varphi(\ell)) \geqslant \min\{F_{\zeta_n,\zeta_{n-1}}(\ell), F_{\zeta_{n-1},\zeta_n}(\ell), F_{\zeta_{n+1},\zeta_n}(\ell)\}.$$

Then $F_{\zeta_n,\zeta_{n+1}}(\varphi(\ell)) \geqslant F_{\zeta_{n-1},\zeta_n}(\ell)$ for all $\ell > 0$. Thus, we obtain

$$F_{\zeta_n,\zeta_{n+1}}\big(\varphi^2(\ell)\big) \geqslant F_{\zeta_{n-1},\zeta_n}(\varphi(\ell)) \geqslant F_{\zeta_{n-2},\zeta_{n-1}}(\ell)$$

for all $\ell > 0$. Consequently, $F_{\zeta_n,\zeta_{n+1}}\big(\varphi^n(\ell)\big) \geqslant F_{\zeta_1,\zeta_0}(\ell)$ for all $\ell > 0$. Since $\lim\limits_{\ell\longrightarrow\infty} F_{\zeta_1,\zeta_0}(\ell) = 1$, there exists ℓ_0 such that $F_{\zeta_1,\zeta_0}(\ell_0) > 1 - \xi$ for $\varrho > 0$ and $\xi \in \top$. Furthermore, since $\lim\limits_{n\to\infty} \varphi^n(\ell) = 0$, there exists a $N_0 \in \mathbb{N}$ such that $\varphi^n(\ell_0) < \varrho$ for $n > N_0$. Therefore, we derive

$$F_{\zeta_n,\zeta_{n+1}}(\varrho) \geqslant F_{\zeta_n,\zeta_{n+1}}(\varphi^n(\ell_0)) \geqslant F_{\zeta_1,\zeta_0}(\ell_0) > 1 - \xi$$

for $n > N_0$, which implies $\lim\limits_{n\to\infty} F_{\zeta_{n+1},\zeta_n}(\ell) = 1$. Next, for any $\varrho > 0$ and $\xi \in \top$, we prove there exists $N(\xi, \varrho)$ with $m > n > N(\varrho, \xi)$ so that $F_{\zeta_n,\zeta_m}(\varrho) > 1 - \xi$. To this end, we proceed by induction that

$$F_{\zeta_{n+k},\zeta_n}(\varrho) \geqslant \Psi^k\big(F_{\zeta_{n+1},\zeta_n}(\varrho - \varphi(\varrho))\big), \tag{2}$$

for all $k \geqslant 1$. Starting with $k = 1$, we have

$$\begin{aligned} F_{\zeta_n,\zeta_{n+1}}(\varrho) &\geqslant \big(F_{\zeta_{n+1},\zeta_n}(\varrho - \varphi(\varrho))\big) \\ &= \Psi\big(F_{\zeta_{n+1},\zeta_n}(\varrho - \varphi(\varrho)), 1\big) \\ &\geqslant \Psi\Big(F_{\zeta_{n+1},\zeta_n}(\varrho - \varphi(\varrho)), F_{\zeta_{n+1},\zeta_n}\big(\varrho - \varphi(\varrho)\big)\Big) \\ &= \Psi^1(F_{\zeta_{n+1},\zeta_n}(\varrho - \varphi(\varrho))). \end{aligned}$$

Therefore, (2) is valid for $k=1$. Now, let's assume (2) holds for $1 \leqslant k \leqslant p$. For $k=p+1$, we obtain

$$\digamma_{\zeta_{n+p+1},\zeta_n}(\varrho) \geqslant \Psi\Big(\digamma_{\zeta_{n+1},\zeta_n}(\varrho-\varphi(\varrho)), \digamma_{\zeta_{n+1},\zeta_{n+p+1}}(\varphi(\varrho))\Big). \tag{3}$$

