Transactions on Fuzzy Sets and Systems (TFSS)
URL: https://sanad.iau.ir/journal/tfss/
Online ISSN: 2821-0131
Vol.5, No.1, (2026), 155-181
DOI: https://doi.org/10.71602/tfss.2026.1217721

Article Type: Original Research Article

Closeness Centrality of Cycle and Corona Product Graphs: Application in Fuzzy Transportation Network

Shaoli Nandi, **Sukumar Mondal**, **Sambhu Charan Barman***

Abstract. Centrality measurement is a useful way to find the most important points (vertices) and connections (edges) in a network. Over time, many researchers have created different types of centrality measures to study and understand how networks work. Closeness centrality, in particular, is crucial for examining biological, social, and transportation networks. The closeness centrality of a node u of a graph is the multiplicative inverse of the sum of the distances from u to each other vertex. We define normalized closeness centrality $C_{NC}(u)$ of a vertex u as $C_{NC}(u) = \frac{n-1}{\sum_{x \in V} \mathfrak{d}(u,x)}$, where $n = |V|$. This centrality measurement is more receivable than degree centrality because it counts direct as well as indirect connections. In this paper, we present some new theoretical results for finding normalized closeness centrality of some corona product graphs like $P_n \odot P_m$, $P_n \odot K_m$, $C_n \odot K_m$, $P_n \odot C_m$, $C_n \odot C_m$, $C_n \odot P_m$, $P_n \odot S_m$, $S_n \odot K_m$, $K_n \odot P_m$ and $K_n \odot K_m$. We also correct the result established by Eballe et al. for finding the vertex closeness centrality of the cycle graph C_n. The corona graph has many applications, including in signed networks, biotechnology, chemistry, and small-world networks. We demonstrate a practical application of our proposed results for identifying influential nodes in small-world networks based on our results and using the corona product graph model. We also present the applications of our studied results in a transportation network by transferring our crisp result to fuzzy membership degree and using the corona product graph model.

AMS Subject Classification 2020: 68R10; 90B10; 05C76; 05C82; 03E72
Keywords and Phrases: Closeness centrality, Cycle graph, Corona product, Small-world network, Transportation network, Crisp set, Fuzzy set, Membership function.

1 Introduction

The closeness centrality measurement is well-known to us for finding the important vertex in a complex network. The high centrality of a vertex gives it a positive influence on the network. It is widely used to study different types of networks [1, 2, 3, 4, 5, 6] such as social, biological, and transportation networks, and it is also applied to identify potential leads in customer data and in bibliometric analysis. The closeness centrality finds the suitable location in the facility location problem and the influence of a brain region in the brain network on other brain regions. Closeness centrality measures the connection between a street and all other neighbouring streets in the road network and also measures their accessibility. Details about centrality measurement are found in [7, 8, 9].

For a graph $G(V,E)$, the *closeness centrality* of a node point/vertex u, denoted by $C_C(u)$, is the multiplicative inverse of the sum of the distances from u to all other nodes of G. The mathematical expression of $C_C(u)$ is defined by $C_C(u) = \frac{1}{\sum_{x \in V} \mathfrak{d}(u,x)}$, where $\mathfrak{d}(u,x)$ is the distance between the two vertices u and x. To

***Corresponding Author:** Sambhu Charan Barman, Email: barman.sambhu@gmail.com, ORCID: 0000-0002-6662-5172
Received: 12 September 2025; **Revised**: 16 November 2025; **Accepted**: 05 January 2026; **Available Online**: 19 February 2026; **Published Online**: 7 May 2026.

How to cite: Nandi S, Mondal S, Barman SC. Closeness centrality of cycle and Corona product graphs: Application in fuzzy transportation network. *Transactions on Fuzzy Sets and Systems.* 2026; 5(1): 155-181. DOI: https://doi.org/10.71602/tfss.2026.1217721

compare the closeness centrality of the vertices of various graphs with several orders, we define normalized closeness centrality. The normalized closeness centrality $C_{NC}(u)$ of u is $C_{NC}(u) = \frac{n-1}{\sum_{x\in V} \eth(u,x)}$ where $n = |V|$. It is more receivable than degree centrality because it counts direct as well as indirect connections. Its aim is to recognize suitable vertices in a network that can reach other vertices more quickly.

1.1 Review of related works

Different centrality measurements were introduced and developed by lots of researchers to find the important nodes or edges in graphs. Closeness centrality is one of them. In 1948, Bavelas [10] first introduced the concept of closeness centrality, and Sabidussi [11] first gave the definition of closeness centrality in 1966. Freeman [12] in 1978, delivered the mathematical expression of closeness centrality. After a few years, Newman [13] generalized the closeness centrality for weighted graphs using Dijkstras shortest paths algorithm. In 2001, Brandes presented a faster algorithm that takes $O(mn)$ time to measure the closeness centrality of any vertex in a network [14]. Okamoto et al. [15] formulated an algorithm to find the top-k vertices in a network according to the highest closeness centrality. Furthermore, Ni et al. [5] studied degree, closeness and betweenness centrality and found application of group centrality measurements to explore macro-disciplinary evolution diachronically. Park et al. [16] have developed an algorithm for calculating the closeness centrality of a workflow-supported social network. In 2013, Yen et al. [17] presented an efficient approach for updating closeness centrality in dynamic networks. In the same year, Kas et al. [18] introduced incremental closeness centrality for dynamically changing social networks. After that Cohen et al. [19] computed classic closeness centrality, at scale. Crescenzi et al. [20] proposed a greedy algorithm for calculating the increment of closeness centrality by adding new edges to it and applied it to real-world networks and synthetic graphs. Also, in 2018, Phuong-Hanh et al. [21] developed an efficient parallel algorithm for computing the closeness centrality in social networks. In 2019, Jin et al. [22] studied parallel computation of hierarchical closeness centrality and applications. Also, Hu et al. [23] presented closeness centrality measures in fuzzy enterprise technology innovation cooperation networks. Again, Shukla et al. [24] designed an efficient parallel algorithm to find closeness centrality in dynamic graphs. In 2021, Skibski [25] studied the closeness centrality via the condorcet principle. In 2021, Eballe et al. investigated the formulation of closeness centrality for certain classes of graphs and in 2023, extended this study to graph products [26, 27]. Nandi [28] determined the closeness centrality of the complete graph, wheel graph, and fan graph in 2022. In the same year, Evans and Chen [29] showed by the shortest path tree approximation method, the inverse of closeness centrality and the logarithm of degree centrality are linearly dependent. After that, Lpez-Rourich and Rodrguez-Prez [30] presented an efficient data transfer by evaluating closeness centrality for dynamic social complex network-inspired routing. In 2023, Liu et al. [31] explored closeness centrality on uncertain graphs. In 2024, a new concept of centrality measurement in fuzzy social networks and a centrality measure using Linguistic Z-graphs were introduced, followed by an exploration of centrality measures in Quantum graphs in 2025 [32, 33].

1.2 Motivation and objective

In many real-life networks such as social, communication, biological, and transportation systems, it is important to know which nodes play the most important roles. To find such nodes, researchers use different centrality measures. One of the most useful measures is closeness centrality, which shows how quickly a node can reach all other nodes in a network. A node with high closeness centrality can spread information faster or connect to others more easily.

Graphs are often used to represent such networks. To study large and complex networks, mathematicians use graph products, which combine two simple graphs to form a bigger one. Two important examples are cycle graphs and corona product graphs. Cycle graphs are used to represent circular connections, such as ring networks or closed routes in transportation. Corona product graphs are useful to model networks that have a main core connected to several smaller sub-networks, such as a central server connected to many local computers.

Although graph products have been widely investigated, only limited attention has been paid to the computation of closeness centrality for cycle graphs and corona product graphs. In particular, Eballe et al. proposed an incorrect formula for the closeness centrality of cycle graphs and later presented a formula for determining the closeness centrality of the corona product of two graphs G and H as follows.

Theorem 1.1. [27] *Let G and H be graphs of orders m and n, respectively, with G connected. Then the closeness centralities of the vertices $u \in G$ and (u, h) of the attached copies H are given by the following expressions:*

$$C_{G\odot H}(u) = \frac{mn+n-1)}{((n+1)\tau_G(u)+mn}$$

$$C_{G\odot H}(u) = \frac{mn+n-1)}{((n+1)\tau_G(u)-deg_H(h)+2mn+m-2}, \text{ where } \tau_G(u) = \sum_{v\in G} d(u,v).$$

In the above formula, Eballe et al. did not provide the formula to find the value of $\tau_G(u)$ for the general graph G. They only proposed the formula of $\tau_G(u)$ for the path graph $G = P_m$. For this reason, they proposed the formula to find the closeness centrality of $P_m \odot \bar{K}_2$ in their studied paper. To the best of our knowledge, there is no single, specific formula for finding the sum of distances from a particular vertex to other vertices in general graphs. So, the problem of finding the closeness centrality of the corona product of different graphs is still open.

Also, understanding closeness centrality in these graphs can help improve the design of communication systems, make transportation routes more efficient, and study how information or influence spreads in different types of networks.

Therefore, this work aims to find and analyze the closeness centrality of cycle and corona product graphs, and to show how these results can be used in real-world applications.

1.3 Result

At first, we correct the result of Eballe et al. for finding the vertex closeness centrality of the cycle graph C_n. After that, we present some theoretical results of the normalized closeness centrality of some corona product graphs. We explore the applications of our studied results in small-world and transportation networks.

1.4 Organization of the paper

Some useful symbols are presented in the next section. We correct the result for finding the vertex closeness centrality of the cycle graph C_n in Section 3. We provide some new theoretical results of normalized vertex closeness centrality of some corona product graphs in Section 4. In Section 5, we present the applications of our studied results in small-world and transportation networks. Section 6 presents the conclusion and future scope of our paper.

2 Symbols

$C_C(u)$	:	vertex closeness centrality of the node point u
$C_{NC}(u)$	:	normalized vertex closeness centrality of the node u
$\mathfrak{d}(u,x)$	:	shortest distance between the nodes u and x
P_n	:	path graph with n node points
C_n	:	cycle graph with n node points
S_n	:	star graph having n node points
K_n	:	complete graph with n node points

3 Rectification of the vertex closeness centrality formula for cycle graphs

Let C_n be a cycle graph and $\{u_\lambda : \lambda = 1, 2, \cdots, n\}$ be its set of nodes. The cycle graphs C_4 and C_5 are shown in Figure 1. In this section, we correct the result presented by Eballe et al. relating to the vertex closeness centrality of the cycle graph C_n. The term 'closeness centrality' in the paper by Eballe et al. is the same as the term 'normalized closeness centrality' in our paper.

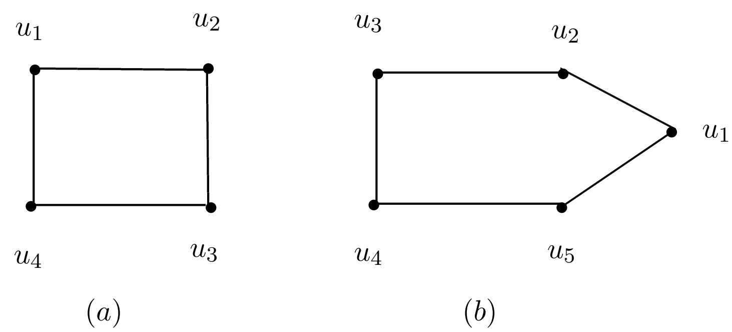

Figure 1: Cycle graph C_4 and C_5

.

We first recall the following result.

Theorem 3.1. [26] *The normalized closeness centrality of any node u of C_n is given by*

$$C_{NC}(u) = \begin{cases} \frac{4(n-1)}{(n+2)^2}, & \text{when } n \text{ is even} \\ \frac{4}{n+1}, & \text{when } n \text{ is odd.} \end{cases}$$

We have found that this result is not true for even values of n. For example, according to the above result, $C_{NC}(u) = \frac{4(4-1)}{(4+2)^2} = \frac{12}{36} = \frac{1}{3}\ \forall u \in C_4$.
But the actual value of $C_{NC}(u)$ is $\frac{3}{4}$ for all $u \in C_4$.

We correct this result below.

Theorem 3.2. *The normalized closeness centrality of any node u of the cycle graph C_n is*

$$C_{NC}(u) = \begin{cases} \frac{4(n-1)}{n^2}, & \text{if } n \text{ is even} \\ \frac{4}{n+1}, & \text{if } n \text{ is odd.} \end{cases}$$

Proof. We prove the above result for even values of n only because the result for odd values of n is already proven in the paper by Eballe et al. To prove this result for the even case, we first label any arbitrary vertex of C_n by u_n and then label the other nodes of C_n as $u_1, u_2, \cdots, u_{\frac{n}{2}-1}, u_{\frac{n}{2}}, u_{\frac{n}{2}+1}, \cdots, u_{n-1}$ in the clockwise direction from the node u_n. Now, if $u = u_1$, then $\sum_{x \in V(C_n)} \mathfrak{d}(u_1, x) = 2[\sum_{\lambda=2}^{\frac{n}{2}} \mathfrak{d}(u_1, u_\lambda)] + \mathfrak{d}(u_1, u_{\frac{n}{2}+1})$
$= 2[1 + 2 + \cdots + (\frac{n}{2} - 1)] + \frac{n}{2}$
$= 2[\frac{\frac{n}{2}(\frac{n}{2}-1)}{2}] + \frac{n}{2}$
$= \frac{n}{2}[(\frac{n}{2} - 1) + 1]$

$$= \frac{n}{2} \cdot \frac{n}{2}$$
$$= \frac{n^2}{4}.$$

Therefore, $C_{NC}(u_1) = \frac{n-1}{\sum_{x \in V} \mathfrak{d}(u_1, x)} = \frac{4(n-1)}{n^2}$.

This result is true for all nodes of C_n, when n is even. Therefore, $C_{NC}(u) = \frac{4(n-1)}{n^2}$, if n is even. □

4 Closeness centrality of some corona product graphs

Several binary graph operations are present in the field of graph theory. These operations always build a new graph. One of them is a corona product. $G_1 \odot G_2$ represents the corona product of two graphs G_1 and G_2. It is created by drawing one copy of G_1 having n_1 nodes and n_1 copies of G_2 having n_2 nodes, and connecting the λ^{th} node of the first graph by an edge to each node of the corresponding copy of the second graph, $\lambda = 1, 2, \cdots, n_1$. This newly formed graph is known as the corona graph of G_1 and G_2. $G_1 \odot G_2$ has $n_1 + n_1 n_2$ nodes and $m_1 + n_1 m_2 + n_1 n_2$ edges, where $|E(G_1)| = m_1$ and $|E(G_2)| = m_2$.

4.1 Closeness centrality of corona product graph $P_n \odot P_m$

The corona product graph $P_n \odot P_m$ is made using the corona product of the path graphs P_n and P_m. The graph $P_n \odot P_m$ has $n + nm$ nodes. Let the nodes of P_n be $u_1, u_2, ..., u_n$, and that of P_m corresponding to the node u_i be $u_{\lambda,1}, u_{\lambda,2}, ..., u_{\lambda,m}$. Figure 2 displays a corona graph $P_n \odot P_4$.

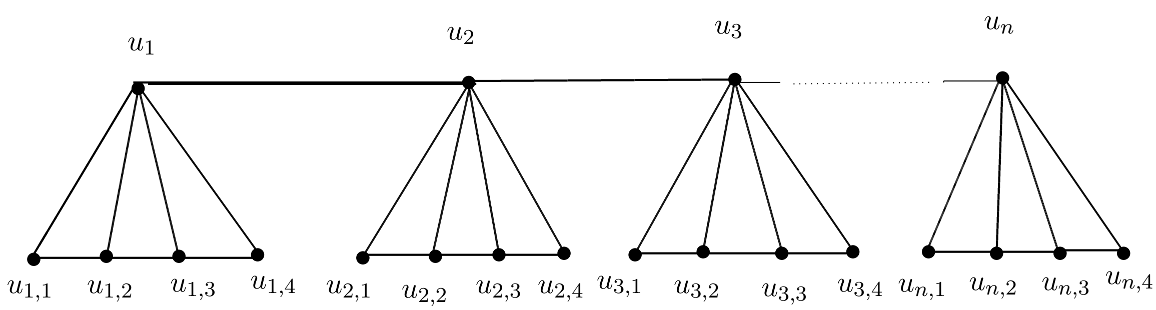

Figure 2: Corona graph $P_n \odot P_4$

.

