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On Lambert multipliers with Kato property between L^p-spaces

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Abstract. In this paper, we investigate the structure of the space of multipliers arising from the range of a composition operator C_φ, which is induced by the conditional expectation between two $L^p(\Sigma)$ spaces. After introducing the necessary preliminaries on measure spaces, L^p-spaces, and the conditional expectation operator, we define weighted conditional multipliers and establish conditions under which they generate bounded operators between different $L^p(\Sigma)$ and $L^q(\Sigma)$ spaces. Using fundamental properties of conditional expectation, we characterize the structural behavior of these multipliers and explore their relation to injectivity and surjectivity moduli of bounded linear operators. The results provide a precise framework for describing conditional multipliers and open new perspectives for the study of composition operators in functional analysis.

Keywords: Conditional expectation, multipliers, multiplication operators, composition operators.

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1. Introduction and preliminaries

In what follows, (X, Σ, μ) will be a complete $\sigma-$finite measure space. $\varphi : X \longrightarrow X$ will be a measurable transformation of X, namely, a mapping from X into itself with the properties that the measure $\mu \circ \varphi^{-1}$ is absolutely continuous with respect to μ, and $\mu \circ \varphi^{-1}$ is finite. We set $h = \frac{d\mu\circ\varphi^{-1}}{d\mu}$. By $\varphi^{-1}(\Sigma)$ we mean the relative completion of the $\sigma-$algebra generated by $\{\varphi^{-1}(F) : F \in \Sigma\}$. By $L^0(\Sigma)$, we denote the linear space of all $\Sigma-$measurable functions on X. For any complete $\sigma-$finite subalgebra $\mathcal{A} \subseteq \Sigma$ with $1 \leqslant p \leqslant \infty$, the L^p-space $L^p(X, \mathcal{A}, \mu|_{\mathcal{A}})$ is abbreviated by $L^p(\mathcal{A})$ and its norm is denoted by $\|.\|_p$. $L^p(\mathcal{A})$ is a Banach subspace of $L^p(\Sigma)$. The support of a measurable function f is defined by $\sigma(f) = \{x \in X : f(x) \neq 0\}$. Equalities and inequalities between measurable

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functions are interpreted in the almost everywhere sense, and equality between sets is interpreted up to a set of measure zero.

Recall that for each complete $\sigma-$finite subalgebra $\mathcal{A} \subseteq \Sigma$, there exists an operator $E = E^{\mathcal{A}}$, called the conditional expectation operator, defined on the set of all non-negative measurable functions f, or on each $f \in L^p(\Sigma)$; $1 \leqslant p \leqslant \infty$. This operator is uniquely determined by the following conditions:

(i) $E(f)$ is $\mathcal{A}$-measurable and integrable;
(ii) If F is any $\mathcal{A}$-measurable set for which $\int_F f d\mu$ exists, we have the functional relation $\int_F f d\mu = \int_F E(f) d\mu$.

The mapping E is a linear operator and, in particular, it is a contraction. When $p = 2$, it is the orthogonal projection of $L^2(\Sigma)$ onto $L^2(\mathcal{A})$ [4]. This operator will play a prominent role in this note, and we list here some of its useful properties:

- If f is an $\mathcal{A}$-measurable function, then $E(fg) = fE(g)$.
- $|E(f)|^p \leqslant E(|f|^p)$.
- $\|E(f)\|_p \leqslant \|f\|_p$.
- If $f \geqslant 0$ then $E(f) \geqslant 0$; if $f > 0$ then $E(f) > 0$.
- $E(|f|^2) = |E(f)|^2$ if and only if $f \in L^0(\mathcal{A})$.

For f and g in $L^0(\Sigma)$, we define $f \star g = fE(g \circ \varphi) + E(f) g \circ \varphi - E(f)E(g \circ \varphi)$.

Let $1 \leqslant p$ and $q \leqslant \infty$. A measurable function $w \in L^0(\Sigma)$ for which $w \star f \in L^q(\Sigma)$ for each $f \in L^p(\Sigma)$, is called weighted conditional multiplier. In other words, $w \in L^0(\Sigma)$ is weighted conditional multiplier if and only if the corresponding $\star$-multiplication operator $K_w^{\varphi} : L^p(\Sigma) \to L^q(\Sigma)$ defined by $K_w^{\varphi} f = w \star f$ is bounded. Note that if w is a $\mathcal{A}$-measurable function or $\mathcal{A} = \Sigma$, then w is a weighted conditional multiplier if and only if the multiplication linear operator $K_w^{\varphi} : L^p(\Sigma) \to L^q(\Sigma)$ is bounded [10].