According to (1), a standard proof by contradiction demonstrates that $\digamma_{\zeta_{n+1},\zeta_{n+2}}(\varrho) \geqslant \digamma_{\zeta_n,\zeta_{n+1}}(\varrho)$. Thus, $\digamma_{\zeta_{n+p+1},\zeta_{n+p}}(\varrho) \geqslant \digamma_{\zeta_n,\zeta_{n+1}}(\varrho)$. Then we obtain

$$\begin{aligned}
\digamma_{\zeta_{n+1},\zeta_{n+p+1}}\Big(\varphi(\varrho)\Big) &\geqslant \min\{\digamma_{\zeta_n,\zeta_{n+1}}(\varrho), \digamma_{\zeta_n,\zeta_{n+p}}(\varrho), \digamma_{\zeta_{n+p},\zeta_{n+p+1}}(\varrho)\} \\
&= \min\{\digamma_{\zeta_n,\zeta_{n+1}}(\varrho), \digamma_{\zeta_n,\zeta_{n+p}}(\varrho)\} \\
&\geqslant \min\Big\{\digamma_{\zeta_n,\zeta_{n+1}}(\varrho-\varphi(\varrho)), \Psi^p\big(\digamma_{\zeta_n,\zeta_{n+1}}(\varrho-\varphi(\varrho))\big)\Big\} \\
&= \Psi^p\Big(\digamma_{\zeta_n,\zeta_{n+1}}(\varrho-\varphi(\varrho))\Big).
\end{aligned} \tag{4}$$

From (3) and (4), we derive

$$\digamma_{\zeta_{n+p+1},\zeta_n}(\varrho) \geqslant \Psi\Big(\digamma_{\zeta_n,\zeta_{n+1}}(\varrho-\varphi(\varrho)), \Psi^p\big(\digamma_{\zeta_n,\zeta_{n+1}}(\varrho-\varphi(\varrho))\big)\Big) = \Psi^{p+1}\Big(\digamma_{\zeta_n,\zeta_{n+1}}(\varrho-\varphi(\varrho))\Big).$$

As a result, we retain (2). Since the t-norm Ψ is H-type, for a given $\xi \in \top$, there exists $\lambda \in \top$ such that for all integers $n \geqslant 1$, $\Psi^n(\ell) > 1-\xi$ when $\ell > 1-\lambda$. Conversely, according to $\lim_{n\to\infty} \digamma_{\zeta_n,\zeta_{n+1}}(\varrho-\varphi(\varrho)) = 1$, there exists a $N_1(\xi,\varrho)$ so that $\digamma_{\zeta_n,\zeta_{n+1}}(\varrho-\varphi(\varrho)) > 1-\lambda$ for all $n > N_1(\xi,\varrho)$. Consequently, we have

$$\digamma_{\zeta_{n+k},\zeta_n}(\varrho) \geqslant \Psi^k\big(\digamma_{\zeta_{n+1},\zeta_n}(\varrho-\varphi(\varrho))\big) \geqslant \Psi^k(1-\lambda) \geqslant 1-\xi$$

for all $k \geqslant 1$. This implies that $\{\zeta_n\}$ is a Cauchy sequence in $\mathbb{A}_0 \subset \mathbb{A}$. Since $\mathbb{A}$ is a closed subset of a complete PMS Ξ, there exists a $\zeta^* \in \mathbb{A}$ such that $\zeta_n \to \zeta^*$. Next, we will demonstrate that ζ^* is a BPP of T. By applying the continuity of T on $\mathbb{A}$, we find that $T\zeta_n$ converges to $T\zeta^*$. Additionally, by the joint continuity of $\digamma$, we can conclude that $\digamma_{\zeta_{n+1},T\zeta_n}(\ell) \to \digamma_{\zeta^*,T\zeta^*}(\ell)$. Furthermore, the sequence $\{\digamma_{\zeta_{n+1},T\zeta_n}(\ell)\}$ is constant and converges to $\digamma_{\mathbb{A},\mathbb{B}}(\ell)$. Since the limit of a sequence is unique, we have $\digamma_{\zeta^*,T\zeta^*}(\ell) = \digamma_{\mathbb{A},\mathbb{B}}(\ell)$. Thus, ζ^* is indeed a BPP of T. ■