Theorem 4.1. *The normalized closeness centrality of any node point v of $P_n \odot P_m$ is given by*

$$C_{NC}(v) = \begin{cases} \frac{2(nm+n-1)}{(n-\lambda+1)[(n-\lambda)+m(n-\lambda+2)]+\lambda[(\lambda-1)+m(\lambda+1)]-2m}, & \text{when } v = u_\lambda \in P_n \text{ and } \lambda = 1, 2, ..., n \\ \frac{2(nm+n-1)}{(n-\lambda+2)[(n-\lambda+1)+m(n-\lambda+3)]+(\lambda+1)[\lambda+m(\lambda+2)]-2(4m+5)}, & \text{when } v = u_{\lambda,\mu} \in P_m, \lambda = 1, 2, ..., n \text{ and } \mu = 1, m \\ \frac{2(nm+n-1)}{(n-\lambda+2)[(n-\lambda+1)+m(n-\lambda+3)]+(\lambda+1)[\lambda+m(\lambda+2)]-2(4m+5)}, & \text{if } v = u_{\lambda,\mu} \in P_m, \lambda = 1, 2, ..., n \text{ and } \mu = 2, 3, ..., m-1. \end{cases}$$

Proof. Consider that the node points of P_n are $u_1, u_2, ..., u_n$, and that of P_m corresponding to the node u_λ are $u_{\lambda,1}, u_{\lambda,2}, ..., u_{\lambda,m}$, $\lambda = 1, 2, ..., n$. If $v = u_\lambda$ be any vertex of P_n then $\sum_{x \in V} \mathfrak{d}(u_\lambda, x) = \sum_{l=\lambda+1}^{n} \mathfrak{d}(u_\lambda, u_l) +$

$\sum_{l=1}^{\lambda-1} \mathfrak{d}(u_\lambda, u_l) + \sum_{\mu=1}^{m} \sum_{l=\lambda}^{n} \mathfrak{d}(u_\lambda, u_{l,\mu}) + \sum_{\mu=1}^{m} \sum_{l=1}^{\lambda-1} \mathfrak{d}(u_\lambda, u_{l,\mu})$
$= [1+2+\cdots+(n-\lambda)] + [1+2+\cdots+(\lambda-1)] + [m+2m+\cdots+m(n-\lambda+1)] + [2m+3m+\cdots+m(\lambda-1+1)]$
$= \frac{(n-\lambda)(n-\lambda+1)}{2} + \frac{\lambda(\lambda-1)}{2} + m\frac{(n-\lambda+1)(n-\lambda+2)}{2} + [m+2m+3m+\cdots+\lambda m] - m$
$= \frac{(n-\lambda)(n-\lambda+1)}{2} + \frac{\lambda(\lambda-1)}{2} + m\frac{(n-\lambda+1)(n-\lambda+2)}{2} + m\frac{\lambda(\lambda+1)}{2} - m$
$= \frac{(n-\lambda+1)[(n-\lambda)+m(n-\lambda+2)]+\lambda[(\lambda-1)+m(\lambda+1)]-2m}{2}$.
Therefore,

$$C_{NC}(v) = \frac{nm+n-1}{\sum_{x\in V} \mathfrak{d}(u,x)} = \frac{2(nm+n-1)}{(n-\lambda+1)[(n-\lambda)+m(n-\lambda+2)]+\lambda[(\lambda-1)+m(\lambda+1)]-2m}.$$

If $v = u_{\lambda,\mu}$ is a vertex of P_m where $\lambda = 1, 2, \cdots, n$; $\mu = 1, m$ then $\sum_{x\in V} \mathfrak{d}(u_{\lambda,\mu}, x) = \sum_{l=\lambda}^{n} \mathfrak{d}(u_{\lambda,\mu}, u_l) + \sum_{l=1}^{\lambda-1} \mathfrak{d}(u_{\lambda,\mu}, u_l) + \sum_{k=1}^{m} \sum_{l=\lambda}^{n} \mathfrak{d}(u_{\lambda,\mu}, u_{l,k}) + \sum_{k=1}^{m} \sum_{l=1}^{\lambda-1} \mathfrak{d}(u_{\lambda,\mu}, u_{l,k})$
$= [1+2+\cdots+\{(n-\lambda)+1\}] + [2+3+\cdots+\{(\lambda-1)+1\}] + [\{1+2(m-2)\} + 3m + 4m + \cdots$
$+ m\{(n-\lambda)+2\}] + [3m+4m+\cdots+m\{(\lambda-1)+2\}]$
$= (2m-3) + \frac{(n-\lambda+1)(n-\lambda+2)}{2} + [1+2+3+\cdots+\lambda] - 1 + [m+2m+3m+4m+\cdots+m(n-\lambda+2)]$
$+ [m+2m+3m+4m+\cdots+m(\lambda+1)] - 3m - 3m$
$= (2m-3) + \frac{(n-\lambda+1)(n-\lambda+2)}{2} + \frac{\lambda(\lambda+1)}{2} + m\frac{(n-\lambda+2)(n-\lambda+3)}{2} + m\frac{(\lambda+1)(\lambda+2)}{2} - 6m - 1$
$= \frac{(n-\lambda+2)}{2}[(n-\lambda+1)+m(n-\lambda+3)] + \frac{(\lambda+1)[\lambda+m(\lambda+2)]}{2} - (4m+4)$
$= \frac{(n-\lambda+2)[(n-\lambda+1)+m(n-\lambda+3)]+(\lambda+1)[\lambda+m(\lambda+2)]-2(4m+4)}{2}$.
Hence,

$$C_{NC}(v) = \frac{nm+n-1}{\sum_{x\in V} \mathfrak{d}(u,x)} = \frac{2(nm+n-1)}{(n-\lambda+2)[(n-\lambda+1)+m(n-\lambda+3)]+(\lambda+1)[\lambda+m(\lambda+2)]-2(4m+4)}.$$

If $v = u_{\lambda,\mu}$ is any vertex of P_m where $\lambda = 1, 2, \cdots, n$; $\mu = 2, 3, \cdots, m-1$ then $\sum_{x\in V} \mathfrak{d}(u_{\lambda,\mu}, x) = \sum_{l=\lambda}^{n} \mathfrak{d}(u_{\lambda,\mu}, u_l) + \sum_{l=1}^{\lambda-1} \mathfrak{d}(u_{\lambda,\mu}, u_l) + \sum_{k=1}^{m} \sum_{l=\lambda}^{n} \mathfrak{d}(u_{\lambda,\mu}, u_{l,k}) + \sum_{k=1}^{m} \sum_{l=1}^{\lambda-1} \mathfrak{d}(u_{\lambda,\mu}, u_{l,k})$
$= [1+2+\cdots+(n-\lambda)+1] + [2+3+\cdots+(\lambda-1)+1] + [\{1+1+2(m-3)\} + 3m + 4m + \cdots$
$+ m\{(n-\lambda)+2\}] + [3m+4m+\cdots+m\{(\lambda-1)+2\}]$
$= (2m-4) + \frac{(n-\lambda+1)(n-\lambda+2)}{2} + [1+2+3+\cdots+\lambda] - 1 + [m+2m+3m+4m+\cdots+m(n-\lambda+2)]$
$+ [m+2m+3m+4m+\cdots+m(\lambda+1)] - 3m - 3m$
$= (2m-4) + \frac{(n-\lambda+1)(n-\lambda+2)}{2} + \frac{\lambda(\lambda+1)}{2} + m\frac{(n-\lambda+2)(n-\lambda+3)}{2} + m\frac{(\lambda+1)(\lambda+2)}{2} - 6m - 1$
$= \frac{(n-\lambda+2)}{2}[(n-\lambda+1)+m(n-\lambda+3)] + \frac{(\lambda+1)[\lambda+m(\lambda+2)]}{2} - (4m+5)$
$= \frac{(n-\lambda+2)[(n-\lambda+1)+m(n-\lambda+3)]+(\lambda+1)[\lambda+m(\lambda+2)]-2(4m+5)}{2}$.

Hence,

$$C_{NC}(v) = \frac{nm+n-1}{\sum_{x\in V} \mathfrak{d}(u,x)} = \frac{2(nm+n-1)}{(n-\lambda+2)[(n-\lambda+1)+m(n-\lambda+3)]+(\lambda+1)[\lambda+m(\lambda+2)]-2(4m+5)}.$$

□

4.2 Closeness centrality of $P_n \odot K_m$

The corona graph $P_n \odot K_m$ is made using the corona product of path graphs P_n and the complete graph K_m. The graph $P_n \odot K_m$ has $n + nm$ nodes. Let the nodes of P_n be $u_1, u_2, ..., u_n$, and that of K_m corresponding to the node u_λ be $u_{\lambda,1}, u_{\lambda,2}, ..., u_{\lambda,m}$. Figure 3 displays a corona graph $P_n \odot K_4$.

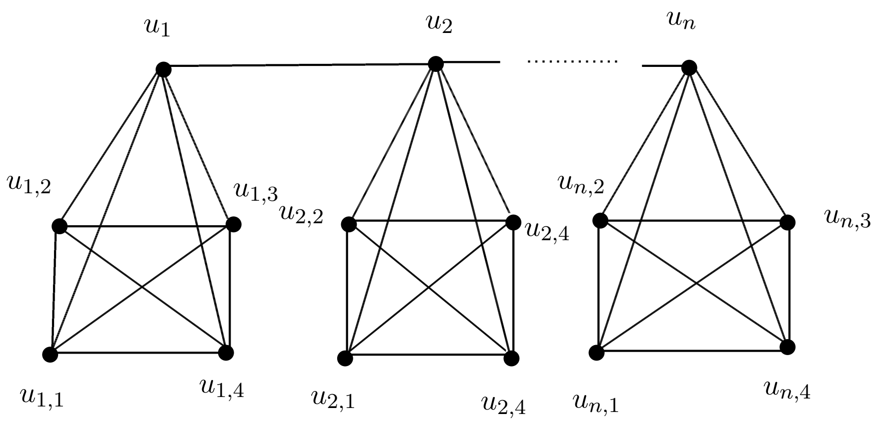

Figure 3: Corona graph $P_n \odot K_4$

.

Theorem 4.2. *The $C_{NC}(v)$ of any node point v of $P_n \odot K_m$ is*

$$C_{NC}(v) = \begin{cases} \frac{2(nm+n-1)}{(n-\lambda+2)[(n-\lambda+1)+m(n-\lambda+3)]+(\lambda+1)[\lambda+m(\lambda+2)]-2(5m+2)}, & \text{if } v = u_{\lambda,\mu} \in K_m, \lambda = 1,2, \cdots, n \text{ and } \mu = 1,2,\cdots,m \\ \frac{2(nm+n-1)}{(n-\lambda+1)[(n-\lambda)+m(n-\lambda+2)]+\lambda[(\lambda-1)+m(\lambda+1)]-2m}, & \text{if } v = u_\lambda \in P_n, \lambda = 1,2,\cdots,n. \end{cases}$$

Proof. Consider that the node points of P_n are $u_1, u_2, ..., u_n$ and that of K_m corresponding to the node u_λ are $u_{\lambda,1}, u_{\lambda,2}, ..., u_{\lambda,m}$, $\lambda = 1,2,...,n$. If $v = u_\lambda$ be any vertex of P_n then $\sum_{x\in V} \mathfrak{d}(u_\lambda, x) = \sum_{l=\lambda+1}^{n} \mathfrak{d}(u_\lambda, u_l) + \sum_{l=1}^{\lambda-1} \mathfrak{d}(u_\lambda, u_l) + \sum_{\mu=1}^{m}\sum_{l=\lambda}^{n} \mathfrak{d}(u_\lambda, u_{l,\mu}) + \sum_{\mu=1}^{m}\sum_{l=1}^{\lambda-1} \mathfrak{d}(u_\lambda, u_{l,\mu})$
$= [1+2+\cdots+(n-\lambda)] + [1+2+\cdots+(\lambda-1)] + [m+2m+\cdots+m(n-\lambda+1)] + [2m+3m+\cdots+m(\lambda-1+1)]$
$= \frac{(n-\lambda)(n-\lambda+1)}{2} + \frac{\lambda(\lambda-1)}{2} + m\frac{(n-\lambda+1)(n-\lambda+2)}{2} + [m+2m+3m+\cdots+\lambda m] - m$
$= \frac{(n-\lambda)(n-\lambda+1)}{2} + \frac{\lambda(\lambda-1)}{2} + m\frac{(n-\lambda+1)(n-\lambda+2)}{2} + m\frac{\lambda(\lambda+1)}{2} - m$
$= \frac{(n-\lambda+1)[(n-\lambda)+m(n-\lambda+2)]+\lambda[(\lambda-1)+m(\lambda+1)]-2m}{2}$.
Therefore,

$$C_{NC}(v) = \frac{nm+n-1}{\sum_{x\in V}\mathfrak{d}(u,x)} = \frac{2(nm+n-1)}{(n-\lambda+1)[(n-\lambda)+m(n-\lambda+2)]+\lambda[(\lambda-1)+m(\lambda+1)]-2m}$$

.
If $v = u_{\lambda,\mu}$ is any vertex of K_m where $\lambda = 1,2,\cdots,n$; $\mu = 1,2,\cdots,m$, then $\sum_{x\in V}\mathfrak{d}(u_{\lambda,\mu}, x) = \sum_{l=\lambda}^{n}\mathfrak{d}(u_{\lambda,\mu}, u_l) + \sum_{l=1}^{\lambda-1}\mathfrak{d}(u_{\lambda,\mu}, u_l) + \sum_{k=1}^{m}\sum_{l=\lambda}^{n}\mathfrak{d}(u_{\lambda,\mu}, u_{l,k}) + \sum_{k=1}^{m}\sum_{l=1}^{\lambda-1}\mathfrak{d}(u_{\lambda,\mu}, u_{l,k})$
$= [1+2+\cdots+\{(n-\lambda)+1\}] + [2+3+\cdots+\{(\lambda-1)+1\}] + [(m-1)+3m+4m+\cdots$
$+ m\{(n-\lambda)+2\}] + [3m+4m+\cdots+\{(\lambda-1)+2\}]$
$= (m-1) + \frac{(n-\lambda+1)(n-\lambda+2)}{2} + [1+2+3+\cdots+\lambda] - 1 + [m+2m+3m+4m+\cdots+m(n-\lambda+2)]$
$+ [m+2m+3m+4m+\cdots+m(\lambda+1)] - 3m - 3m$
$= (m-1) + \frac{(n-\lambda+1)(n-\lambda+2)}{2} + \frac{\lambda(\lambda+1)}{2} + m\frac{(n-\lambda+2)(n-\lambda+3)}{2} + m\frac{(\lambda+1)(\lambda+2)}{2} - 6m - 1$
$= \frac{(n-\lambda+2)}{2}[(n-\lambda+1)+m(n-\lambda+3)] + \frac{(\lambda+1)[\lambda+m(\lambda+2)]}{2} - (5m+2)$
$= \frac{(n-\lambda+2)[(n-\lambda+1)+m(n-\lambda+3)]+(\lambda+1)[\lambda+m(\lambda+2)]-2(5m+2)}{2}$.
Hence,

$$C_{NC}(v) = \frac{nm+n-1}{\sum_{x\in V}\mathfrak{d}(u,x)} = \frac{2(nm+n-1)}{(n-\lambda+2)[(n-\lambda+1)+m(n-\lambda+3)]+(\lambda+1)[\lambda+m(\lambda+2)]-2(5m+2)}.$$

□

4.3 Closeness centrality $C_n \odot K_m$

The graph $C_n \odot K_m$ is obtained by the corona product of the cycle graph C_n and the complete graph K_m. It has $n + nm$ nodes. Let the nodes of C_n be $u_1, u_2, ..., u_n$, and that of K_m corresponding to the node u_λ be $u_{\lambda,1}, u_{\lambda,2}, ..., u_{\lambda,m}$. A corona graph $C_3 \odot K_4$ is shown in Figure 4.