In next section, weighted conditional multipliers acting between two different $L^p(\Sigma)$ spaces are characterized by using some properties of conditional expectation operator. The algebra of all bounded linear operators from Banach space X into a Banach space Y is denoted by $\mathcal{B}(X, Y)$. Let $T \in \mathcal{B}(X, Y)$. The injectivity and surjectivity modulus of T are defined by $j(T) = \inf \{\|T(x)\| : x \in X, \|x\| = 1\}$ and $\kappa(T) = \sup\{r \geqslant 0 : TU_X \supset r \cdot U_Y\}$, respectively, where U_X and U_Y denote the closed unit ball in X and Y. It is said to be that T is bounded below if $j(T) > 0$. By $\mathrm{R}^{\infty}(T)$ and $\mathrm{N}^{\infty}(T)$, we mean the linear subspaces of X, $\bigcap_{n=1}^{\infty} RanT^n$ and $\bigcup_{n=1}^{\infty} KerT^n$, respectively for $T \in \mathcal{B}(X)$ [8].

2. Some properties of Lambert multipliers

In this section, we bring some facts and definitions, which will be used later.

Definition 2.1 Let $T_u : L^2(\Sigma) \longrightarrow L^2(\Sigma)$. Define

$$W = \left\{ u \in L^0(\Sigma) : T_u \textit{ is bounded on } L^2(\Sigma) \right\}.$$

We already know one important property of function in W, namely that $E(|u|^2)$ is bounded. However, since $|E(u)|^2 \leqslant E|u|^2$, $u \in W$ implies that $E(u)$ is bounded. Therefore, if a function is both $\mathcal{A}-$measurable and in W, then it must be bounded. Our next Lemma states that the converse also holds.

Lemma 2.2 $W \bigcap L^0(\mathcal{A}) = L^{\infty}(\mathcal{A})$.

Proof. Let $s \in L^{\infty}(\mathcal{A})$. Since s is $\mathcal{A}-$measurable, then $T_s f = sf$ for $f \in L^2(\Sigma)$. Also, we know $L^2(\mathcal{A}) \subset L^2(\Sigma)$ and we get

$$\|T_s f\|_2^2 = \int_X |T_s f|^2 d\mu = \int_X |sf|^2 d\mu \leqslant \|s\|_\infty^2 \int_X |f|^2 d\mu = \|s\|_\infty^2 \|f\|_2^\infty$$

so that $T_s f \in L^2(\Sigma)$ for all f. Hence, $s \in W \bigcap L^0(\mathcal{A})$. The converse we proved in the remarks leading up to the lemma. ■

Theorem 2.3 T_u is normal if and only if $u \in L^\infty(\mathcal{A})$.

Proof. Assume T_u is normal. Then, for any $f \in L^2(\Sigma)$, $T_u T_u^* f = T_u^* T_u f$. Now, we have

$$T_u^* T_u f = E(u)E(\bar{u}f) \tag{1}$$

and

$$T_u^* T_u f = E(f)E(|u|^2) + E(u)E(\bar{u}f) - E(\bar{u})E(u)E(f). \tag{2}$$

Therefore, we conclude from (4) and (5) that $E(u)E(|u|^2) = E(f)|E(u)|^2$. The last equation holds for every L^2 function, so it must hold for any strictly positive $\mathcal{A}-$ measurable L^2 function s with $E(s)E(|u|^2) = E(s)|E(u)|^2$. Now, by letting $E(s) = s$, we have $sE(|u|^2) = s|E(u)|^2$. Since $s > 0$, $|E(u)|^2 = E(|u|^2)$. However, we saw that this is equivalent to u being $\mathcal{A}-$ measurable. By Lemma 2.2, $u \in L^\infty(\mathcal{A})$. Conversely, suppose that $u \in L^\infty(\mathcal{A})$. Then, for $f \in L^2(\Sigma)$, $T_u f = uf$ and $T_u^* f = \bar{u}f$. Therefore, $T_u T_u^* f = T_u(\bar{u}f) = u(\bar{u}f) = |u|^2 f$ and $T_u^* T_u f = T_u^*(uf) = \bar{u}(uf) = |u|^2 f$. Hence, $T_u T_u^* = T_u^* T_u$. Thus, T_u is normal. ■

Theorem 2.4 T_u is self-adjoint if and only if $u \in L^\infty(\mathcal{A})$ is real-valued.