***Example* 2.4** Let $\Xi = \{(1,0),(0,1),(-1,0),(0,-1)\}$ and consider η as the Euclidean metric. Thus, (Ξ,η) is a complete metric space. Let $\mathbb{A} = \{(1,0),(0,1)\}$ and $\mathbb{B} = \{(-1,0),(0,-1)\}$ be two subsets of Ξ. It can be noted that $\eta(\mathbb{A},\mathbb{B}) = \sqrt{2}$ with $\mathbb{A} = \mathbb{A}_0$ and $\mathbb{B} = \mathbb{B}_0$. Consider a function $T : \mathbb{A} \to \mathbb{B}$ defined by $T(\zeta,\varsigma) = (-\zeta,-\varsigma)$. It can be observed that T is continuous and $T(\mathbb{A}_0) \subset \mathbb{B}_0$. Next, let us define the mapping $\digamma : \Xi \times \Xi \to D^+$ by $\digamma_{\zeta,\varsigma}(\ell) = \chi(\ell - \eta(\zeta,\varsigma))$ for $\zeta,\varsigma \in \Xi$ and $\ell > 0$, where

$$\chi(\ell) = \begin{cases} 0 & if \quad \ell \leqslant 0 \\ 1 & if \quad \ell > 0 \end{cases}.$$

As a result, $(\Xi,\digamma,\Psi)$ with $\Psi(a,b) = \min\{a,b\}$ is a complete PMS. Furthermore, if we take $\varphi(\ell) = \ell$, it can be demonstrated that the other hypotheses of Theorem 2.3 are also satisfied. Therefore, T has a BPP.

Corollary 2.5 Consider $(\Xi, \digamma, \Psi)$ a PMS, a mapping continuous $T : \Xi \to \Xi$ with $T(\Xi) \subset \Xi$ and assume Φ is all functions $\varphi : \mho \longrightarrow \mho$ satisfying $\varphi(r) < r$ and $\lim_{n\to\infty} \varphi^n(r) = 0$ for all $r > 0$ and $\digamma_{T\zeta,T\varsigma}(\varphi(\ell)) \geqslant \min\{\digamma_{\zeta,\varsigma}(\ell), \digamma_{\zeta,T\zeta}(\ell), \digamma_{\varsigma,T\varsigma}(\ell)\}$ for all $\zeta, \varsigma \in \Xi$. Then T has an FP in Ξ.

3. New coupled BPP theorems in PMS

In this section, we present a new coupled best proximity point theorem in PMS.

Definition 3.1 Let $(\Xi, \digamma, \Psi)$ be a PMS, $\mathbb{A}$ and $\mathbb{B}$ be two nonempty subsets of Ξ, $T : \mathbb{A} \times \mathbb{A} \to \mathbb{B}$ be a mapping and $\digamma_{\mathbb{A},\mathbb{B}}(\ell) = \sup\{\digamma_{a,b}(\ell) : a \in \mathbb{A}, b \in \mathbb{B}\}$. An element $(\zeta, \varsigma) \in \Xi\times\Xi$ is named a coupled BPP for T if $\digamma_{\zeta,T(\zeta,\varsigma)}(\ell) = \digamma_{\mathbb{A},\mathbb{B}}(\ell)$ and $\digamma_{\varsigma,T(\varsigma,\zeta)}(\ell) = \digamma_{\mathbb{A},\mathbb{B}}(\ell)$.

Let

$$\mathbb{A}_0 = \{a \in \mathbb{A} : \digamma_{a,b}(\ell) = \digamma_{\mathbb{A},\mathbb{B}}(\ell) \text{ for some } b \in \mathbb{B}\},$$
$$\mathbb{B}_0 = \{b \in \mathbb{B} : \digamma_{a,b}(\ell) = \digamma_{\mathbb{A},\mathbb{B}}(\ell) \text{ for some } a \in \mathbb{A}\}.$$

Also, denote for simplicity $\Xi \times \cdots \times \Xi$ by Ξ^n, where Ξ is a non-empty set and $n \in \mathbb{N}$.

Lemma 3.2 Let $(\Xi, \digamma, \Psi)$ be a PMS. Then

1. $(\Xi^n, \triangle_\digamma, \Psi)$ is a PMS with

$$\triangle_{\digamma_{(\zeta_1,\cdots,\zeta_n),(\varsigma_1,\cdots,\varsigma_n)}}(\ell) = \min[\digamma_{\zeta_1,\varsigma_1}(\ell), \digamma_{\zeta_2,\varsigma_2}(\ell), \cdots, \digamma_{\zeta_n,\varsigma_n}(\ell)],$$

2. The mapping $f : \Xi^n \to \Xi$ has a n-tuple BPP iff $F : \Xi^n \to \Xi^n$ defined by

$$F(\zeta_1, \zeta_2, \cdots, \zeta_n) = (f(\zeta_1, \zeta_2, \cdots, \zeta_n), f(\zeta_2, \cdots, \zeta_n, \zeta_1), \cdots, f(\zeta_n, \zeta_1, \cdots, \zeta_{n-1}))$$

has a BPP in Ξ^n.