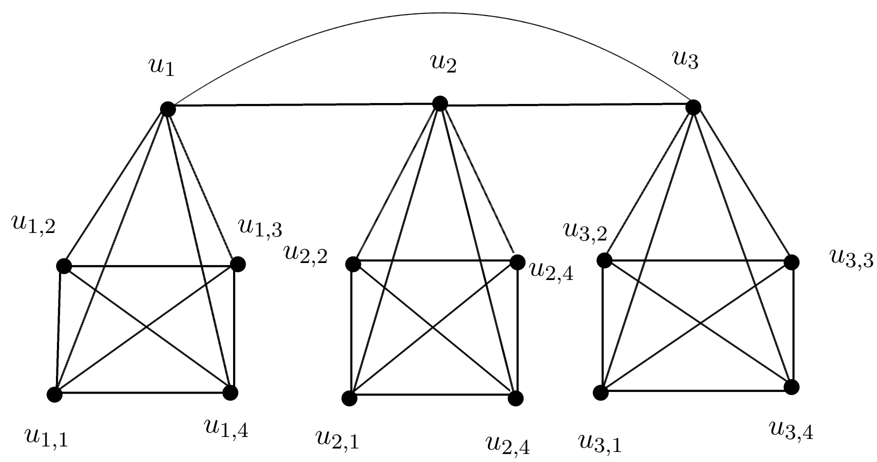

Figure 4: Corona graph $C_3 \odot K_4$

.

Theorem 4.3. *The $C_{NC}(v)$ of any node point v of $C_n \odot K_m$ is given by*

$$C_{NC}(v) = \begin{cases} \frac{4(nm+n-1)}{m(n+4)^2+(n+2)^2-4(5m+2)}, & \text{if } v = u_{\lambda,\mu} \in K_m, \lambda = 1,2,\cdots,n;\ \mu = 1,2,\cdots,m, \\ & \text{and } n \text{ is even} \\ \frac{4(nm+n-1)}{(n+3)[(n+1)+m(n+5)]-4(5m+2)}, & \text{if } v = u_{\lambda,\mu} \in K_m, \lambda = 1,2,\cdots,n;\ \mu = 1,2,\cdots,m, \\ & \text{and } n \text{ is odd} \\ \frac{4(nm+n-1)}{n^2+mn(n+4)}, & \text{if } v = u_\lambda \in C_n, \lambda = 1,2,\cdots,n \text{ and } n \text{ is even} \\ \frac{4(nm+n-1)}{m(n^2+4n-1)+n^2-1}, & \text{if } v = u_\lambda \in C_n, \lambda = 1,2,\cdots,n \text{ and } n \text{ is odd.} \end{cases}$$

Proof. Suppose that the node points of C_n are $u_1, u_2, ..., u_n$, and that of K_m corresponding to the node u_λ are $u_{\lambda,1}, u_{\lambda,2}, ..., u_{\lambda,m}$, $\lambda = 1, 2, ..., n$. If $v = u_\lambda$ is any vertex of C_n and n is even then $\sum_{x\in V} \mathfrak{d}(u_\lambda, x) = \sum_{\lambda=1,k\neq\lambda}^{n} \mathfrak{d}(u_\lambda, u_k) + \sum_{k=1}^{n}\sum_{\mu=1}^{m} \mathfrak{d}(u_\lambda, u_{k,\mu})$.

We know from Theorem 3.1, $\sum_{\lambda=1,k\neq\lambda}^{n} \mathfrak{d}(u_\lambda, u_k) = \frac{n^2}{4}$.

Now, $\sum_{k=1}^{n}\sum_{\mu=1}^{m} \mathfrak{d}(u_n, u_{k,\mu}) = \sum_{\mu=1}^{m} \mathfrak{d}(u_n, u_{n,\mu}) + \sum_{\mu=1}^{m} \mathfrak{d}(u_n, u_{\frac{n}{2},\mu}) + 2[\sum_{\mu=1}^{m} \mathfrak{d}(u_n, u_{1,\mu})$
$+ \sum_{\mu=1}^{m} \mathfrak{d}(u_n, u_{2,\mu}) + \cdots + \sum_{\mu=1}^{m} \mathfrak{d}(u_n, u_{\frac{n}{2}-1,\mu})]$
$= m + m(\frac{n}{2} + 1) + 2(2m + 3m + \cdots + \frac{n}{2} \cdot m)$
$= \frac{mn}{2} + 2m[1 + 2 + 3 + \cdots + \frac{n}{2}]$
$= \frac{mn}{2} + 2m[\frac{\frac{n}{2}(\frac{n}{2}+1)}{2}]$

$= \frac{mn}{2} + \frac{mn(n+2)}{4}$
$= \frac{mn}{2}(1 + \frac{n+2}{2})$
$= \frac{mn(n+4)}{4}$.
Therefore, $\sum_{k=1}^{n}\sum_{\mu=1}^{m} \mathfrak{d}(u_n, u_{k,\mu}) = \frac{mn(n+4)}{4}$.
This result is true for all u_λ, $\lambda = 1, 2, \cdots, n$.
Therefore, $\sum_{k=1}^{n}\sum_{\mu=1}^{m} \mathfrak{d}(u_\lambda, u_{k,\mu}) = \frac{mn(n+4)}{4}$.
Hence, $\sum_{x\in V} \mathfrak{d}(u_\lambda, x) = \frac{n^2}{4} + \frac{mn(n+4)}{4} = \frac{n^2+mn(n+4)}{4}$.
Therefore, the normalized closeness centrality of v is $C_{NC}(v) = \frac{nm+n-1}{\sum_{x\in V} \mathfrak{d}(u_\lambda,x)} = \frac{4(nm+n-1)}{n^2+mn(n+4)}$.

If $v = u_\lambda$ is an arbitrary node of C_n, and n is odd, then
$\sum_{x\in V} \mathfrak{d}(u_\lambda, x) = \sum_{\lambda=1,k\neq\lambda}^{n} \mathfrak{d}(u_\lambda, u_k) + \sum_{k=1}^{n}\sum_{\mu=1}^{m} \mathfrak{d}(u_\lambda, u_{k,\mu})$.
Again, we know from Theorem 3.1, $\sum_{\lambda=1,k\neq\lambda}^{n} \mathfrak{d}(u_\lambda, u_k) = \frac{(n-1)(n+1)}{4}$.
Again, $\sum_{k=1}^{n}\sum_{\mu=1}^{m} \mathfrak{d}(u_n, u_{k,\mu}) = \sum_{\mu=1}^{m} \mathfrak{d}(u_n, u_{n,\mu}) + 2[\sum_{l=1}^{\frac{n-1}{2}}\sum_{\mu=1}^{m} \mathfrak{d}(u_n, u_{l,\mu})]$
$= m + 2(2m + 3m + \cdots + \frac{n+1}{2} \cdot m)$
$= 2m[1 + 2 + 3 + \cdots + \frac{n+1}{2}] - m$
$= 2m[\frac{\frac{n+1}{2}(\frac{n+1}{2}+1)}{2}] - m$
$= \frac{m(n+1)(n+3)}{4} - m$
$= \frac{m(n^2+4n+3)-4m}{4}$
$= \frac{m(n^2+4n-1)}{4}$.
This result is true for all u_λ, $\lambda = 1, 2, \cdots, n$.
Therefore, $\sum_{k=1}^{n}\sum_{\mu=1}^{m} \mathfrak{d}(u_\lambda, u_{k,\mu}) = \frac{m(n^2+4n-1)}{4}$.

Hence, $\sum_{x\in V} \mathfrak{d}(u_\lambda, x) = \frac{(n-1)(n+1)}{4} + \frac{m(n^2+4n-1)}{4} = \frac{m(n^2+4n-1)+n^2-1}{4}$.
Therefore, the normalized closeness centrality of v is $C_{NC}(v) = \frac{nm+n-1}{\sum_{x\in V} \mathfrak{d}(u_\lambda,x)} = \frac{4(nm+n-1)}{m(n^2+4n-1)+n^2-1}$.

If $v = u_{\lambda,\mu}$ is any vertex of K_m adjacent to u_λ where $\lambda = 1, 2, \cdots, n$; $\mu = 1, 2, \cdots, m$ and n is even then $\sum_{x\in V} \mathfrak{d}(u_{\lambda,\mu}, x) = \sum_{l=1}^{n} \mathfrak{d}(u_{\lambda,\mu}, u_l) + \sum_{k=1}^{m} \mathfrak{d}(u_{\lambda,\mu}, u_{\lambda,k}) + \sum_{l=1,\lambda\neq l}^{n}\sum_{k=1}^{m} \mathfrak{d}(u_{\lambda,\mu}, u_{l,k})$. As n is even, so from $u_{\lambda,\mu}$, $2m$ vertices of K_m are at a distance 3, $2m$ vertices of K_m are at a distance $4, \cdots, 2m$ vertices of K_m are at a distance $\frac{n}{2}+1$ and m vertices of K_m are at a distance $\frac{n}{2}+2$ and $m-1$ vertices of K_m are at a distance 1.
Now, $\sum_{x\in V} \mathfrak{d}(u_{\lambda,\mu}, x) = [1+2+2+3+3\cdots+\frac{n}{2}+\frac{n}{2}+(\frac{n}{2}+1)]+(m-1)+[2m\cdot 3+2m\cdot 4+\cdots+2m\cdot(\frac{n}{2}+1)+m(\frac{n}{2}+2)]$
$= [1 + 2 + 3 + \cdots + \frac{n}{2} + (\frac{n}{2}+1)] + [2 + 3 + \cdots + \frac{n}{2}] + (m-1) + [3m + 4m + \cdots + m\{\frac{n}{2}+1\} + m\{\frac{n}{2}+2\}] + [3m + 4m + \cdots + m\{\frac{n}{2}+1\}]$
$= (m-1) + [1 + 2 + \cdots + (\frac{n}{2}+1)] + [1 + 2 + 3 + \cdots + \frac{n}{2}] - 1 + [m + 2m + 3m + 4m + \cdots + m\{\frac{n}{2}+1\} + m\{\frac{n}{2}+2\}] - 3m + [m + 2m + 3m + 4m + \cdots + m\{\frac{n}{2}+1\}] - 3m$
$= (m-1) + \frac{(\frac{n}{2}+1)(\frac{n}{2}+2)}{2} + \frac{\frac{n}{2}(\frac{n}{2}+1)}{2} - 1 + m\frac{(\frac{n}{2}+2)(\frac{n}{2}+3)}{2} + m\frac{(\frac{n}{2}+1)(\frac{n}{2}+2)}{2} - 6m$
$= \frac{(n+2)(n+4)}{4} + \frac{n(n+2)}{4} + \frac{m(n+4)(n+6)}{4} + \frac{m(n+2)(n+4)}{4} - (6m + 1 - m + 1)$
$= \frac{(n+2)(n+4+n)}{4} + \frac{m(n+4)(n+6+n+2)}{4} - (5m+2)$
$= \frac{m(n+4)^2+(n+2)^2}{4} - (5m+2)$
$= \frac{m(n+4)^2+(n+2)^2-4(5m+2)}{4}$.

Hence, the normalized closeness centrality of v is $C_{NC}(v) = \frac{nm+n-1}{\sum_{x\in V} \mathfrak{d}(u,x)} = \frac{4(nm+n-1)}{m(n+4)^2+(n+2)^2-4(5m+2)}$.

If $v = u_{\lambda,\mu}$ is any vertex of K_m where $\lambda = 1, 2, \cdots, n$; $\mu = 1, 2, \cdots, m$ and n is odd then
$\sum_{x\in V} \mathfrak{d}(u_{\lambda,\mu}, x) = \sum_{l=1}^{n} \mathfrak{d}(u_{\lambda,\mu}, u_l) + \sum_{k=1}^{m} \mathfrak{d}(u_{\lambda,\mu}, u_{\lambda,k}) + \sum_{l=1,\lambda\neq l}^{n}\sum_{k=1}^{m} \mathfrak{d}(u_{\lambda,\mu}, u_{l,k})$. As n is odd, so from

$u_{\lambda,\mu}$, $2m$ vertices of K_m are at a distance 3, $2m$ vertices of K_m are at a distance $4, \cdots, 2m$ vertices of K_m are at a distance $\frac{n+3}{2}$ and $m-1$ vertices of K_m are at a distance 1.
Now, $\sum_{x\in V} \mathfrak{d}(u_{\lambda,\mu}, x) = [1+2+2+3+3\cdots+\frac{n+1}{2}+\frac{n+1}{2}] + (m-1) + [2m\cdot 3 + 2m\cdot 4 + \cdots + 2m\cdot(\frac{n+1}{2}+1)]$
$= (m-1) + 2\{1+2+\cdots+\frac{n+1}{2}\} - 1 + 2\{m+2m+3m+4m+\cdots+m(\frac{n+1}{2}+1)\} - 6m$
$= 2\cdot\frac{\frac{n+1}{2}(\frac{n+1}{2}+1)}{2} + 2m\cdot\frac{(\frac{n+1}{2}+1)(\frac{n+1}{2}+2)}{2} + m - 1 - 6m - 1$
$= \frac{n+1}{2}(\frac{n+1}{2}+1) + m(\frac{n+1}{2}+1)(\frac{n+1}{2}+2) - (5m+2)$
$= \frac{(n+1)(n+3)}{4} + m\frac{(n+3)(n+5)}{4} - (5m+2)$
$= \frac{(n+3)[(n+1)+m(n+5)]-4(5m+2)}{4}$.

Therefore, $C_{NC}(v) = \frac{nm+n-1}{\sum_{x\in V}\mathfrak{d}(u,x)} = \frac{4(nm+n-1)}{(n+3)[(n+1)+m(n+5)]-4(5m+2)}$. □

4.4 Closeness centrality of corona product graph $P_n \odot C_m$

The graph $P_n \odot C_m$ is obtained by the corona product of the path graphs P_n and n copies of the cycle graph C_m. The graph $P_n \odot C_m$ has $n + nm$ nodes. Let the nodes of P_n be $u_1, u_2, ..., u_n$, and that of C_m corresponding to the node u_λ be $u_{\lambda,1}, u_{\lambda,2}, ..., u_{\lambda,m}$. Figure 5 shows a corona graph $P_n \odot C_4$.

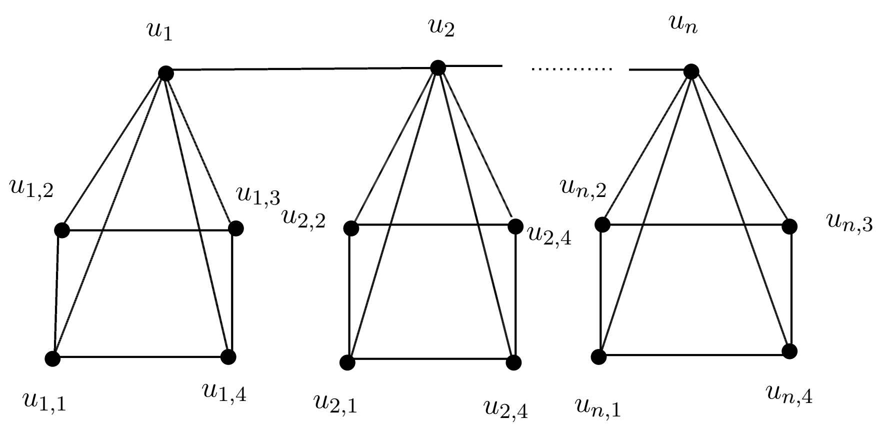

Figure 5: Corona graph $P_n \odot C_4$

.