Proof. Assume T_u is self-adjoint. Then, T_u is normal, and by Theorem 2.3, $u \in L^\infty(\mathcal{A})$. Therefore, we only show that u is real-valued. Let $f \in L^2(\Sigma)$. Then $T_u^* f = T_u f$ can be written by

$$E(\bar{u}f) + E(\bar{u})\,(f - E(f)) = uE(f) + fE(u) - E(u)E(f).$$

Since u is $\mathcal{A}-$ measurable, $\bar{u}f = uf$ which implies $(u - \bar{u})f = 0$. This last equality holds for any L^2 function. In particular, it holds for strictly positive $s \in L^2(\mathcal{A})$. Therefore, $u = \bar{u}$. Conversely, suppose $u \in L^\infty(\mathcal{A})$ and is real-valued. For $f \in L^2(\Sigma)$, $T_u^* f = uf = T_u f$. Hence, T_u is self-adjoint. ■

Theorem 2.5 Suppose $T_u : L^p(\Sigma) \to L^p(\Sigma)$ is bounded linear operator. Then T_u is bounded below linear operator if and only if $|E(u)| \geqslant \delta$ a.e. on $\sigma(E(u))$ for some $\delta > 0$.

Proof. Suppose $|E(u)| \geqslant \delta$ on $\sigma(E(u))$ for some $\delta > 0$ and f is an arbitrary element of $L^p(\Sigma)$. Since T_u is bounded linear operator, it follows from Theorem 2.1 of [5] that $E(|u|^p) \in L^\infty(\mathcal{A})$. Now, we have $\|uE(f)\|_p \leqslant \|E(|u|^p)\|_\infty^{\frac{1}{p}} \|f\|_p$ and $\|E(u)E(f)\|_p \leqslant$

$\|E(|u|^p)\|_\infty^{\frac{1}{p}}\|f\|_p$. Hence,

$$\begin{aligned}\|T_u f\|_p &= \|uE(f) + fE(u) - E(u)E(f)\|_p\\ &\geqslant \|fE(u)\|_p - \|uE(f)\|_p - \|E(u)E(f)\|_p\\ &\geqslant \left(\delta - 2\|E(|u|^p)\|_\infty^{\frac{1}{p}}\right)\|f\|_p.\end{aligned}$$

Thus,

$$\|T_u\| = \inf\left\{\frac{\|T_u f\|_p}{\|f\|_p} : f \in L^p(\Sigma),\ f \neq 0\right\} \geqslant \left(\delta - 2\|E(|u|^p)\|_\infty^{\frac{1}{p}}\right).$$

Here, if δ is chosen such that $\delta > 2\|E(|u|^p)\|_\infty^{\frac{1}{p}}$, then $j(T_u) > 0$; i.e., T_u is bounded below. The proof of "only if" part is followed as in [10]. Assume that T_u is bounded below. Then there exists a constant $k > 0$ such that

$$\|T_u f\|_{L^p(\sigma(E(u)),\Sigma,\mu)} \geqslant k\|f\|_{L^p(\sigma(E(u)),\Sigma,\mu)} \qquad (f \in L^p(\sigma(E(u)),\Sigma,\mu)).$$

Let $\delta = \frac{k}{2}$ and put $U = \{x \in X : |E(u)x| < \delta\}$. Suppose $\mu(U) > 0$. Since (X,Σ,μ) is $\sigma-$finite measure space, we can find a set $Q \in \Sigma$ such that $Q \subseteq U$ and $0 < \mu(Q) < \infty$. Then the characteristic function χ_Q lies in $L^p(\sigma(E(u)),\Sigma,\mu)$ and satisfies

$$\begin{aligned}\|T_u|_{\sigma(E(u))}\chi_Q\|^p_{L^p(\sigma(E(u)),\Sigma,\mu)} &= \int_{\sigma(E(u))} |uE(\chi_Q) + \chi_Q E(u) - E(u)E(\chi_Q)|^p d\mu\\ &\leqslant \int_{\sigma(E(u))} (|E(u)|\chi_Q + |E(u)|\chi_Q + |E(u)|\chi_Q)^p d\mu\\ &= \int_{\sigma(E(u))} (3|E(u)|)^p \chi_Q d\mu \leqslant (3\delta)^p \int_{\sigma(E(u))} \chi_Q d\mu\\ &= (3\delta)^p \|\chi_Q\|^p_{L^p(\sigma(E(u)),\Sigma,\mu)}.\end{aligned}$$