3. $(\Xi, \digamma, \Psi)$ is $\digamma$-complete iff $(\Xi^n, \triangle_\digamma, \Psi)$ is $\digamma$-complete.

Proof. 1. Clearly, $\triangle_\digamma$ satisfies in $(PM1)$ and $(PM2)$. We show that $\triangle_\digamma$ satisfies in $(PM3)$. For every $(\zeta_i), (\varsigma_i)\,(z_i) \subset \Xi$ for $1 \leqslant i \leqslant n$, suppose that

$$\min\{\digamma_{\zeta_1,\varsigma_1}(\ell + \wp), ..., \digamma_{\zeta_n,\varsigma_n}(\ell + \wp)\} = \digamma_{\zeta_i,\varsigma_i}(\ell + \wp).$$

Then we obtain

$$\begin{aligned}
\triangle_{\digamma_{(\zeta_1,\cdots,\zeta_n),(\varsigma_1,\cdots,\varsigma_n)}}(\ell + \wp) &= \min\{\digamma_{\zeta_1,\varsigma_1}(\ell + \wp), ..., \digamma_{\zeta_n,\varsigma_n}(\ell + \wp)\} \\
&= \digamma_{\zeta_i,\varsigma_i}(\ell + \wp) \geqslant \Psi(\digamma_{\zeta_i,z_i}(\ell), \digamma_{z_i,\varsigma_i}(\wp)) \\
&\geqslant \Psi(\min(\digamma_{\zeta_1,z_1}(\ell), ..., \digamma_{\zeta_n,z_n}(\ell)), \min(\digamma_{z_1,\varsigma_1}(\wp), ..., \digamma_{z_n,\varsigma_n}(\wp))) \\
&= \Psi(\triangle_{\digamma_{(\zeta_1,\cdots,\zeta_n),(z_1,\cdots,z_n)}}(\ell), \triangle_{\digamma_{(z_1,\cdots,z_n),(\varsigma_1,\cdots,\varsigma_n)}}(\wp)).
\end{aligned}$$

Therefore, $\triangle_\digamma$ satisfies in $(PM3)$. The proofs of statements 2 and 3 are straightforward and left to the reader. ■

It is important to note that Lemma 3.2 establishes a two-way relationship. Therefore, we can derive n-tuple FP results from FP theorems, and vice versa. Now, set $n = 2$ in

Lemma 3.2. This leads to the following theorem.

Theorem 3.3 Consider a PMS $(\Xi, \digamma, \Psi)$ with two nonempty subsets $\mathbb{A}$ and $\mathbb{B}$ of Ξ. Also, assume $T : \mathbb{A} \times \mathbb{A} \to \mathbb{B}$ is a mapping such that $T(\mathbb{A}_0 \times \mathbb{A}_0) \subset \mathbb{B}_0$, T is continuous, and there exist $\zeta_0, \zeta_1, \varsigma_1, \varsigma_2 \in \mathbb{A}_0$ so that $\digamma_{\zeta_1, T(\zeta_0,\varsigma_0)}(\ell) = \digamma_{\mathbb{A},\mathbb{B}}(\ell)$ and $\digamma_{\varsigma_1, T(\varsigma_0,\zeta_0)}(\ell) = \digamma_{\mathbb{A},\mathbb{B}}(\ell)$. Moreover, suppose that $(\mathbb{A}, \mathbb{B})$ have the PPM-property, Φ is all functions $\varphi : \mho \longrightarrow \mho$ satisfying $\varphi(r) < r$ and $\lim\limits_{n\to\infty} \varphi^n(r) = 0$ for all $r > 0$. Assume

$$\digamma_{T(\zeta,\varsigma),T(u,v)}(\varphi(\ell)) \geqslant \min(\digamma_{\zeta,u}(\ell), \digamma_{\varsigma,v}(\ell)) \tag{5}$$

for all $(\zeta, \varsigma), (u.v) \in \Xi^2$. Then T has a coupled BPP in $\Xi \times \Xi$.