Theorem 4.4. *The $C_{NC}(v)$ of an arbitrary node v of $P_n \odot C_m$ is*

$$= \begin{cases} \frac{2(nm+n-1)}{(n-\lambda+2)\{(n-\lambda+1)+m(n-\lambda+3)\}+(\lambda+1)\{\lambda+m(\lambda+2)\}-2(4m+5)}, & \text{if } v = u_{\lambda,\mu} \in C_m, \lambda = 1,2,\cdots,n; \\ & \mu = 1,2,\cdots,m \\ \frac{2(nm+n-1)}{(n-\lambda+1)[(n-\lambda)+m(n-i+2)]+\lambda[(\lambda-1)+m(\lambda+1)]-2m}, & \text{if } v = u_\lambda \in P_n, \lambda = 1,2,\cdots,n. \end{cases}$$

Proof. Suppose that the node points of P_n are $u_1, u_2, ..., u_n$ and that of C_m corresponding to the node u_λ are $u_{\lambda,1}, u_{\lambda,2}, ..., u_{\lambda,m}$, $\lambda = 1, 2, ..., n$. If $v = u_\lambda$ is any vertex of P_n then $\sum_{x\in V}\mathfrak{d}(u_\lambda, x) = \sum_{l=\lambda+1}^{n}\mathfrak{d}(u_\lambda, u_l) + \sum_{l=1}^{\lambda-1}\mathfrak{d}(u_\lambda, u_l) + \sum_{\mu=1}^{m}\sum_{l=\lambda}^{n}\mathfrak{d}(u_\lambda, u_{l,\mu}) + \sum_{\mu=1}^{m}\sum_{l=1}^{\lambda-1}\mathfrak{d}(u_\lambda, u_{l,\mu})$

$= [1+2+\cdots+(n-\lambda)]+[1+2+\cdots+(\lambda-1)]+[m+2m+\cdots+m(n-\lambda+1)]+[2m+3m+\cdots+m(\lambda-1+1)]$
$= \frac{(n-\lambda)(n-\lambda+1)}{2} + \frac{\lambda(\lambda-1)}{2} + m\frac{(n-\lambda+1)(n-\lambda+2)}{2} + [m+2m+3m+\cdots+\lambda m] - m$
$= \frac{(n-\lambda)(n-\lambda+1)}{2} + \frac{\lambda(\lambda-1)}{2} + m\frac{(n-\lambda+1)(n-\lambda+2)}{2} + m\frac{\lambda(\lambda+1)}{2} - m$
$= \frac{(n-\lambda+1)[(n-\lambda)+m(n-\lambda+2)]+\lambda[(\lambda-1)+m(\lambda+1)]-2m}{2}.$

Therefore, the normalized closeness centrality of v is

$$C_{NC}(v) = \frac{nm+n-1}{\sum_{x\in V}\mathfrak{d}(u,x)} = \frac{2(nm+n-1)}{(n-\lambda+1)[(n-\lambda)+m(n-\lambda+2)]+\lambda[(\lambda-1)+m(\lambda+1)]-2m}.$$

If $v = u_{\lambda,\mu}$ is any vertex of λ^{th} C_m, where $\lambda = 1,2,\cdots,n$; $\mu = 1,2,\cdots,m$ then
$\sum_{x\in V}\mathfrak{d}(u_{\lambda,\mu},x) = \sum_{l=\lambda}^{n}\mathfrak{d}(u_{\lambda,\mu},u_l) + \sum_{l=1}^{\lambda-1}\mathfrak{d}(u_{\lambda,\mu},u_l) + \sum_{k=1}^{m}\sum_{l=\lambda}^{n}\mathfrak{d}(u_{\lambda,\mu},u_{l,k}) + \sum_{k=1}^{m}\sum_{l=1}^{\lambda-1}\mathfrak{d}(u_{\lambda,\mu},u_{l,k})$.
Now, $\sum_{x\in V}\mathfrak{d}(u_{\lambda,\mu},x) = [1+2+\cdots+\{(n-\lambda)+1\}] + [2+3+\cdots+\{(\lambda-1)+1\}] + [\{1+1++2(m-3)\} + 3m+4m+\cdots+m\{(n-\lambda)+2\}] + [3m+4m+\cdots+\{(\lambda-1)+2\}]$
$= \frac{(n-\lambda+1)(n-\lambda+2)}{2} + [1+2+3+\cdots+\lambda] - 1 + [m+2m+3m+4m+\cdots+m(n-\lambda+2)]$
$+ [m+2m+3m+4m+\cdots+m(\lambda+1)] - m - 3m - 4$
$= \frac{(n-\lambda+1)(n-\lambda+2)}{2} + \frac{\lambda(\lambda+1)}{2} + m\frac{(n-\lambda+2)(n-\lambda+3)}{2} + m\frac{(\lambda+1)(\lambda+2)}{2} - (4m+5)$
$= \frac{(n-\lambda+2)[(n-\lambda+1)+m(n-\lambda+3)]+(\lambda+1)[\lambda+m(\lambda+2)]}{2} - (4m+5)$
$= \frac{(n-\lambda+2)\{(n-\lambda+1)+m(n-\lambda+3)\}+(\lambda+1)\{\lambda+m(\lambda+2)\}-2(4m+5)}{2}.$
Hence, the normalized closeness centrality of v is

$$C_{NC}(v) = \frac{nm+n-1}{\sum_{x\in V}\mathfrak{d}(u,x)} = \frac{2(nm+n-1)}{(n-\lambda+2)\{(n-\lambda+1)+m(n-\lambda+3)\}+(\lambda+1)\{\lambda+m(\lambda+2)\}-2(4m+5)}.$$

□

4.5 Closeness centrality of $C_n \odot C_m$

The corona graph $C_n \odot C_m$ is made using the corona product of cycle graphs C_n and C_m. It has $n+nm$ nodes. Let the nodes of C_n be $u_1, u_2, ..., u_n$, and that of C_m corresponding to the node u_λ be $u_{\lambda,1}, u_{\lambda,2}, ..., u_{\lambda,m}$. A corona graph $C_3 \odot C_3$ is shown in Figure 6.

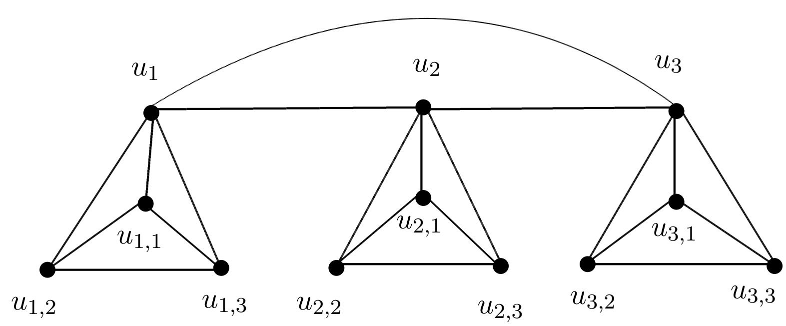

Figure 6: Corona graph $C_3 \odot C_3$

.

Theorem 4.5. *The $C_{NC}(v)$ of an arbitrary node v of the corona product graph $C_n \odot C_m$ is*

$$C_{NC}(v)=\begin{cases}\frac{4(nm+n-1)}{n^2+m(n+2)^2-4m}, & \textit{if } v=u_\lambda\in C_n, \lambda=1,2,\cdots,n \textit{ and } n \textit{ is even}\\ \frac{4(nm+n-1)}{m(n^2+4n-1)+n^2-1}, & \textit{if } v=u_\lambda\in C_n,\ \lambda=1,2,\cdots,n \textit{ and } n \textit{ is odd}\\ \frac{4(nm+n-1)}{m(n+4)^2+(n+2)^2-4(4m+5)}, & \textit{if } v=u_{\lambda,\mu}\in C_m, \lambda=1,2,\cdots,n; \ n \textit{ is even;}\\ & \quad \mu=1,2,\cdots,m\\ \frac{4(nm+n-1)}{(n+3)[(n+1)+m(n+5)]-4(4m+5)}, & \textit{if } v=u_{\lambda,\mu}\in C_m, \lambda=1,2,\cdots,n; \ n \textit{ is odd;}\\ & \quad \mu=1,2,\cdots,m\end{cases}$$

Proof. Suppose that the node points of C_n are $u_1, u_2, ..., u_n$ and that of C_m corresponding to the node u_λ are $u_{\lambda,1}, u_{\lambda,2}, ..., u_{\lambda,m}$, $\lambda = 1, 2, ..., n$. If $v = u_1$ is a vertex of C_n and n is even then $\sum_{x\in V}\mathfrak{d}(u_1, x) = \sum_{\lambda=1}^{n}\mathfrak{d}(u_1, u_\lambda) + \sum_{\mu=1}^{m}\mathfrak{d}(u_1, u_{1,\mu}) + 2[\sum_{l=2}^{\frac{n}{2}}\sum_{\mu=1}^{m}\mathfrak{d}(u_1, u_{l,\mu})] + \sum_{\mu=1}^{m}\mathfrak{d}(u_1, u_{\frac{n}{2}+1,\mu})$.
We know from Theorem 3.1, $\sum_{\lambda=1}^{n}\mathfrak{d}(u_1, u_\lambda) = \frac{n^2}{4}$.
Therefore, $\sum_{x\in V}\mathfrak{d}(u_1, x) = \frac{n^2}{4} + m + 2[m + 2m + 3m + \cdots + m\cdot\frac{n}{2}] + m(\frac{n}{2}+1)$
$= \frac{n^2}{4} + 2[m + 2m + 3m + \cdots + \frac{nm}{2}] + m(\frac{n}{2}+1) - m$
$= \frac{n^2}{4} + 2m[\frac{\frac{n}{2}(\frac{n}{2}+1)}{2}] + \frac{m(n+2)}{2} - m$
$= \frac{n^2}{4} + \frac{mn(n+2)}{4} + \frac{m(n+2)}{2} - m$
$= \frac{n^2+mn^2+2mn+2mn+4m-4m}{4}$
$= \frac{n^2+mn(n+4)}{4}$.
This result is true for all u_λ, $\lambda = 1, 2, \cdots, n$.
Therefore, the normalized closeness centrality of u_λ is $C_{NC}(u_\lambda) = \frac{nm+n-1}{\sum_{x\in V}\mathfrak{d}(u_\lambda,x)} = \frac{4(nm+n-1)}{n^2+mn(n+4)}$.

Again, if $v = u_1$, $u_1 \in C_n$ and n is odd, then $\sum_{x\in V}\mathfrak{d}(u_1, x) = \sum_{\lambda=1}^{n}\mathfrak{d}(u_1, u_\lambda) + \sum_{\mu=1}^{m}\mathfrak{d}(u_1, u_{1,\mu}) + 2[\sum_{l=2}^{\frac{n+1}{2}}\sum_{\mu=1}^{m}\mathfrak{d}(u_1, u_{l,\mu})]$.

We know from Theorem 3.1, $\sum_{\lambda=1}^{n}\mathfrak{d}(u_1, u_\lambda) = \frac{(n-1)(n+1)}{4}$.
Therefore, $\sum_{x\in V}\mathfrak{d}(u_1, x) = \frac{(n-1)(n+1)}{4} + m + 2[2m + 3m + \cdots + m(\frac{n+1}{2})]$
$= \frac{(n-1)(n+1)}{4} + 2[m + 2m + 3m + \cdots + m(\frac{n+1}{2})] - m$
$= \frac{(n-1)(n+1)}{4} + 2m\frac{\frac{n+1}{2}(\frac{n+1}{2}+1)}{2} - m$
$= \frac{(n-1)(n+1)}{4} + \frac{m(n+1)}{2}(\frac{n+1}{2}+1) - m$
$= \frac{(n-1)(n+1)}{4} + m\{\frac{(n+1)^2}{4} + \frac{(n+1)}{2} - 1\}$
$= \frac{(n-1)(n+1)}{4} + \frac{m(n^2+4n-1)}{4}$
$= \frac{m(n^2+4n-1)+n^2-1}{4}$.
This result is true for all u_λ, $\lambda = 1, 2, \cdots, n$.
Therefore, the normalized closeness centrality of u_λ is $C_{NC}(u_\lambda) = \frac{nm+n-1}{\sum_{x\in V}\mathfrak{d}(u_\lambda,x)} = \frac{4(nm+n-1)}{m(n^2+4n-1)+n^2-1}$.

If $v = u_{\lambda,\mu}$ is any vertex of λ^{th} C_m, where $\lambda = 1, 2, \cdots, n$; $\mu = 1, 2, \cdots, m$ and n is even then $\sum_{x\in V}\mathfrak{d}(u_{\lambda,\mu}, x) = \sum_{\lambda=1}^{n}\mathfrak{d}(u_{\lambda,\mu}, u_\mu) + \sum_{k=1}^{m}\mathfrak{d}(u_{\lambda,\mu}, u_{\lambda,k}) + \sum_{l=1,\lambda\neq l}^{n}\sum_{k=1}^{m}\mathfrak{d}(u_{\lambda,\mu}, u_{l,k})$.
From $u_{\lambda,\mu}$, u_λ is situated at a distance $1, 2$ vertices of C_n are situated at a distance 2, another two vertices of C_n are situated at a distance $3\cdots$, another two vertices of C_n are situated at a distance $\frac{n}{2}$ and 1 vertices of C_n are situated at a distance $\frac{n}{2}+1$.
Also from $u_{\lambda,\mu}$, 2 sets of m vertices of C_m are situated at a distance 3, another 2 sets of m vertices of C_m

are situated at a distance $4, \cdots$, another 2 sets of m vertices of C_m are situated at a distance $\frac{n}{2}+1$ and m vertices of C_m are at a distance $\frac{n}{2}+2$.
Now, $\sum_{x\in V} \mathfrak{d}(u_{\lambda,\mu},x) = [1+2+2+3+3\cdots+\frac{n}{2}+\frac{n}{2}+(\frac{n}{2}+1)]+[2+2(m-3)]+[3m+3m+4m+4m+\cdots+m(\frac{n}{2}+1)+m(\frac{n}{2}+1)+m(\frac{n}{2}+2)]$
$= [1+2+3+\cdots+\frac{n}{2}+(\frac{n}{2}+1)]+[2+3+\cdots+\frac{n}{2}]+(2m-4)+[3m+4m+\cdots+m\{\frac{n}{2}+1\}+m\{\frac{n}{2}+2\}]+[3m+4m+\cdots+m\{\frac{n}{2}+1\}]$
$= 2m-4+[1+2+\cdots+(\frac{n}{2}+1)]+[1+2+3+\cdots+\frac{n}{2}]-1+[m+2m+3m+4m+\cdots+m\{\frac{n}{2}+1\}+m\{\frac{n}{2}+2\}]-3m+[m+2m+3m+4m+\cdots+m\{\frac{n}{2}+1\}]-3m$
$= 2m-4+\frac{(\frac{n}{2}+1)(\frac{n}{2}+2)}{2}+\frac{\frac{n}{2}(\frac{n}{2}+1)}{2}-1+m\frac{(\frac{n}{2}+2)(\frac{n}{2}+3)}{2}+m\frac{(\frac{n}{2}+1)(\frac{n}{2}+2)}{2}-6m$
$= \frac{(n+2)(n+4)}{8}+\frac{n(n+2)}{8}+\frac{m(n+4)(n+6)}{8}+\frac{m(n+2)(n+4)}{8}-6m-1+2m-4$
$= \frac{(n+2)(n+4+n)}{8}+\frac{m(n+4)(n+6+n+2)}{8}-(4m+5)$
$= \frac{m(n+4)^2+(n+2)^2}{4}-(4m+5)$
$= \frac{m(n+4)^2+(n+2)^2-4(4m+5)}{4}$.

Hence, the normalized closeness centrality of v is $C_{NC}(v) = \frac{nm+n-1}{\sum_{x\in V}\mathfrak{d}(u,x)} = \frac{4(nm+n-1)}{m(n+4)^2+(n+2)^2-4(4m+5)}$.

If $v = u_{\lambda,\mu}$ is any vertex of λ^{th} C_m, where $\lambda = 1,2,\cdots,n$; $\mu = 1,2,\cdots,m$; n is odd, then $\sum_{x\in V}\mathfrak{d}(u_{\lambda,\mu},x) = \sum_{\lambda=1}^{n}\mathfrak{d}(u_{\lambda,\mu},u_\lambda)+\sum_{k=1}^{m}\mathfrak{d}(u_{\lambda,\mu},u_{\lambda,k})+\sum_{l=1,\lambda\neq l}^{n}\sum_{k=1}^{m}\mathfrak{d}(u_{\lambda,\mu},u_{l,k})$.
$= [1+2+2+3+3\cdots+\frac{n+1}{2}+\frac{n+1}{2}]+[2+2(m-3)]+[3m+3m+4m+4m+\cdots+m(\frac{n+1}{2}+1)+m(\frac{n+1}{2}+1)]$
$= 2[1+2+3+\cdots+\frac{n+1}{2}]-1+2m-4+2m[1+2+3+4+\cdots+\{\frac{n+1}{2}+1\}]-6m$
$= 2m-4+2\cdot\frac{\frac{n+1}{2}(\frac{n+1}{2}+1)}{2}+2m\cdot\frac{(\frac{n+1}{2}+1)(\frac{n+1}{2}+1+1)}{2}-6m-1$
$= \frac{(n+1)(n+3)}{4}+\frac{m(n+3)(n+5)}{4}-(6m+1)+2m-4$
$= \frac{(n+3)[(n+1)+m(n+5)]-4(4m+5)}{4}$.

Hence, the normalized closeness centrality of v is $C_{NC}(v) = \frac{nm+n-1}{\sum_{x\in V}\mathfrak{d}(u,x)} = \frac{4(nm+n-1)}{(n+3)[(n+1)+m(n+5)]-4(4m+5)}$. □

4.6 Closeness centrality of $C_n \odot P_m$

The corona graph $C_n \odot P_m$ is made using the corona product of cycle graph C_n and the path graph P_m. The graph $C_n \odot P_m$ has $n+nm$ nodes. Let the nodes of C_n be $u_1, u_2, ..., u_n$, and that of P_m corresponding to the node u_λ be $u_{\lambda,1}, u_{\lambda,2}, ..., u_{\lambda,m}$. A corona graph $C_3 \odot P_4$ is shown in Figure 7.