This is contrary to the choice of k. Therefore, $\mu(U) = 0$; i.e., $|E(u)| \geqslant \delta$ a.e. on $\sigma(E(u))$. ■

Remark 1 *By the following theorem, a bounded operator $T_u : L^p(\Sigma) \to L^p(\Sigma)$ is one-to-one and has closed range if and only if $|E(u)| \geqslant \delta$ on $\sigma(E(u))$ for some $\delta > 0$.*

Theorem 2.6 [9] An operator $T \in \mathcal{B}(X,Y)$ is bounded below if and only if it is one-to-one and has closed range. T is onto if and only if $\kappa(T) > 0$.

Theorem 2.7 [9] Let $T \in \mathcal{B}(X,Y)$. Then $j(T) = \kappa(T^*)$ and $\kappa(T) = j(T^*)$.

Definition 2.8 Let $u \in L^0(\Sigma)$ be conditionable. For $1 \leqslant p \leqslant \infty$, the mapping $R_u : L^p(\Sigma) \to L^p(\mathcal{A})$ defined by $R_u f = E(uf)$ is called a weighted conditional expectation operator with respect to $\mathcal{A}$ (or WCE operator) if uf is conditionable for every $f \in L^p(\Sigma)$ and $E(uf) \in L^p(\mathcal{A})$.

Theorem 2.9 [4, Theorem 2.1.2] Let $1 < p < \infty$ and $\frac{1}{p} + \frac{1}{q} = 1$. If $R_u : L^p(\Sigma) \to L^p(\mathcal{A})$ is a bounded WCE operator, then $E(|u|^q) \in L^\infty(\mathcal{A})$ and $\|R_u\| = \|E(|u|^q)\|_\infty^{\frac{1}{q}}$.

Theorem 2.10 Let $1 < p < \infty$ and $\frac{1}{p} + \frac{1}{q} = 1$. Suppose $T_u : L^p(\Sigma) \to L^p(\Sigma)$ and R_u are bounded linear operators. If $|E(u)| \geqslant \delta$ on $\sigma(E(u))$ for some $\delta > 0$, then $T_u^{*(n)}$ is one-to-one and has closed range for every $n \in \mathbb{N}$.

Proof. By [5, Proposition 3.1], $T_u^{*n} f = \left(\overline{E(u)}\right)^{n-1} \left\{ nE(\bar{u}f) + \overline{E(u)}(f - nE(f)) \right\}$. Let $f \in L^p(\Sigma)$ be an arbitrary element. Since $|E(u)| \geqslant \delta$ on $\sigma(E(u))$ for some $\delta > 0$, then $\|(E(\bar{u}))^n f\|_p \geqslant \delta^n \|f\|_p^p$. By Theorem 2.9,

$$\|(E(\bar{u}))^{n-1} E(\bar{u}f)\|_p \leqslant \|E(|u|^p)\|_\infty^{\frac{n-1}{p}} \|E(|u|^q)\|_\infty^{\frac{1}{p}} \|f\|_p$$

and

$$\|(E(\bar{u}))^{n-1} E(f)\|_p \leqslant \|E(|u|^p)\|_\infty^{\frac{n-1}{p}} \|f\|_p.$$

Hence,

$$\begin{aligned} \|T_u^{*n} f\|_p &\geqslant \|(E(\bar{u}))^n f\|_p - n\|(E(\bar{u}))^{n-1} E(\bar{u}f)\|_p - n\|(E(\bar{u}))^{n-1} E(f)\|_p \\ &\geqslant \left(\delta^n - n\|E(|u|^p)\|_\infty^{\frac{n-1}{p}} \left\{ \|E(|u|^q)\|_\infty^{\frac{1}{q}} + 1 \right\} \right) \|f\|_p. \end{aligned}$$