Proof. Define $\triangle_\digamma : \Xi^2 \times \Xi^2 \to D^+$ by $\triangle_{\digamma_{(\zeta_1,\zeta_2),(\varsigma_1,\varsigma_2)}}(\ell) = \min[\digamma_{\zeta_1,\varsigma_1}(\ell), \digamma_{\zeta_2,\varsigma_2}(\ell)]$ and $F : \Xi^2 \to \Xi^2$ by $F(\zeta, \varsigma) = (T(\zeta, \varsigma), T(\varsigma, \zeta))$. By Lemma 3.2, $(\Xi^2, \triangle_\digamma, \Psi)$ is a complete PMS. Moreover, $(\zeta, \varsigma) \in \Xi^2$ is a coupled BPP of T iff it is a BPP of F. In contrast, from (5), we obtain

$$\begin{aligned}
\triangle_{\digamma_{F(\zeta,\varsigma),F(u,v)}}(\varphi(\ell)) &= \triangle_{\digamma_{(T(\zeta,\varsigma),T(\varsigma,\zeta)),(T(u,v),T(v,u))}}(\varphi(\ell)) \\
&= \min[\digamma_{T(\zeta,\varsigma),T(u,v)}(\varphi(\ell)), \digamma_{T(\varsigma,\zeta),T(v,u)}(\varphi(\ell))] \\
&= \digamma_{T(\zeta,\varsigma),T(u,v)}(\varphi(\ell)) \\
&\geqslant \min[\digamma_{\zeta,u}(\ell), \digamma_{\varsigma,v}(\ell)] \\
&= \triangle_{\digamma_{(\zeta,\varsigma),(u,v)}}(\ell).
\end{aligned}$$

Now, by Theorem 2.3, F has a BPP and by Lemma 3.2, T has a coupled BPP. ■

Corollary 3.4 Let $(\Xi, \digamma, \Psi)$ be a PMS and $T : \Xi \times \Xi \to \Xi$ be a mapping such that $T(\Xi^2) \subset \Xi$ and T is continuous. Also, suppose Φ is all functions $\varphi : \mho \longrightarrow \mho$ satisfying $\varphi(r) < r$ and $\lim\limits_{n\to\infty} \varphi^n(r) = 0$ for all $r > 0$, and $\digamma_{T(\zeta,\varsigma),T(u,v)}(\varphi(\ell)) \geqslant \min(\digamma_{\zeta,u}(\ell), \digamma_{\varsigma,v}(\ell))$ for all $(\zeta, \varsigma), (u.v) \in \Xi^2$. Then T has a coupled FP in $\Xi \times \Xi$.

4. Application

In mathematics and sciences like physics, chemistry, and biology, some problems are modeled with integral equations. Finding exact solutions can be challenging, so iterative methods are often used instead [30–32].

Consider $\hbar$ with $||\zeta||_\infty = \max_{\ell\in\Upsilon} |\zeta(\ell)|$ for $\zeta \in \hbar$. Alternatively, the Banach space $\hbar$ can be endowed with Bielecki norm $||\zeta||_{\mathbb{B}} = \max_{\ell\in\Upsilon}\{|\zeta(\ell)|e^{-\tau\ell}\}$ for all $\zeta \in \hbar$ and $\tau > 0$, and the induced metric $\eta_{\mathbb{B}}(\zeta, \varsigma) = ||\zeta - \varsigma||_{\mathbb{B}}$ for all $\zeta, \varsigma \in \hbar$. Define $\eta_{\mathbb{B}} : \Xi \times \Xi \to \mho$ by $\eta_{\mathbb{B}}(\zeta, \varsigma) = \max_{\ell\in\Upsilon}\{|\zeta(\ell) - \varsigma(\ell)|e^{-||\zeta||_{\mathbb{B}}\ell}\}$. Set $\digamma : \hbar \times \hbar \to D^+$ by $\digamma_{\zeta,\varsigma}(\ell) = \chi(\ell - \eta_{\mathbb{B}}(\zeta, \varsigma))$ for $\zeta, \varsigma \in \hbar$ and $\ell > 0$, where

$$\chi(\ell) = \begin{cases} 0 & if \quad \ell \leqslant 0 \\ 1 & if \quad \ell > 0 \end{cases} .$$