Theorem 4.6. *The $C_{NC}(v)$ of an arbitrary node v of the corona product $C_n \odot P_m$ of two graphs C_n and P_m is*

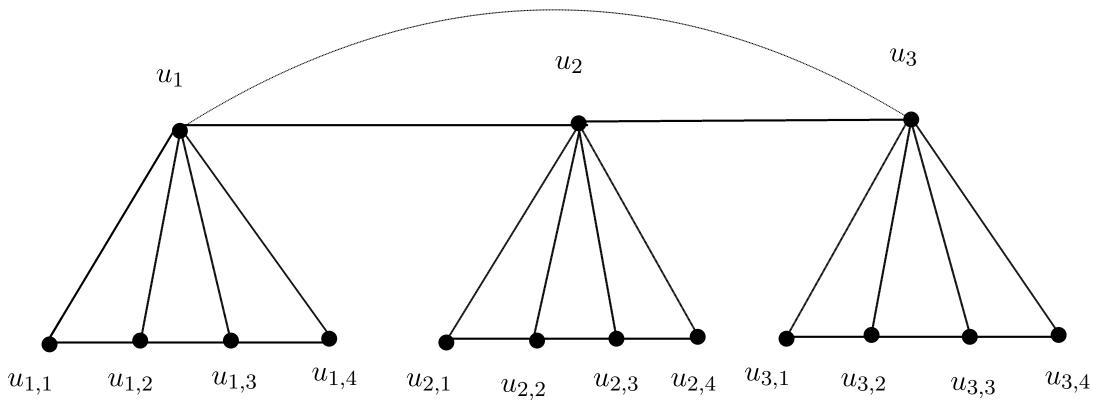

Figure 7: Corona graph $C_3 \odot P_4$

.

$$= \begin{cases} \frac{4(nm+n-1)}{n^2+mn(n+4)}, & \text{if } v = u_\lambda \in C_n, \lambda = 1,2,\cdots,n \text{ and } n \text{ is even} \\ \frac{4(nm+n-1)}{m(n^2+4n-1)+n^2-1}, & \text{if } v = u_\lambda \in C_n,\ \lambda = 1,2,\cdots,n,\ n \text{ is odd} \\ \frac{4}{m(n+4)^2+(n+2)^2-4(4m+4)}, & \text{if } v = u_{\lambda,\mu} \in P_m, \lambda = 1,2,\cdots,n;\ n \text{ is even and } \mu = 1, m \\ \frac{4}{m(n+4)^2+(n+2)^2-4(4m+5)}, & \text{if } v = u_{\lambda,\mu} \in P_m, \lambda = 1,2,\cdots,n;\ n \text{ is even} \\ & \text{and } \mu = 2,3,\cdots,m-1 \\ \frac{4(nm+n-1)}{(n+3)[(n+1)+m(n+5)]-4(4m+4)}, & \text{if } v = u_{\lambda,\mu} \in P_m, \lambda = 1,2,\cdots,n;\ n \text{ is odd and } \mu = 1, m \\ \frac{4(nm+n-1)}{(n+3)[(n+1)+m(n+5)]-4(4m+5)}, & \text{if } v = u_{\lambda,\mu} \in P_m, \lambda = 1,2,\cdots,n;\ n \text{ is odd} \\ & \text{and } \mu = 2,3,\cdots,m-1. \end{cases}$$

Proof. Suppose that the nodes of C_n are $u_1, u_2, ..., u_n$ and that of P_m corresponding to the node u_λ are $u_{\lambda,1}, u_{\lambda,2}, ..., u_{\lambda,m}$, $\lambda = 1, 2, ..., n$. If $v = u_1$, $u_1 \in V(C_n)$ and n is even, then
$\sum_{x\in V} \mathfrak{d}(u_1, x) = \sum_{\lambda=1}^{n} \mathfrak{d}(u_1, u_\lambda) + \sum_{\mu=1}^{m} \mathfrak{d}(u_1, u_{1,\mu}) + 2[\sum_{\mu=1}^{m} \sum_{l=2}^{\frac{n}{2}} \mathfrak{d}(u_1, u_{l,\mu})] + \sum_{\mu=1}^{m} \mathfrak{d}(u_1, u_{\frac{n}{2}+1,\mu})$.
We know from Theorem 3.2, $\sum_{\lambda=1}^{n} \mathfrak{d}(u_1, u_\lambda) = \frac{n^2}{4}$.

$$\begin{aligned} \text{Therefore, } \sum_{x\in V} \mathfrak{d}(u_1, x) &= \frac{n^2}{4} + m + 2[m + 2m + 3m + \cdots + m \cdot \frac{n}{2}] + m(\frac{n}{2} + 1) \\ &= \frac{n^2}{4} + 2[m + 2m + 3m + \cdots + \frac{nm}{2}] + m(\frac{n}{2} + 1) - m \\ &= \frac{n^2}{4} + 2m[\frac{\frac{n}{2}(\frac{n}{2}+1)}{2}] + \frac{m(n+2)}{2} - m \\ &= \frac{n^2}{4} + \frac{mn(n+2)}{4} + \frac{m(n+2)}{2} - m \\ &= \frac{n^2+mn^2+2mn+2mn+4m-4m}{4} \end{aligned}$$

$= \frac{n^2+mn(n+4)}{4}$.
This result is true for all u_λ, $\lambda = 1, 2, \cdots, n$.

Therefore, $C_{NC}(v) = \frac{nm+n-1}{\sum_{x\in V} \mathfrak{d}(u,x)} = \frac{4(nm+n-1)}{n^2+mn(n+4)}$, $v \in V(C_n)$ and n is even.

If $v = u_1$, $u_1 \in V(C_n)$ and n is odd, then
$\sum_{x\in V} \mathfrak{d}(u_1, x) = \sum_{\lambda=1}^{n} \mathfrak{d}(u_1, u_\lambda) + \sum_{\mu=1}^{m} \mathfrak{d}(u_1, u_{1,\mu}) + 2[\sum_{\mu=1}^{m} \sum_{l=2}^{\frac{n+1}{2}} \mathfrak{d}(u_1, u_{l,\mu})]$.

We know from Theorem 3.2, $\sum_{\lambda=1}^{n} \mathfrak{d}(u_1, u_\lambda) = \frac{(n-1)(n+1)}{4}$.

$$\begin{aligned}
\text{Therefore, } \sum_{x\in V} \mathfrak{d}(u_1, x) &= \tfrac{(n-1)(n+1)}{4} + m + 2[2m + 3m + \cdots + m(\tfrac{n+1}{2})] \\
&= \tfrac{(n-1)(n+1)}{4} + 2[m + 2m + 3m + \cdots + m(\tfrac{n+1}{2})] - m \\
&= \tfrac{(n-1)(n+1)}{4} + 2m\tfrac{\frac{n+1}{2}(\frac{n+1}{2}+1)}{2} - m \\
&= \tfrac{(n-1)(n+1)}{4} + \tfrac{m(n+1)}{2}(\tfrac{n+1}{2} + 1) - m \\
&= \tfrac{(n-1)(n+1)}{4} + m\{\tfrac{(n+1)^2}{4} + \tfrac{(n+1)}{2} - 1\} \\
&= \tfrac{(n-1)(n+1)}{4} + \tfrac{m(n^2+4n-1)}{4} \\
&= \tfrac{m(n^2+4n-1)+n^2-1}{4}.
\end{aligned}$$

This result is true for all u_λ, $\lambda = 1, 2, \cdots, n$.

Therefore, $C_{NC}(v) = \frac{nm+n-1}{\sum_{x\in V} \mathfrak{d}(u,x)} = \frac{4(nm+n-1)}{m(n^2+4n-1)+n^2-1}$, $v \in V(C_n)$ and n is odd.

If $v = u_{\lambda,\mu}$ is any vertex of λ^{th} P_m, where $\lambda = 1, 2, \cdots, n$; $\mu = 1, m$ and n is even then $\sum_{x\in V} \mathfrak{d}(u_{\lambda,\mu}, x) = \sum_{\lambda=1}^{n} \mathfrak{d}(u_{\lambda,\mu}, u_\lambda) + \sum_{k=1}^{m} \mathfrak{d}(u_{\lambda,\mu}, u_{\lambda,k}) + \sum_{l=1,\lambda\neq l}^{n} \sum_{k=1}^{m} \mathfrak{d}(u_{\lambda,\mu}, u_{l,k})$. As n is even, so, there are two sets of m vertices of P_m are at a distance 3, another two sets of m vertices of P_m are at a distance $4, \cdots$, two sets of m vertices of P_m are at a distance $\frac{n}{2}+1$ and m vertices of P_m are at a distance $\frac{n}{2}+2$ from $u_{\lambda,\mu}$.

Now, $\sum_{x\in V} \mathfrak{d}(u_{\lambda,\mu}, x) = [1 + 2 + 2 + 3 + 3 \cdots + \frac{n}{2} + \frac{n}{2} + (\frac{n}{2} + 1)] + [1 + 2(m-2)] + [3m + 3m + 4m + 4m + \cdots + m(\frac{n}{2} + 1) + m(\frac{n}{2} + 1) + m(\frac{n}{2} + 2)]$
$= [1 + 2 + 3 + \cdots + \frac{n}{2} + (\frac{n}{2} + 1)] + [2 + 3 + \cdots + \frac{n}{2}] + (2m-3) + [3m + 4m + \cdots + m\{\frac{n}{2} + 1\} + m\{\frac{n}{2} + 2\}] + [3m + 4m + \cdots + m\{\frac{n}{2} + 1\}]$
$= (2m-3) + [1 + 2 + \cdots + (\frac{n}{2} + 1)] + [1 + 2 + 3 + \cdots + \frac{n}{2}] - 1 + [m + 2m + 3m + 4m + \cdots + m\{\frac{n}{2} + 1\} + m\{\frac{n}{2} + 2\}] - 3m + [m + 2m + 3m + 4m + \cdots + m\{\frac{n}{2} + 1\}] - 3m$
$= (2m-3) + \frac{(\frac{n}{2}+1)(\frac{n}{2}+2)}{2} + \frac{\frac{n}{2}(\frac{n}{2}+1)}{2} - 1 + m\frac{(\frac{n}{2}+2)(\frac{n}{2}+3)}{2} + m\frac{(\frac{n}{2}+1)(\frac{n}{2}+2)}{2} - 6m$
$= \frac{(n+2)(n+4)}{8} + \frac{n(n+2)}{8} + \frac{m(n+4)(n+6)}{8} + \frac{m(n+2)(n+4)}{8} - (4m+4)$
$= \frac{(n+2)(n+4+n)}{8} + \frac{m(n+4)(n+6+n+2)}{8} - (4m+4)$
$= \frac{m(n+4)^2+(n+2)^2}{4} - (4m+4)$
$= \frac{m(n+4)^2+(n+2)^2-4(4m+4)}{4}$.

Hence, the normalized closeness centrality of v is $C_{NC}(v) = \frac{nm+n-1}{\sum_{x\in V} \mathfrak{d}(u,x)} = \frac{4(nm+n-1)}{m(n+4)^2+(n+2)^2-4(4m+4)}$.

Similarly, if $v = u_{\lambda,\mu}$ is any vertex of λ^{th} P_m where $\lambda = 1, 2, \cdots, n$; $\mu = 2, 3, \cdots, m-1$ and n is even then $\sum_{x\in V} \mathfrak{d}(u_{\lambda,\mu}, x) = \sum_{\lambda=1}^{n} \mathfrak{d}(u_{\lambda,\mu}, u_\lambda) + \sum_{k=1}^{m} \mathfrak{d}(u_{\lambda,\mu}, u_{\lambda,k}) + \sum_{l=1,\lambda\neq l}^{n} \sum_{k=1}^{m} \mathfrak{d}(u_{\lambda,\mu}, u_{l,k})$
$= [1+2+2+3+3\cdots+\frac{n}{2}+\frac{n}{2}+(\frac{n}{2}+1)]+[2+2(m-3)]+[3m+3m+4m+4m+\cdots+m(\frac{n}{2}+1)+m(\frac{n}{2}+1)+m(\frac{n}{2}+2)]$.
From the above result, we get

$$\begin{aligned}
\sum_{x\in V} \mathfrak{d}(u_{\lambda,\mu}, x) &= (2m-4) + \tfrac{(n+2)(n+4)}{8} + \tfrac{n(n+2)}{8} + \tfrac{m(n+4)(n+6)}{8} + \tfrac{m(n+2)(n+4)}{8} - 1 - 6m \\
&= \tfrac{m(n+4)^2+(n+2)^2-4(4m+5)}{4}.
\end{aligned}$$

Therefore, the normalized closeness centrality of v is $C_{NC}(v) = \frac{nm+n-1}{\sum_{x\in V}\mathfrak{d}(u,x)} = \frac{4(nm+n-1)}{m(n+4)^2+(n+2)^2-4(4m+5)}$.

If $v = u_{\lambda,\mu}$ is any vertex of λ^{th} P_m where $\lambda = 1,2,\cdots,n$; $\mu = 1, m$ and n is odd then
$\sum_{x\in V}\mathfrak{d}(u_{\lambda,\mu},x) = \sum_{\lambda=1}^{n}\mathfrak{d}(u_{\lambda,\mu},u_\lambda) + \sum_{k=1}^{m}\mathfrak{d}(u_{\lambda,\mu},u_{\lambda,k}) + \sum_{l=1,\lambda\neq l}^{n}\sum_{k=1}^{m}\mathfrak{d}(u_{\lambda,\mu},u_{l,k})$.
Since n is odd, there are two sets of m vertices of P_m are at a distance 3, two sets of m vertices of P_m are at a distance 4, and so on, up to distance $\frac{n+1}{2}+1$.
Now, $\sum_{x\in V}\mathfrak{d}(u_{\lambda,\mu},x) = [1+2+2+3+3\cdots+\frac{n+1}{2}+\frac{n+1}{2}] + [1+2(m-2)] + [3m+3m+4m+4m+\cdots+m(\frac{n+1}{2}+1)+m(\frac{n+1}{2}+1)]$
$= (2m-3) + 2\{1+2+\cdots+\frac{n+1}{2}\} - 1 + 2\{m+2m+3m+4m+\cdots+m(\frac{n+1}{2}+1)\} - 6m$
$= 2\cdot\frac{\frac{n+1}{2}(\frac{n+1}{2}+1)}{2} + 2m\cdot\frac{(\frac{n+1}{2}+1)(\frac{n+1}{2}+2)}{2} + 2m - 3 - 6m - 1$
$= \frac{n+1}{2}(\frac{n+1}{2}+1) + m(\frac{n+1}{2}+1)(\frac{n+1}{2}+2) - (4m+4)$
$= \frac{(n+1)(n+3)}{4} + m\frac{(n+3)(n+5)}{4} - (4m+4)$
$= \frac{(n+3)[(n+1)+m(n+5)]-4(4m+4)}{4}$.
Therefore, the normalized closeness centrality of v is $C_{NC}(v) = \frac{nm+n-1}{\sum_{x\in V}\mathfrak{d}(u,x)} = \frac{4(nm+n-1)}{(n+3)[(n+1)+m(n+5)]-4(4m+4)}$.