Thus,

$$j(T_u^{*n} f) \geqslant \left(\delta^n - n\|E(|u|^p)\|_\infty^{\frac{n-1}{p}} \left\{ \|E(|u|^q)\|_\infty^{\frac{1}{q}} + 1 \right\} \right).$$

If we assume that $\delta > \left(n\|E(|u|^p)\|_\infty^{\frac{n-1}{p}} \left\{ \|E(|u|^q)\|_\infty^{\frac{1}{q}} + 1 \right\} \right)^{\frac{1}{n}}$, then $j(T_u^{*n} f) > 0$. Consequently, by Theorem 2.6, T_u^{*n} is one-to-one and has closed range. ■

Corollary 2.11 T_u^n is onto if and only if $|E(u)| \geqslant \delta$ a.e. on $\sigma(E(u))$ for some $\delta > 0$.

Proof. By Theorems 2.6 and 2.7, it is trivial. ■

Theorem 2.12 [9] For $T \in \mathcal{B}(X)$ with closed range, the following conditions are equivalent:

(i) $Ker(T) \subset R^\infty(T)$;
(ii) $N^\infty(T) \subset Ran(T)$;
(iii) $N^\infty(T) \subset R^\infty(T)$;
(iv) $Ker(T) \subset \overline{R^\infty(T)}$.

Lemma 2.13 Let $1 < p < \infty$ and $T_u : L^p(\Sigma) \to L^p(\Sigma)$ be a bounded linear operator. Then $Ker(T_u) = L^p(X \setminus \sigma(E(u)))$.

Proof. For given $f \in L^p(\Sigma)$, let $f \in Ker(T_u)$. Then $T_u f = 0$; i.e., $uE(f) + fE(u) - E(u)E(f) = 0$. Taking the conditional E of both sides equation gives $E(u)E(f) = 0$. Thus,

$$\int_{\sigma(E(u))} f d\mu = \int_{\sigma(E(u))} E(f) d\mu = 0.$$

Consequently, $f \in L^p(X \setminus \sigma(E(u))$. Conversely, let $f \in L^p(X \setminus \sigma(E(u))$. Then $fE(u) = 0$ which implies

$$\int_{\sigma(E(u))} f d\mu = \int_{\sigma(E(u))} E(f) d\mu = 0.$$

Hence, we have $E(u)E(f) = 0$ and

$$|\int_{\sigma(u)} E(f)d\mu| \leqslant \int_{\sigma(u)} |E(f)|d\mu \leqslant \int_{\sigma(E(u))} |E(f)|d\mu = 0,$$

which yields $uE(f) = 0$. Then $T_u f = 0$; i.e., $f \in Ker(T_u)$. ■

Definition 2.14 Let $T \in \mathcal{B}(X)$. T is said to be Kato if $Ran(T)$ is closed and T satisfies any of the conditions of Theorem 2.12.

Theorem 2.15 Let $1 < p < \infty$ and $\frac{1}{p} + \frac{1}{q} = 1$. For bounded linear operators R_u and $T_u : L^p(\Sigma) \to L^p(\Sigma)$, T_u is Kato if $|E(u)| \geqslant \delta$ a.e. on $\sigma(E(u))$ for some $\delta > 0$.

Proof. By Corollary 2.11, $R^\infty(T_u^n) = L^p(X, \Sigma, \mu)$ and by Lemma 2.13, $Ker(T_u) = L^p(X \setminus \sigma(E(u)))$. Hence, by Theorem 2.12(i), T_u is Kato. ■

Theorem 2.16 [5] Let $1 \leqslant q < p < \infty$. If $T_u : L^p(\Sigma) \to L^q(\Sigma)$ is a bounded linear operator and $|E(u)| \geqslant \delta$ a.e. on $\sigma(E(u))$ for some $\delta > 0$, then T_u is one-to-one and has closed range.

Theorem 2.17 [5] Let $1 \leqslant p < q < \infty$. If $T_u : L^p(\Sigma) \to L^q(\Sigma)$ is a bounded linear operator and $|E(u)| \geqslant \delta$ a.e. on $\sigma(E(u))$ for some $\delta > 0$, then T_u is one-to-one and has closed range.