Then $(\hbar, \digamma, \Psi)$ with $\Psi(a, b) = \min\{a, b\}$ is a complete PMS. We study the following nonlinear Fredholm integral equation:

$$u(\ell) = \vartheta(\ell) + \int_0^1 K(\ell, \wp, u(\wp))d\wp, \tag{6}$$

where $\vartheta : \Upsilon \to \mho$ and $K : \Upsilon^2 \times \mho \to \mho$ are continuous. Take $T : \hbar \to \hbar$ by

$$T\zeta(\ell) = \vartheta(\ell) + \int_0^1 K(\ell, \wp, u(\wp))d\wp, \quad \zeta \in \hbar.$$

Theorem 4.1 Let $(\hbar, \digamma, \Psi)$ be a complete PMS with $\Psi(a, b) = \min\{a, b\}$ and $T : \hbar \to \hbar$ be a operator with $T\zeta(\ell) = \vartheta(\ell) + \int_0^1 K(\ell, \wp, u(\wp))d\wp$. Consider an operator $K \in C(\Upsilon^2 \times \mathbb{R}, \mathbb{R})$ such that

(i) K is continuous;
(ii) there exists $\tau > 0$ such that $|K(\ell, \imath, \zeta(\imath)) - K(\ell, \imath, \varsigma(\imath))| \leqslant \frac{1}{2}|\zeta(\imath) - \varsigma(\imath)|$ for all $\zeta, \varsigma \in \hbar$ and $\imath \in \Upsilon$.

Then, (6) has a solution in $\hbar$.

Proof. According to the definition of T, we have

$$\begin{aligned}
\eta_{\mathbb{B}}(T\zeta, T\varsigma) &= \sup_{\ell\in\Upsilon}\{|\int_0^\ell K(\ell, \imath, \zeta(\imath))d\imath - \int_0^\ell K(\ell, \imath, \varsigma(\imath))d\imath|\ e^{-\|\zeta\|_{\mathbb{B}}\ell}\} \\
&\leqslant \sup_{\ell\in\Upsilon}\{\int_0^\ell |K(\ell, \imath, \zeta(\imath)) - K(\ell, \imath, \varsigma(\imath))|e^{-\|\zeta\|_{\mathbb{B}}\ell}\ d\imath\} \\
&\leqslant \sup_{\ell\in\Upsilon}\{\int_0^\ell \frac{1}{2}|\zeta(\imath) - \varsigma(\imath)|e^{-\|\zeta\|_{\mathbb{B}}\ell}\ d\imath\} \\
&\leqslant (||\zeta - \varsigma||_{\mathbb{B}}) \sup_{\ell\in\Upsilon}\{\int_0^\ell \frac{1}{2}d\imath\} \\
&\leqslant \eta_{\mathbb{B}}(\zeta, \varsigma).
\end{aligned}$$

Let $\varphi(\ell) = \ell$. Then we obtain

$$\begin{aligned}
\digamma_{T\zeta,T\varsigma}(\ell) &= \xi(\ell - \eta_{\mathbb{B}}(T\zeta, T\varsigma)) \\
&\geqslant \xi(\ell - \eta_{\mathbb{B}}(\zeta, \varsigma)) \\
&= \digamma_{\zeta,\varsigma}(\ell) \\
&\geqslant \min\{\digamma_{\zeta,\varsigma}(\ell), \digamma_{\zeta,T\zeta}(\ell), \digamma_{\varsigma,T\varsigma}(\ell)\}.
\end{aligned}$$

Therefore, all conditions of Corollary 2.5 are satisfied and T has an FP; that is, there exists a solution of (6) in Ξ. ■

Now, we will apply our findings by examining the system of integral equations:

$$\begin{cases} \zeta(\ell) = \int_a^b \mathcal{G}(\ell, \wp)K(\wp, \zeta(\wp), \varsigma(\wp))d\wp, \\ \varsigma(\ell) = \int_a^b \mathcal{G}(\ell, \wp)K(\wp, \varsigma(\wp), \zeta(\wp))d\wp \end{cases} \tag{7}$$

for all $\ell \in \Upsilon$, where $b > a$, $\mathcal{G} \in C(\Upsilon \times \Upsilon, \mho)$ and $K \in C(\Upsilon \times \mathbb{R} \times \mathbb{R}, \mathbb{R})$.