Similarly, if $v = u_{\lambda,\mu}$ is any vertex of λ^{th} P_m where $\lambda = 1,2,\cdots,n$; $\mu = 2,3,\cdots,m-1$ and n is odd then
$\sum_{x\in V}\mathfrak{d}(u_{\lambda,\mu},x) = \sum_{\lambda=1}^{n}\mathfrak{d}(u_{\lambda,\mu},u_\lambda) + \sum_{k=1}^{m}\mathfrak{d}(u_{\lambda,\mu},u_{\lambda,k}) + \sum_{l=1,\lambda\neq l}^{n}\sum_{k=1}^{m}\mathfrak{d}(u_{\lambda,\mu},u_{l,k})$.
$= [1+2+2+3+3\cdots+\frac{n+1}{2}+\frac{n+1}{2}] + [2+2(m-3)] + [3m+3m+4m+4m+\cdots+m(\frac{n+1}{2}+1)+m(\frac{n+1}{2}+1)]$
From the above result, we get,
$\sum_{x\in V}\mathfrak{d}(u_{\lambda,\mu},x) = 2\cdot\frac{\frac{n+1}{2}(\frac{n+1}{2}+1)}{2} + 2m\cdot\frac{(\frac{n+1}{2}+1)(\frac{n+1}{2}+2)}{2} + 2m - 4 - 6m - 1$
$= \frac{(n+1)(n+3)}{4} + m\frac{(n+3)(n+5)}{4} - (4m+5)$
$= \frac{(n+3)[(n+1)+m(n+5)]-4(4m+5)}{4}$.
Therefore, the normalized closeness centrality of v is $C_{NC}(v) = \frac{nm+n-1}{\sum_{x\in V}\mathfrak{d}(u,x)} = \frac{4(nm+n-1)}{(n+3)[(n+1)+m(n+5)]-4(4m+5)}$.
□

4.7 Closeness centrality of $P_n \odot S_m$

The corona graph $P_n \odot S_m$ is made using the corona product of path graphs P_n and the star graph S_m. It has $n + nm$ nodes. Let the nodes of P_n be $u_1, u_2, ..., u_n$, and that of S_m corresponding to the node u_λ be $u_{\lambda,1}, u_{\lambda,2}, ..., u_{\lambda,m}$, where $u_{\lambda,1}$ is the central vertex of S_m. Figure 8 shows a corona graph $P_4 \odot S_4$.

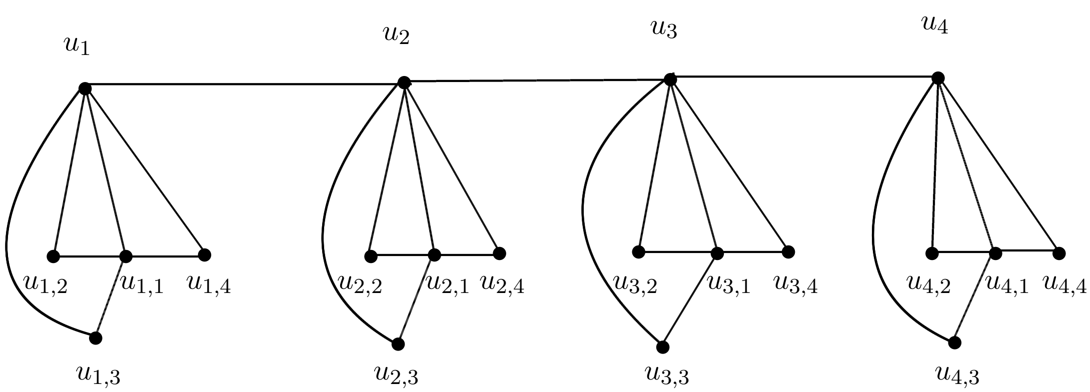

Figure 8: Corona graph $P_4 \odot S_4$

.

Theorem 4.7. *The $C_{NC}(v)$ of an arbitrary node v of $P_n \odot S_m$ is*

$$= \begin{cases} \frac{2(nm+n-1)}{(n-\lambda+1)[(n-\lambda)+m(n-\lambda+2)]+\lambda[(\lambda-1)+m(\lambda+1)]-2m}, & \text{if } v = u_\lambda \in P_n, \lambda = 1,2,\cdots,n \\ \frac{2(nm+n-1)}{(n-\lambda+2)[(n-\lambda+1)+m(n-\lambda+3)]+(\lambda+1)[\lambda+m(\lambda+2)]-2(5m+2)}, & \text{if } v = u_{\lambda,\mu} \text{ is the central vertex of the star graph } S_m,\ \lambda = 1,2,\cdots,n;\ \mu = 1 \\ \frac{2(nm+n-1)}{(n-\lambda+2)[(n-\lambda+1)+m(n-\lambda+3)]+(\lambda+1)[\lambda+m(\lambda+2)]-2(4m+4)}, & \text{if } v = u_{\lambda,\mu} \text{ is the non-central vertex of the star graph } S_m,\ \lambda = 1,2,\cdots,n;\ \mu = 2,3,\cdots,m. \end{cases}$$

Proof. Suppose that the node points of P_n are $u_1, u_2, ..., u_n$ and that of S_m corresponding to the node u_λ are $u_{\lambda,1}, u_{\lambda,2}, ..., u_{\lambda,m}$, where $u_{\lambda,1}$ is the central vertex of S_m and $\lambda = 1, 2, ..., n$. If $v = u_\lambda$ is any vertex of P_n then $\sum_{x\in V} \mathfrak{d}(u_\lambda, x) = \sum_{l=\lambda+1}^{n} \mathfrak{d}(u_\lambda, u_l) + \sum_{l=1}^{\lambda-1} \mathfrak{d}(u_\lambda, u_l) + \sum_{\mu=1}^{m}\sum_{l=\lambda}^{n} \mathfrak{d}(u_\lambda, u_{l,\mu}) + \sum_{\mu=1}^{m}\sum_{l=1}^{\lambda-1} \mathfrak{d}(u_\lambda, u_{l,\mu})$
$= [1+2+\cdots+(n-\lambda)] + [1+2+\cdots+(\lambda-1)] + [m+2m+\cdots+m(n-\lambda+1)] + [2m+3m+\cdots+m(\lambda-1+1)]$
$= \frac{(n-\lambda)(n-\lambda+1)}{2} + \frac{\lambda(\lambda-1)}{2} + m\frac{(n-\lambda+1)(n-\lambda+2)}{2} + [m+2m+3m+\cdots+\lambda m] - m$
$= \frac{(n-\lambda)(n-\lambda+1)}{2} + \frac{\lambda(\lambda-1)}{2} + m\frac{(n-\lambda+1)(n-\lambda+2)}{2} + m\frac{\lambda(\lambda+1)}{2} - m$
$= \frac{(n-\lambda+1)[(n-\lambda)+m(n-\lambda+2)]+\lambda[(\lambda-1)+m(\lambda+1)]-2m}{2}$.

Therefore, $C_{NC}(v) = \frac{nm+n-1}{\sum_{x\in V}\mathfrak{d}(u,x)} = \frac{2(nm+n-1)}{(n-\lambda+1)[(n-\lambda)+m(n-\lambda+2)]+\lambda[(\lambda-1)+m(\lambda+1)]-2m}$.

If $v = u_{\lambda,\mu}$ is the central vertex of λ^{th} S_m where $\lambda = 1, 2, \cdots, n$; $\mu = 1$ then $\sum_{x\in V} \mathfrak{d}(u_{\lambda,\mu}, x) = \sum_{k=1,k\neq\mu}^{m} \mathfrak{d}(u_{\lambda,\mu}, u_{\lambda,k}) + \sum_{l=\lambda}^{n} \mathfrak{d}(u_{\lambda,\mu}, u_l) + \sum_{l=1}^{\lambda-1} \mathfrak{d}(u_{\lambda,\mu}, u_l) + \sum_{k=1}^{m}\sum_{l=\lambda+1}^{n} \mathfrak{d}(u_{\lambda,\mu}, u_{l,k})$
$+ \sum_{k=1}^{m}\sum_{l=1}^{\lambda-1} \mathfrak{d}(u_{\lambda,\mu}, u_{l,k})$
$= (m-1) + [1+2+\cdots+\{(n-\lambda)+1\}] + [2+3+\cdots+\{(\lambda-1)+1\}] + [3m+4m+\cdots$
$+ m\{(n-\lambda)+2\}] + [3m+4m+\cdots+\{(\lambda-1)+2\}]$
$= (m-1) + \frac{(n-\lambda+1)(n-\lambda+2)}{2} + [1+2+3+\cdots+\lambda] - 1 + [m+2m+3m+4m+\cdots+m(n-\lambda+2)]$
$+ [m+2m+3m+4m+\cdots+m(\lambda+1)] - 3m - 3m$
$= (m-1) + \frac{(n-\lambda+1)(n-\lambda+2)}{2} + \frac{\lambda(\lambda+1)}{2} + m\frac{(n-\lambda+2)(n-\lambda+3)}{2} + m\frac{(\lambda+1)(\lambda+2)}{2} - 6m - 1$
$= \frac{(n-\lambda+2)[(n-\lambda+1)+m(n-\lambda+3)]+(\lambda+1)[\lambda+m(\lambda+2)]}{2} + m - 1 - 6m - 1$
$= \frac{(n-\lambda+2)[(n-\lambda+1)+m(n-\lambda+3)]+(\lambda+1)[\lambda+m(\lambda+2)]-2(5m+2)}{2}$.

Hence, $C_{NC}(v) = \frac{nm+n-1}{\sum_{x\in V}\mathfrak{d}(u,x)} = \frac{2(nm+n-1)}{(n-\lambda+2)[(n-\lambda+1)+m(n-\lambda+3)]+(\lambda+1)[\lambda+m(\lambda+2)]-2(5m+2)}$.

If $v = u_{\lambda,\mu}$ is the non-central vertex of S_m where $\lambda = 1, 2, \cdots, n$; $\mu = 2, \cdots, m$ then $\sum_{x\in V} \mathfrak{d}(u_{\lambda,\mu}, x) = \sum_{k=1,k\neq\mu}^{m} \mathfrak{d}(u_{\lambda,\mu}, u_{\lambda,k}) + \sum_{l=\lambda}^{n} \mathfrak{d}(u_{\lambda,\mu}, u_l) + \sum_{l=1}^{\lambda-1} \mathfrak{d}(u_{\lambda,\mu}, u_l)$
$+ \sum_{k=1}^{m}\sum_{l=\lambda+1}^{n} \mathfrak{d}(u_{\lambda,\mu}, u_{l,k}) + \sum_{k=1}^{m}\sum_{l=1}^{\lambda-1} \mathfrak{d}(u_{\lambda,\mu}, u_{l,k})$
$= [1+2(m-2)] + \frac{(n-\lambda+1)(n-\lambda+2)}{2} + \frac{\lambda(\lambda+1)}{2} + m\frac{(n-\lambda+2)(n-\lambda+3)}{2} + m\frac{(\lambda+1)(\lambda+2)}{2} - 6m - 1$
[Using the result of the previous paragraph]
$= \frac{(n-\lambda+2)[(n-\lambda+1)+m(n-\lambda+3)]+(\lambda+1)[\lambda+m(\lambda+2)]}{2} + 2m - 3 - 6m - 1$
$= \frac{(n-\lambda+2)[(n-\lambda+1)+m(n-\lambda+3)]+(\lambda+1)[\lambda+m(\lambda+2)]-2(4m+4)}{2}$.

Hence, $C_{NC}(v) = \frac{nm+n-1}{\sum_{x\in V}\mathfrak{d}(u,x)} = \frac{2(nm+n-1)}{(n-\lambda+2)[(n-\lambda+1)+m(n-\lambda+3)]+(\lambda+1)[\lambda+m(\lambda+2)]-2(4m+4)}$. □

4.8 Closeness centrality of corona product $S_n \odot K_m$ of two graphs S_n and K_m

The corona graph $S_n \odot K_m$ is made using the corona product of star graphs S_n and the complete graph K_m. It has $n + nm$ nodes. Let the nodes of S_n be $u_1, u_2, ..., u_n$, and that of K_m corresponding to the node u_λ be $u_{\lambda,1}, u_{\lambda,2}, ..., u_{\lambda,m}$. A corona graph $S_3 \odot K_4$ is shown in Figure 9.

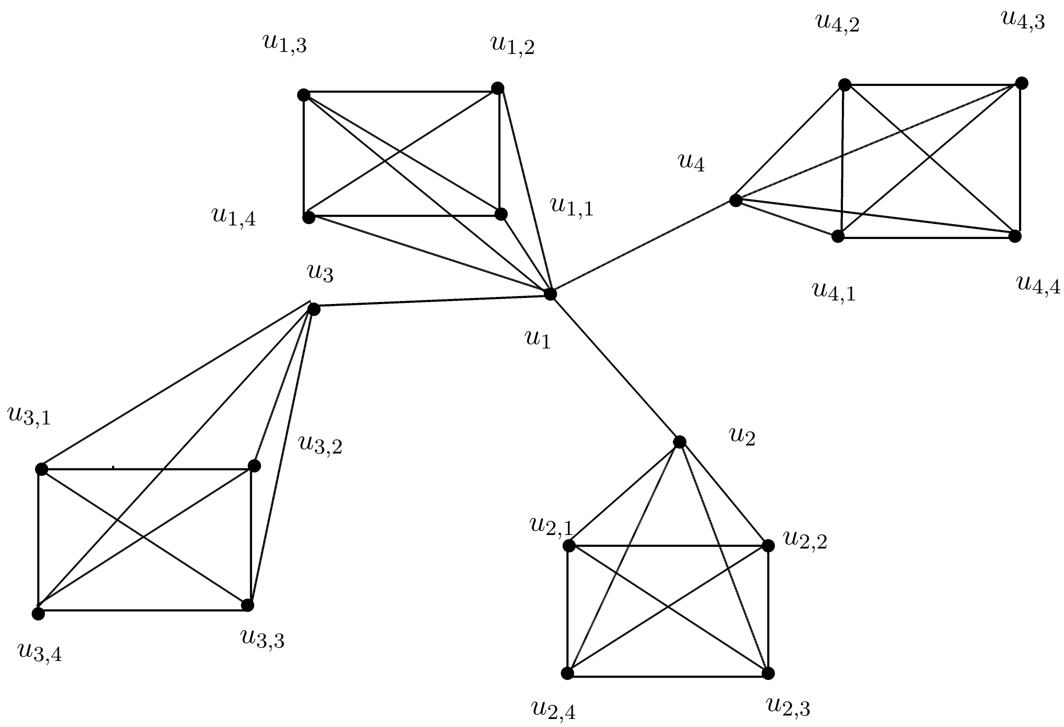

Figure 9: Corona graph $S_4 \odot K_4$

.

Theorem 4.8. *The normalized closeness centrality $C_{NC}(u)$ of arbitrary node u of $S_n \odot K_m$ is*

$$= \begin{cases} \frac{nm+n-1}{2mn+n-m-1}, & \text{if } u \text{ is central vertex of } S_n \\ \frac{nm+n-1}{3mn+2n-3m-3}, & \text{if } u \text{ is non-central vertex of } S_n \\ \frac{nm+n-1}{4mn+3n-4m-4}, & \text{if } u = u_{\lambda,\mu} \in K_m,\ \lambda = 2,3,\cdots,n;\ \mu = 1,2,\cdots,m \\ \frac{nm+n-1}{3mn-2m+2n-2}, & \text{if } u = u_{\lambda,\mu} \in K_m,\ \lambda = 1;\ \mu = 1,2,\cdots,m. \end{cases}$$

Proof. Suppose that the nodes of S_n are $u_1, u_2, ..., u_n$ and that of K_m corresponding to the node u_λ are $u_{\lambda,1}, u_{\lambda,2}, ..., u_{\lambda,m}$, $\lambda = 1, 2, ..., n$. If $u = u_1$ then $\sum_{x \in V} \mathfrak{d}(u_1, x) = \sum_{\lambda=1}^{n} \mathfrak{d}(u_1, u_\lambda) + [\sum_{\mu=1}^{m} \mathfrak{d}(u_1, u_{1,\mu}) + \sum_{\mu=1}^{m} \mathfrak{d}(u_1, u_{2,\mu}) + \cdots + \sum_{\mu=1}^{m} \mathfrak{d}(u_1, u_{n,\mu})]$.
$= (n-1) + m + [2m + 2m + \cdots + upto\ (n-1)^{th}\ term]$
$= (n-1) + m + (n-1) \cdot 2m$
$= n - 1 + m(1 + 2n - 2)$
$= 2mn + n - m - 1$

Therefore, the normalized closeness centrality of u is $C_{NC}(u) = \frac{nm+n-1}{\sum_{x\in V}\mathfrak{d}(u,x)} = \frac{nm+n-1}{2mn+n-m-1}$.

If $u = u_\lambda$ is a non-central node of S_n then $\sum_{x\in V}\mathfrak{d}(u_\lambda,x) = \sum_{k=1}^{n}\mathfrak{d}(u_\lambda,u_k) + [\sum_{\mu=1}^{m}\sum_{l=\lambda}^{n}\mathfrak{d}(u_\lambda,u_{l,\mu}) + \sum_{\mu=1}^{m}\sum_{l=2}^{\lambda-1}\mathfrak{d}(u_\lambda,u_{l,\mu})] + \sum_{\mu=1}^{m}\mathfrak{d}(u_\lambda,u_{1,\mu})$.
$= [1+2(n-2)] + m + [3m+3m+\cdots+ upto\ (n-2)^{th}\ term] + 2m$
$= 2n-3+3m(n-2)+3m$
$= 3mn-3m+2n-3$
Therefore, the normalized closeness centrality of u is $C_{NC}(u) = \frac{nm+n-1}{\sum_{x\in V}\mathfrak{d}(u_\lambda,x)} = \frac{nm+n-1}{3mn-3m+2n-3}$.