3. Characterization of Lambert composition multipliers

Let $1 \leqslant p, q \leqslant \infty$ and $u \in L^0(\Sigma)$. We define Lambert weighted composition operator $K_u^\varphi : L^p(\Sigma) \longrightarrow L^q(\Sigma)$ by $K_u^\varphi f = uE(f \circ \varphi) + (f \circ \varphi)E(u) - E(u)E(f \circ \varphi)$.

Theorem 3.1 Let $1 \leqslant p < \infty$, $\varphi^{-1}(\mathcal{A}) \subseteq \mathcal{A}$, $u \in L^0(\Sigma)$ and $h \in L^0(\mathcal{A})$. Suppose $E^{\mathcal{A}}E^{\varphi^{-1}\Sigma}$ is an orthogonal projection. Then K_u^φ is normal if and only if

(i) $h\Big\{E^{\mathcal{A}\cap\varphi^{-1}\Sigma}(|u^2|) + E^{\varphi^{-1}\Sigma}(uE^{\mathcal{A}}(\overline{u}) - E^{\mathcal{A}}(\overline{u})E^{\mathcal{A}}(u))\Big\} \circ \varphi^{-1} = uh \circ \varphi E^{\mathcal{A}\cap\varphi^{-1}\Sigma}(\overline{u})$,

(ii) $\mathcal{A} \cap \varphi^{-1}\Sigma \cap \sigma(u) = \Sigma \cap \sigma(u)$.

Proof. The method of proof is same in case weighted composition operators stated in [3]. First note that for every $f \in L^p(X, \Sigma, \mu)$, $K_u^\varphi f = T_u C_\varphi f$. Then $(K_u^\varphi)^* f = C_\varphi^* T_u^* f$. As we can see $C_\varphi^* f = h(E^{\varphi^{-1}\Sigma} f) \circ \varphi^{-1}$ [1, 8], hence

$$(K_u^\varphi)^* f = hE^{\varphi^{-1}\Sigma}\Big\{E^{\mathcal{A}}(\overline{u}f) + \overline{E^{\mathcal{A}}}(u)f - \overline{E^{\mathcal{A}}}(u)E^{\mathcal{A}}(f)\Big\} \circ \varphi^{-1}.$$

Since $E^{\mathcal{A}}E^{\varphi^{-1}\Sigma}$ is an orthogonal projection, it follows from [2, Corollary3] that $E^{\mathcal{A}}E^{\varphi^{-1}\Sigma} = E^{\mathcal{A}\cap\varphi^{-1}\Sigma}$. Consequently, a computation similar to the one carried above shows that

$$(K_u^\varphi)^* K_u^\varphi f = hE^{\varphi^{-1}\Sigma}\Big\{E^{\mathcal{A}}(|u^2|)E^{\mathcal{A}}(f\circ\varphi) + E^{\mathcal{A}}(u)E^{\mathcal{A}}(\overline{u}f\circ\varphi) + u\overline{E^{\mathcal{A}}}(u)E^{\mathcal{A}}(f\circ\varphi)$$
$$+ \overline{E^{\mathcal{A}}}(u)E^{\mathcal{A}}(u)f\circ\varphi - 3\overline{E^{\mathcal{A}}}(u)E^{\mathcal{A}}(u)E^{\mathcal{A}}(f\circ\varphi)\Big\}\circ\varphi^{-1} \tag{3}$$

and

$$K_u^\varphi(K_u^\varphi)^* f = \Big(u - E^{\mathcal{A}}(u)\Big)h\circ\varphi E^{\mathcal{A}\cap\varphi^{-1}\Sigma}\Big\{E^{\mathcal{A}}(\overline{u}f) + \overline{E^{\mathcal{A}}(u)}f - \overline{E^{\mathcal{A}}(u)}E^{\mathcal{A}}(f)\Big\}$$
$$+ h\circ\varphi E^{\mathcal{A}}(u)E^{\varphi^{-1}\Sigma}\Big\{E^{\mathcal{A}}(\overline{u}f) + \overline{E^{\mathcal{A}}(u)}f - \overline{E^{\mathcal{A}}(u)}E^{\mathcal{A}}(f)\Big\}. \tag{4}$$