Let $\hbar$ be the Banach space of all real continuous functions defined on Υ with the norm $||\zeta||_\infty = \max_{\ell\in\Upsilon} |\zeta(\ell)|$ for $\zeta \in \hbar$ and induced $b-$metric $\eta(\zeta, \varsigma) = ||\zeta - \varsigma||^2$ for all $\zeta, \varsigma \in \hbar$. Note that η is complete b-metric with $\wp = 2$. Set $F : \hbar \times \hbar \to D^+$ by $F_{\zeta,\varsigma}(\ell) = \chi(\ell - \eta(\zeta, \varsigma))$ for $\zeta, \varsigma \in \hbar$ and $\ell > 0$. Then $(\hbar, F, \Psi)$ with $\Psi(a, b) = \min\{a, b\}$ is a complete PMS.

Theorem 4.2 Let $(\hbar, F, \Psi)$ be a complete PMS. Consider an operator $T : \hbar \times \hbar \to \hbar$ defined by $f(\zeta, \varsigma)\ell = \int_a^b \mathcal{G}(\ell, \wp)K(\wp, \zeta(\wp), \varsigma(\wp))d\wp$, where $\mathcal{G} \in C(\Upsilon \times \Upsilon, \mho)$ and $K \in C(\Upsilon \times \mathbb{R} \times \mathbb{R}, \mathbb{R})$ are two operators satisfying the following conditions:

(i) $||K||_\infty = \max_{\wp\in\Upsilon,\, \zeta,\varsigma\in\hbar} |K(\wp, \zeta(\wp), \varsigma(\wp))| < \infty$;

(ii) for all $\zeta, \varsigma \in \hbar$ and all $\wp \in \Upsilon$, we obtain

$$||K(\wp, \zeta(\wp), \varsigma(\wp)) - K(\wp, u(\wp), v(\wp))|| \leqslant \max\{\|\zeta(\wp) - u(\wp)\|^2, \|\varsigma(\wp) - v(\wp)\|^2\};$$

(iii) $\max_{\ell\in\Upsilon} \int_a^b \mathcal{G}(\ell, \wp)d\wp < 1$.

Then, 7 has a solution in $\hbar \times \hbar$.

Proof. For all $\zeta, \varsigma \in \hbar$, we consider $\eta(\zeta, \varsigma) = \max_{\ell\in\Upsilon}(|\zeta(\ell) - \varsigma(\ell)|^2)$. As we mentioned above $(\hbar, F, \Psi)$ is a complete PMS. Therefore, for all $\zeta, \varsigma \in \hbar$, we have

$$\begin{aligned}
\eta(T(\zeta,\varsigma), T(u, v)) &\leqslant \max_{\ell\in\Upsilon} \int_a^b \mathcal{G}(\ell, \wp)|K(\wp, \zeta(\wp)\varsigma(\wp)) - K(\wp, u(\wp), v(\wp))|d\wp \\
&\leqslant \max_{\ell\in\Upsilon} \int_a^b \mathcal{G}(\ell, \wp) \max\{\|\zeta(\wp) - u(\wp)\|^2, \|\varsigma(\wp) - v(\wp)\|^2\}d\wp \\
&= \max\{\eta(\zeta, u), \eta(\varsigma, v)\} \max_{\ell\in\Upsilon} \int_a^b \mathcal{G}(\ell, \wp)d\wp \\
&\leqslant \max\{\eta(\zeta, u), \eta(\varsigma, v)\}
\end{aligned}$$

Let $\varphi(\ell) = \ell$. Thus, for any $\ell > 0$, we obtain

$$\begin{aligned}
F_{T(\zeta,\varsigma),T(u,v)}(\ell) &= \chi(\ell - \eta(T(\zeta, \varsigma), T(u, v)) \\
&\geqslant \chi(\ell - \max\{\|\zeta(\wp) - u(\wp)\|^2, \|\varsigma(\wp) - v(\wp)\|^2\}) \\
&= \min\{F_{\zeta,u}(\ell), F_{\varsigma,v}(\ell)\}
\end{aligned}$$

for all $\zeta, \varsigma \in \hbar$. By Corollary 3.4, when we use the function $\varphi(r) = r$ for all $r > 0$ along with the parameters $\zeta, \varsigma \in \hbar$ and $\ell > 0$, T has a coupled FP. This FP serves as the solution to the system of integral equations (7). ■

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