If $u = u_{\lambda,\mu}$ where $\lambda = 2,\cdots,n$; $\mu = 1,2,\cdots,m$ then $\sum_{x\in V}\mathfrak{d}(u_{\lambda,\mu},x) = \sum_{k=1,k\neq\mu}^{m}\mathfrak{d}(u_{\lambda,\mu},u_{\lambda,k}) + \sum_{k=1}^{n}\mathfrak{d}(u_{\lambda,\mu},u_k) + \sum_{k=1}^{m}\mathfrak{d}(u_{\lambda,\mu},u_{1,k}) + [\sum_{k=1}^{m}\sum_{l=\lambda+1}^{n}\mathfrak{d}(u_{\lambda,\mu},u_{l,k}) + \sum_{k=1}^{m}\sum_{l=2}^{\lambda-1}\mathfrak{d}(u_{\lambda,\mu},u_{l,k})]$
$= (m-1) + [1+3(n-2)+2] + 3m + [4m(n-2)]$
$= m+3n-4+m(4n-8+3)$
$= 4mn-4m+3n-4$.

Hence, the normalized closeness centrality of u is $C_C(u) = \frac{nm+n-1}{\sum_{x\in V}\mathfrak{d}(u,x)} = \frac{nm+n-1}{4mn-4m+3n-4}$.

If $u = u_{1,\mu}$ where $\mu = 1,2,\cdots,m$ then $\sum_{x\in V}\mathfrak{d}(u_{1,\mu},x) = \sum_{k=1,k\neq\mu}^{m}\mathfrak{d}(u_{1,\mu},u_{1,k}) + \sum_{p=1}^{n}\mathfrak{d}(u_{1,\mu},u_p) + \sum_{k=1}^{m}\sum_{l=2}^{n}\mathfrak{d}(u_{1,\mu},u_{l,k})$
$= (m-1) + [1+2(n-1)] + [3m(n-1)]$
$= m+2n-2+3mn-3m$
$= 3mn-2m+2n-2$.

Hence, the normalized closeness centrality of u is $C_{NC}(u) = \frac{nm+n-1}{\sum_{x\in V}\mathfrak{d}(u,x)} = \frac{nm+n-1}{3mn-2m+2n-2}$. □

4.9 Closeness centrality of $K_n \odot P_m$

The corona graph $K_n \odot P_m$ is made using the corona product of the complete graphs S_n and the path graph P_m. It has $n+nm$ nodes. Let the nodes of K_n be $u_1, u_2, ..., u_n$, and that of P_m corresponding to the node u_λ be $u_{\lambda,1}, u_{\lambda,2}, ..., u_{\lambda,m}$. Figure 10 displays a corona graph $K_3 \odot P_4$.

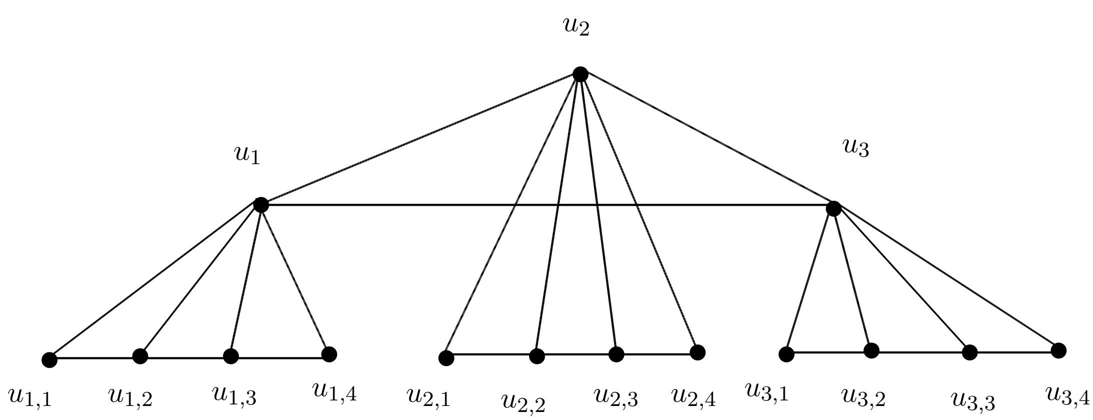

Figure 10: Corona graph $K_3 \odot P_4$

.

Theorem 4.9. *The $C_{NC}(v)$ of an arbitrary node v of $K_n \odot P_m$ is given by*

$$C_{NC}(u) = \begin{cases} \frac{nm+n-1}{2mn+n-m-1}, & \text{if } v = u_\lambda \in K_n \text{ and } \lambda = 1, 2, \cdots, n \\ \frac{nm+n-1}{3mn-m+2n-4}, & \text{if } v = u_{\lambda,\mu} \in P_m,\ \lambda = 1, 2, \cdots, n \text{ and } \mu = 1, m \\ \frac{nm+n-1}{3mn-m+2n-5}, & \text{if } v = u_{\lambda,\mu} \in P_m,\ \lambda = 1, 2, \cdots, n \text{ and } \mu = 2, 3, \cdots, m-1. \end{cases}$$

.

Proof. Suppose that the nodes of K_n are $u_1, u_2, ..., u_n$, and that of P_m corresponding to the node u_λ are $u_{\lambda,1}, u_{\lambda,2}, ..., u_{\lambda,m},\ \lambda = 1, 2, ..., n$. If $v = u_\lambda$ then $\sum_{x\in V} \mathfrak{d}(u_\lambda, x) = \sum_{k=1}^{n} \mathfrak{d}(u_\lambda, u_k) + \sum_{\mu=1}^{m} \mathfrak{d}(u_\lambda, u_{\lambda,\mu}) + [\sum_{\mu=1}^{m} \sum_{l=\lambda+1}^{n} \mathfrak{d}(u_\lambda, u_{l,\mu}) + \sum_{\mu=1}^{m} \sum_{l=1}^{\lambda-1} \mathfrak{d}(u_\lambda, u_{l,\mu})]$
$= (n-1) + m + [2m + 2m + \cdots + upto\ (n-1)^{th}\ term]$
$= (n-1) + m + 2m(n-1)$
$= n - 1 + m(1 + 2n - 2)$
$= 2mn + n - m - 1$

Therefore, the normalized closeness centrality of v is $C_{NC}(v) = \frac{nm+n-1}{\sum_{x\in V} \mathfrak{d}(u_i,x)} = \frac{nm+n-1}{2mn+n-m-1}$.

If $v = u_{\lambda,\mu}$, where $\lambda = 1, 2, \cdots, n$; $\mu = 1, m$ then $\sum_{x\in V} \mathfrak{d}(u_{\lambda,\mu}, x) = \sum_{k=1}^{m} \mathfrak{d}(u_{\lambda,\mu}, u_{\lambda,k}) + [\sum_{k=1}^{m} \sum_{l=\lambda+1}^{n} \mathfrak{d}(u_{\lambda,\mu}, u_{l,k}) + \sum_{k=1}^{m} \sum_{l=1}^{\lambda-1} \mathfrak{d}(u_{\lambda,\mu}, u_{l,k})] + [\sum_{l=\lambda}^{n} \mathfrak{d}(u_{\lambda,\mu}, u_l) + \sum_{l=1}^{\lambda-1} \mathfrak{d}(u_{\lambda,\mu}, u_l)]$
$= [1 + 2(m-2)] + 3m(n-1) + [1 + 2(n-1)]$
$= 3mn - m + 2n - 4.$

Hence, the normalized closeness centrality of v is $C_{NC}(v) = \frac{nm+n-1}{\sum_{x\in V} \mathfrak{d}(u,x)} = \frac{nm+n-1}{3mn-m+2n-4}$.

If $v = u_{\lambda,\mu}, \mu = 2, 3, \cdots, m-1, \lambda = 1, 2, \cdots, n$, then
$\sum_{x\in V} \mathfrak{d}(u_{\lambda,\mu}, x)$
$= \sum_{k=1}^{m} \mathfrak{d}(u_{\lambda,\mu}, u_{\lambda,k}) + [\sum_{k=1}^{m} \sum_{l=\lambda+1}^{n} \mathfrak{d}(u_{\lambda,\mu}, u_{l,k}) + \sum_{k=1}^{m} \sum_{l=1}^{\lambda-1} \mathfrak{d}(u_{\lambda,\mu}, u_{l,k})] + [\sum_{l=\lambda}^{n} \mathfrak{d}(u_{\lambda,\mu}, u_l) + \sum_{l=1}^{\lambda-1} \mathfrak{d}(u_{\lambda,\mu}, u_l)]$
$= [2 + 2(m-3)] + 3m(n-1) + [1 + 2(n-1)]$
$= 3mn - m + 2n - 5.$

Hence, $C_{NC}(v) = \frac{nm+n-1}{\sum_{x\in V} \mathfrak{d}(u,x)} = \frac{nm+n-1}{3mn-m+2n-5}$. □

4.10 Closeness centrality of corona product graph $K_n \odot K_m$

The graph $K_n \odot K_m$ is made by the corona product of the complete graph K_n and n copies of the complete graph K_m. The graph $K_n \odot K_m$ has $n + nm$ nodes. Let the nodes of K_n be $u_1, u_2, ..., u_n$, and that of K_m corresponding to the node u_i be $u_{i,1}, u_{i,2}, ..., u_{i,m}$. A Corona graph $K_3 \odot K_3$ is shown in Figure 11.

Theorem 4.10. *The normalized closeness centrality of any vertex u of the corona graph $K_n \odot K_m$ is given by*

$$C_{NC}(u) = \begin{cases} \frac{nm+n-1}{2mn-m+n-1}, & \text{if } u = u_i \in K_n, i = 1, 2, ..., n \\ \frac{nm+n-1}{3mn-2m+2n-2}, & \text{if } u = u_{i,j} \in K_m, i = 1, 2, ..., n \text{ and } j = 1, 2, 3, ..., m. \end{cases}$$

Proof. Consider that the nodes of K_n are $u_1, u_2, ..., u_n$ and that of K_m corresponding to the node u_i are $u_{i,1}, u_{i,2}, ..., u_{i,m},\ i = 1, 2, ..., n$. If $u = u_i$ is any vertex of K_n then $\sum_{x\in V} d(u_i, x) = [d(u_i, u_1) + d(u_i, u_2) + \cdots + d(u_i, u_{i-1}) + d(u_i, u_{i+1}) + \cdots + d(u_i, u_n)] + [\sum_{j=1}^{m} d(u_i, u_{i,j}) + \sum_{j=1}^{m} \sum_{i\neq k,k=1}^{n} d(u_i, u_{k,j})]$

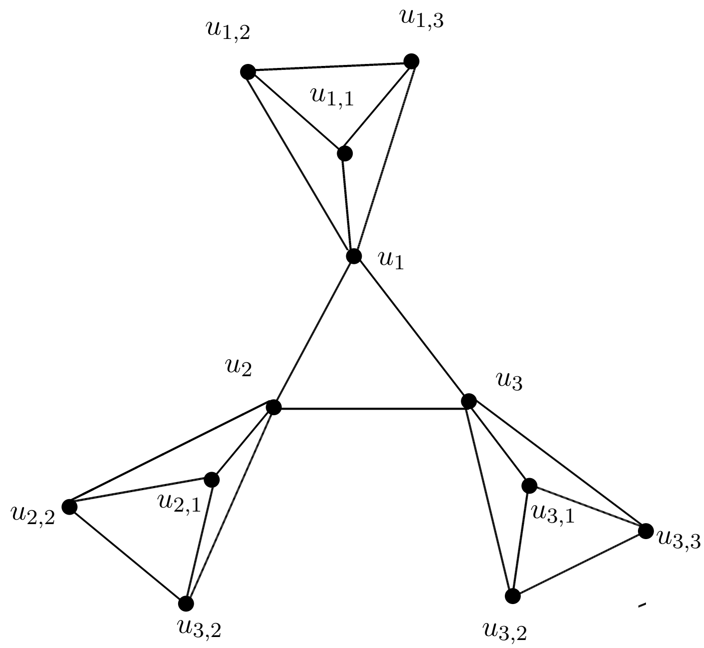

Figure 11: Corona graph $K_3 \odot K_3$

.

$= n - 1 + [m + 2m(n-1)]$
$= 2mn - 2m + m + n - 1$
$= 2mn - m + n - 1.$

Therefore, the normalized closeness centrality of u is $C_{NC}(u) = \frac{nm+n-1}{\sum_{x \in V} d(u,x)} = \frac{(nm+n-1)}{2mn-m+n-1}$.

If $u = u_{i,j}$ is a vertex of K_m, where $i = 1, 2, \cdots, n$; $j = 1, 2, \cdots, m$ then
$\sum_{x \in V} d(u_{i,j}, x) = [d(u_{i,j}, u_i) + \sum_{i \neq l, l=1}^{n} d(u_{i,j}, u_l)] + [\sum_{k=1}^{m} d(u_{i,j}, u_{i,k}) + \sum_{k=1}^{m} \sum_{l \neq i, l=1}^{n} d(u_{i,j}, u_{l,k})]$
$= [1 + 2(n-1)] + [m - 1 + 3m(n-1)]$
$= 2n - 1 + 3mn - 3m + m - 1$
$= 3mn - 2m + 2n - 2.$
Hence, the normalized closeness centrality of u is $C_{NC}(u) = \frac{nm+n-1}{\sum_{x \in V} d(u,x)} = \frac{nm+n-1}{3mn-2m+2n-2}$. □

5 Real applications

In this section, we present two real applications of our studied results.

5.1 Application in small-world network

Consider a small-world network: a type of network, where nodes are not typically directly connected but can still be reached from any other node through a short chain of intermediate nodes. These networks have high clustering, meaning nodes form tight-knit groups, while also maintaining short average path lengths between them. Small-world networks are common in social networks, the internet, and neural networks, and are essential for understanding connectivity patterns and information flow in complex systems. In 2015, Lv et al. [34] introduced recursive corona product graphs as a novel model for small-world networks. They denoted a g^{th} generation of recursive corona graph by $C_q(g+1)$, where $C_q(g+1) = C_q(g) \odot K_q, g \geq 0, k \geq 2$, with the initial condition $C_q(0) = K_q$. They examined various characteristics of their proposed corona product graph model, including order and size, degree distribution, average path length, and clustering coefficient. Additionally, they established results for all quantities in the recursive corona graphs that align with those found in real-world networks. Now, if we define a g^{th} generation of the recursive corona graph $C_q(g+1)$, where $C_q(g+1) = K_q \odot C_q(g)$, and keep all other conditions the same, then it also represents a graph model of a small-world network. Obviously, in the above two recursive corona graphs, if we take a complete graph, a path graph, a cycle graph, or a star graph as $C_q(g)$ (arbitrary graph), then we can readily identify the influential nodes in this type of small-world network up to the 2nd generation, using the results presented in this paper.

5.2 Application in transportation network

We can also apply our results for corona product graphs in real-life problems like selecting an emergency hub in a transportation network by transferring closeness centrality into fuzzy membership [35, 36], and integrating it with other fuzzy factors (that are criteria or variables that cannot be defined or measured precisely with crisp) such as cost, reliability, etc. through fuzzy reasoning operators (fuzzy AND, fuzzy OR, etc.). This produces a decision process that is both graph-theoretically informed and uncertainty-aware. We consider a transportation network which is similar to a corona product graph $C_n \odot P_m$, where cycle represents a main ring road, and paths represent feeder roads. We explain it through the following steps with the help of a small example.

Step 0: Setup (graph and notation) Graph: Consider a corona product $C_5 \odot P_3$ (see Figure 12) as a model of a transportation network. Node types (by symmetry): Let cycle vertices be $v_1, v_2, \ldots, v_5$, middle node and end nodes of the ith attached path be, respectively, $a_{i,2}$ and $\{a_{i,1}, a_{i,3}\}$. So, the total nodes in $C_5 \odot P_3$ is 20.