Now, assume that K_u^φ is normal; i.e., $K_u^\varphi(K_u^\varphi)^* f = (K_u^\varphi)^* K_u^\varphi f$ for every $f \in L^2(\Sigma)$. This is equivalent to the equations (3) and (4) are equal. Therefore, for every $f \in L^2(\mathcal{A}\cap\varphi^{-1}\Sigma)$, the equality of (3) and (4) is equivalent to

$$h\Big\{E^{\mathcal{A}\cap\varphi^{-1}\Sigma}(|u^2|) + E^{\varphi^{-1}\Sigma}(uE^{\mathcal{A}}(\overline{u}) - E^{\mathcal{A}}(\overline{u})E^{\mathcal{A}}(u))\Big\}\circ\varphi^{-1}f$$
$$= \Big(u - E^{\mathcal{A}}(u)\Big)h\circ\varphi E^{\mathcal{A}\cap\varphi^{-1}\Sigma}(\overline{u})f + h\circ\varphi E^{\mathcal{A}}(u)E^{\mathcal{A}\cap\varphi^{-1}\Sigma}(\overline{u})f$$
$$= uh\circ\varphi E^{\mathcal{A}\cap\varphi^{-1}\Sigma}(\overline{u})f. \tag{5}$$

Now, we prove (ii). Since K_u^φ is normal, then $\overline{R(K_u^\varphi)}$ is a reducing subspace for K_u^φ. By (4) and $h\circ\varphi > 0$, we have

$$\overline{R(K_u^\varphi)} = \overline{R(K_u^\varphi(K_u^\varphi)^*)}$$
$$= \Big\{f \in L^2(X,\Sigma,\mu) : \sigma(f) \subseteq \sigma\Big((u - E^{\mathcal{A}}(u))E^{\mathcal{A}\cap\varphi^{-1}\Sigma}\{E^{\mathcal{A}}(\overline{u}f) + \overline{E^{\mathcal{A}}(u)}f$$
$$- \overline{E^{\mathcal{A}}(u)}E^{\mathcal{A}}(f)\} + E^{\mathcal{A}}(u)E^{\varphi^{-1}\Sigma}\{E^{\mathcal{A}}(\overline{u}f) + \overline{E^{\mathcal{A}}(u)}f - \overline{E^{\mathcal{A}}(u)}E^{\mathcal{A}}(f)\}\Big)\Big\} \tag{6}$$

Conversely, suppose (i) and (ii) hold. Since $h\circ\varphi > 0$, then

$$\sigma\Big(h\Big\{E^{\mathcal{A}\cap\varphi^{-1}\Sigma}(|u^2|) + E^{\varphi^{-1}\Sigma}(uE^{\mathcal{A}}(\overline{u}) - E^{\mathcal{A}}(\overline{u})E^{\mathcal{A}}(u))\Big\}\circ\varphi^{-1}\Big) = \sigma\Big(uE^{\mathcal{A}\cap\varphi^{-1}\Sigma}(\overline{u})\Big) = \sigma(u). \tag{7}$$

Let F be an arbitrary $\mathcal{A}\cap\varphi^{-1}\Sigma-$ measurable set with finite measure. Then

$$h\circ\varphi uE^{\mathcal{A}\cap\varphi^{-1}\Sigma}(\overline{u}\chi_F) = h\circ\varphi uE^{\mathcal{A}\cap\varphi^{-1}\Sigma}(\overline{u})\chi_F$$
$$= h\Big\{E^{\mathcal{A}\cap\varphi^{-1}\Sigma}(|u^2|) + E^{\varphi^{-1}\Sigma}(uE^{\mathcal{A}}(\overline{u}) - E^{\mathcal{A}}(\overline{u})E^{\mathcal{A}}(u))\Big\}\circ\varphi^{-1}\chi_F. \tag{8}$$

By (ii), (8) holds for any finite measure and $\Sigma-$measurable subset F of $\sigma(u)$. In addition, by (7), (8) holds for all $\Sigma-$measurable sets F of finite measure, whereby $(K_u^\varphi)^* K_u^\varphi\chi_F = K_u^\varphi(K_u^\varphi)^*\chi_F$ for all such F. Thus, K_u^φ is normal. ∎

4. Conclusion

In conclusion, the investigation of the spectral radius and the polar decomposition of conditional expectation operators and T_u operator can be regarded as one of the important and appealing problems for future research. A detailed study of these properties plays a significant role in understanding the spectral structure and operator-theoretic behavior of this class of operators. Moreover, the obtained results may pave the way for the development of new applications in functional analysis and operator theory, particularly in function spaces.

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