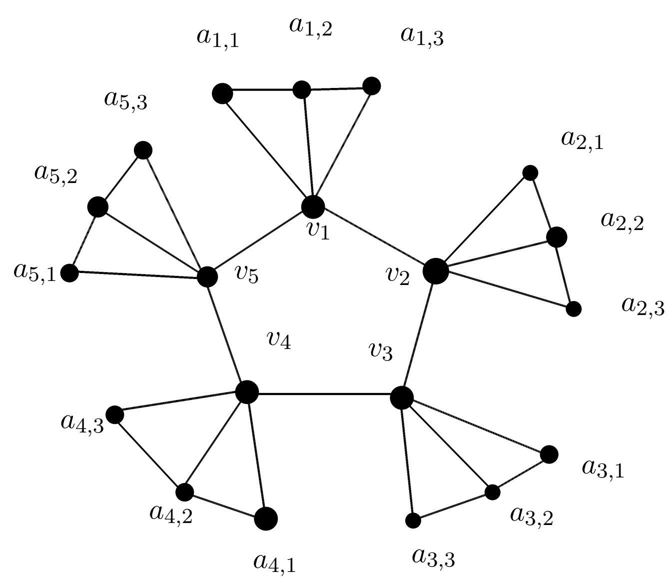

Figure 12: Corona graph $C_5 \odot P_3$

.

Step 1: Computation of crisp closeness centrality. We can easily compute C_{NC} with the help of the formula mentioned in subsection 4.6. For this graph (Figure 12), we will obtain exact values: Hare $n = 5, m = 3$. For cycle vertex: $C_{NC}(v_i) = 19/39 \approx 0.4871794872, i = 1, 2, ..., 5$.
For the attached middle vertex of the path P_3: $C_{NC}(a_{i,2}) = 19/55 \approx 0.3454545455, i = 1, 2, ..., 5$.
For the attached end vertex of the path P_3: $C_{NC}(a_{i,j}) = 1956 \approx 0.3392857143, i = 1, 2, ..., 5; j = 1, 3$

Step 2: Conversion of centrality to fuzzy membership (accessibility degree). We normalize C_{NC} linearly to $[0,1]$ using the following formula.

$\mu_{\text{centrality}}(x) = \dfrac{C_{\text{NC}}(x) - C_{\min}}{C_{\max} - C_{\min}}$, where $C_{\max}$ and $C_{\min}$ are, respectively, the maximum and minimum value of C_{NC} for all $v \in C_5 \odot P_3$. Here, $C_{\max} = \dfrac{19}{39}$ and $C_{\min} = \dfrac{19}{56}$. Also, $\mu_{\text{centrality}}(\text{for cycle node } v_i) = 1.000000$, $\mu_{\text{centrality}}(\text{mid node } a_{i,2} \text{ of the attached path}) \approx 0.041711$ and $\mu_{\text{centrality}}(\text{end nodes of the attached path}) = 0.000000$.

Interpretation: cycle nodes are maximally accessible; attached mid nodes are weakly accessible; attached end nodes are effectively not accessible on this normalized scale.

Step 3: Creation of fuzzy cost membership (low-cost degree). To choose or measure the setup cost for each node (lower cost higher membership), we use linear invertible mapping from $\mu_{\text{lowcost}}(x) : V \to [0,1]$ such that $\mu_{\text{lowcost}}(x) = \frac{C_{\text{Maxcost}} - \text{cost}(x)}{C_{\text{Maxcost}} - C_{\text{Mincost}}}$ so that the cheapest node has membership 1 and the most expensive 0, where $C_{Maxcost}$ and $C_{Mincost}$ are, respectively, the maximum and minimum costs among all costs of nodes. For instance, let costs of nodes be– cycle node cost is 60 (units), attached middle node's cost of path is 30, and attached end nodes' cost of path is 25. So, $C_{Mincost} = 25$, and $C_{Maxcost} = 60$. So, $C_{Maxcost} - C_{Mincost} = 35$. So, $\mu_{\text{lowcost}}(\text{cycle node}) = (60 - 60)/35 = 0.000000$, $\mu_{\text{lowcost}}(\text{mid node of attached path}) = (60 - 30)/35 = 30/35 = 0.8571428571$ and $\mu_{\text{lowcost}}(\text{end nodes of attached path}) = (60 - 25)/35 = 35/35 = 1.000000$.

Step 4: We combine fuzzy criteria to produce a good-hub degree in the following two cases.

Option A : Consider a strict fuzzy AND (min operator, conservative rule) Rule: good hub = central AND low-cost, i.e,, $\mu_{\text{good}}(x) = \min\{\mu_{\text{centrality}}(x), \mu_{\text{lowcost}}(x)\}$.
So, for the cycle nodes: $\mu_{\text{good}}(x) = \min\{1.0000,\ 0.0000\} = 0.0000.$,
for the middle node of the attached path: $\mu_{\text{good}}(x) = \min\{0.041711,\ 0.857143\} = 0.041711$, and
for the end nodes of the attached path: $\mu_{\text{good}}(x) = \min\{0.0000,\ 1.0000\} = 0.0000$.

Interpretation (min): the small centrality of non-cycle nodes kills their chance; only nodes with both reasonable centrality and low cost get non-zero membership. Here, middle nodes get a small nonzero score; cycle and end nodes score zero under strict AND.

Option B: Weighted aggregation (trade-off approach).

Many planners prefer a soft trade-off instead of a hard AND. Here, we consider a simple weighted score: $S(x) = w_c \cdot \mu_{\text{centrality}}(x) + w_{\text{cost}} \cdot \mu_{\text{lowcost}}(x), w_c + w_{cost} = 1$, where w_cw and w_{cost} are weights that represent the relative importance (or preference strength) assigned to the centrality criterion, and the low cost criterion, respectively, in the combined score $S(x)$.

For instances, let weights $w_c = 0.7$ (centrality more important), and $w_{\text{cost}} = 0.3$. Therefore, the value of $S(x)$ will be as follows.

$S(\text{for each cycle node}) = 0.7 \cdot 1.0000 + 0.3 \cdot 0.0000 = 0.7000$,
$S(\text{middle nodes of the attached path}) = 0.7{\cdot}0.041711 + 0.3{\cdot}0.857142857 \approx 0.0291977 + 0.2571429 = 0.2863406$, and $S(\text{end nodes of the attached path}) = 0.7 \cdot 0.0000 + 0.3 \cdot 1.0000 = 0.3000$.

Interpretation (weighted): ranking by score: cycle nodes (0.7000) >end nodes (0.3000) > mid nodes (0.2863). Thus, the cycle node is recommended as an emergency hub when centrality is emphasized; cheap ends are second-best; cost-friendly middle nodes are close to ends depending on weight.

Step 5: Final decision and recommendation

- If our decision rule is conservative (must be both central and cheap), use fuzzy AND (min). That tends to eliminate nodes with very low centrality (which may be undesirable for emergency hubs). In our example, the middle nodes only had tiny membership; cycle nodes and end nodes got zero because they were expensive in this toy cost assignment.
- If we want a trade-off (realistic planning), use a weighted aggregation. Choose weights to reflect policy (e.g., emergency response may weight accessibility higher, so, pick $w_c \geq 0.6$). With $w_c = 0.7$, we

selected a cycle node as the best hub.

Limitations: Although the corona product graph model is useful for studying complex networks, it has some limitations in solving real-life problems.

1. The structure of corona graphs is often too idealized and regular, which may not accurately represent the irregular and dynamic nature of real-world networks.

2. It does not easily capture weighted or directed relationships that are common in practical systems such as transportation, communication, or biological networks.

3. The model assumes static connections, while many real-life networks evolve over time, making the corona product less suitable for dynamic network analysis.

4. The proposed model is based on hypothetical examples. We did not use practical data.

6 Conclusion

Closeness centrality is one of the important variants of centrality measurement that is used to recognize the characteristic of a vertex in a network. It is possible to find the significant or influential vertex in biological networks, social networks, transportation networks, etc. In our paper, we correct the result presented by Eballe et al. for finding the vertex closeness centrality of the cycle graph C_n. We also present some new theoretical results for finding the normalized closeness centrality of some corona product graphs. We also present the applications of our studied results in small-world and transportation networks. In the future, we shall try to formulate the closeness centrality of more complex graphs as well as the corona product of general graphs with the help of the results presented in this paper. We also try to apply our studied results to solve different types of real-life problems such as the small-world network, the transportation problem based on the corona product graph model. Our proposed model has some limitations such as static connections, simple structure, undirected relations and hypothetical data. In addition, we have a plan to include a case study using real-world data in our future research to strengthen the practical applicability of the proposed model.

Acknowledgements: "We would like to express our sincere gratitude to the honorable editors and reviewers for their valuable comments and constructive suggestions, which significantly improved the quality of this paper. We also thank our colleagues and friends for their constant support and encouragement."

Conflict of Interest: "The authors declare no conflict of interest."

References

[1] Otte E, Rousseau R. Social network analysis: a powerful strategy, also for the information sciences. *Journal of Information Science*. 2002; 28(6): 441-453. DOI: https://doi.org/10.1177/016555150202800601

[2] Park K, Yilmaz A. A social network an analysis approach to analyze road networks. *Proceedings of the ASPRS Annual Conference, 26-30 Apr 2010, San Diego, USA, Bethesda (MD): American Society for Photogrammetry and Remote Sensing*. 2010. p.754-759.

[3] Kim DY. Closeness centrality: A Social network perspective. *Journal of International and Interdisciplinary Business Research*. 2019; 6: 115-122. DOI: https://doi.org/10.58809/EHGJ2799

[4] Wuchty S, Sadler P. Centers of complex network. *Journal of Theoretical Biology* 2003; 223(1): 45-53. DOI: https://doi.org/10.1016/S0022-5193(03)00071-7

[5] Ni C, Sugimoto CR, Jiang J. Degree, closeness and betweenness: Application of group centrality measurements to explore macro-disciplinary evolution diachronically. *Proceedings of the 13th International Conference of the International Society for Scientometrics and Informetrics; 4-7 Jul 2011; Durban, South Africa*. 2011. p.605-616.

[6] Kiss C, Bichler M. Identification of influencers- Measuring influence in customer networks. *Decision Support Systems*. 2008; 46(1): 233-253. DOI: https://doi.org/10.1016/j.dss.2008.06.007

[7] Das K, Samanta S, Pal M. Study on centrality measures in social networks: a survey. *Social Network Analysis and Mining*. 2018; 8: 13. DOI: https://doi.org/10.1007/s13278-018-0493-2

[8] Freeman LC. A set of measures of centrality based on betweenness. *Sociometry* . 1977; 40(1): 35-41. DOI: https://doi.org/10.2307/3033543

[9] Brandes U. On variants of shortest-path betweenness centrality and their generic computation. *Social Networks*. 2008; 30(2): 136145. DOI: https://doi.org/10.1016/j.socnet.2007.11.001

[10] Bavelas A. A mathematical model for group structures. *Applied Anthropology*. 1948; 7(3): 16-30.

[11] Sabidussi G. The centrality index of a graph. *Psychometrika*. 1966; 31(4): 581603. DOI: https://doi.org/10.1007/BF02289527

[12] Freeman LC. Centrality in social networks conceptual clarification. *Social Networks*. 1978; 1(3): 215-239. DOI: https://doi.org/10.1016/0378-8733(78)90021-7

[13] Newman MEJ. Scientific collaboration networks I, Network construction and fundamental results. *Physical Review E*. 2001; 64(1): 016131. DOI: https://doi.org/10.1103/PhysRevE.64.016131

[14] Brandes U. A faster algorithm for betweenness centrality. *The Journal of Mathematical Sociology*. 2001; 25(2): 163177. DOI: https://doi.org/10.1080/0022250X.2001.9990249

[15] Okamoto K, Chen W, Li XY. Ranking of Closeness Centrality for Large-Scale Social Networks. In: Preparata, F.P., Wu, X., Yin, J. (eds.) *Frontiers in Algorithmics. FAW 2008. Lecture Notes in Computer Science, vol. 5059.* Berlin, Germany: Springer; 2008. p.186-195. DOI: https://doi.org/10.1007/978-3-540-69311-6_21

[16] Park S, Park M, Kim H, kim K, et al. A closeness centrality analysis algorithm for workflow-supported social networks. 15*th International Conference on Advanced Communications Technology (ICACT), 27-30 Jan 2013, PyeongChang, Korea.* 2013. p.158-161.

[17] Yen CC, Yeh MY, Chen MS. An efficient approach to updating closeness centrality and average path length in dynamic networks. *IEEE 13th International Conference on Data Mining, 7-10 Dec 2013, Dallas, TX, USA.* 2013. p.867 - 876. DOI: https://doi.org/10.1109/ICDM.2013.135

[18] Kas M, Carley KM, Carley LR. Incremental closeness centrality for dynamically changing social networks. *Proceedings of the IEEE/ACM International Conference on Advances in Social Networks Analysis and Mining (ASONAM 2013), 25-28 Aug 2013; Niagara Falls, ON, Canada.* 2013. p.1250-1258. DOI: https://doi.org/10.1145/2492517.2500270

[19] Cohen E, Delling D, Pajor T, Werneck RF. Computing classic closeness centrality, at scale. *In Proceedings of the second ACM conference on Online social networks, 1-2 Oct 2014. Dublin, Ireland.* 2014. p.37-50. DOI: https://doi.org/10.48550/arXiv.1409.0035

[20] Crescenzi P, D'angelo G, Severini L, Velaj Y. Greedily improving our own closeness centrality in a network. *ACM Transactions on Knowledge Discovery from Data*. 2016; 11(1): 1-32. DOI: https://doi.org/10.1145/2953882

[21] Phuong-Hanh D, Hai CN, Khoa NK, Ngoc-Hoa N. An efficient parallel algorithm for computing the closeness centrality in social networks. *Proceedings of the 9th International Symposium on Information and Communication Technology. 6-7 Dec 2018, Da Nang, Vietnam.* New York: Association for Computing Machinery; 2018. p.456-462. DOI: https://doi.org/10.1145/3287921.3287981

[22] Jin H, Qian C, Yu D, et al. Parallel computation of hierarchical closeness centrality and applications. *World Wide Web*. 2019; 22 (6): 30473064. DOI: https://doi.org/10.1007/s11280-018-0605-y

[23] Hu R, Liao L, Chen C, Zhang G. Closeness centrality measures in fuzzy enterprise technology innovation cooperation networks. *Fuzzy Information and Engineering*. 2019; 11(4): 494-505. DOI: https://doi.org/10.1080/16168658.2020.1764465

[24] Shukla K, Regunta SC, Tondomker SH, Kothapalli K. Efficient parallel algorithms for betweenness- and closeness-centrality in dynamic graphs. *Proceedings of the 34th ACM International Conference on Supercomputing, 29 June-2 July 2020, Barcelona, Spain.* New York: Association for Computing Machinery; 2020. p.110. DOI: https://doi.org/10.1145/3392717.3392743

[25] Skibski, O. Closeness centrality via the condorcet principle. *Social Networks*. 2021; 74: 1318. DOI: https://doi.org/10.48550/arXiv.2112.00494

[26] Eballe RG, Cabahug IS. Closeness centrality of some graph families. *International Journal of Contemporaray Mathematical Sciences*. 2021; 16(4): 127-134. DOI: https://doi.org/10.12988/ijcms.2021.91609

[27] Eballe RG, Balingit CMR, Cabahug IS, et al. Closeness centrality in graph products. *Advances and Applications in Discrete Mathematics* 2023; 39 (1): 29-41. DOI: https://.doi.org/10.17654/0974165823034

[28] Nandi S. Closeness centrality of complete, fan and wheel graphs. *Journal of Research in Applied Mathematics*. 2022; 8(9): 44-48.

[29] Evans ST, Chen B. Linking the network centrality measures closeness and degree. *Communication Physics*. 2022; 5(1): 172. DOI: https://doi.org/10.1038/s42005-022-00949-5

[30] Lpez-Rourich MA, Rodrguez-Prez FJ. Efficient data transfer by evaluating closeness centrality for dynamic social complex network-inspired routing. *Applied Sciences* 2023; 13(19): 10766. DOI: https://doi.org/10.3390/app131910766

[31] Liu Z, Ye J, Zou Z. Closeness centrality on uncertain graphs. *ACM Transactions on the Web*. 2023; 17(4): 1-29. DOI: https://doi.org/10.1145/3604912

[32] Mahapatra R, Samanta S, Pal M. New concept of centrality measurement in fuzzy social networks. *Journal of Intelligent and Fuzzy Systems*. 2024; 46(4): 1 - 15. DOI: https://doi.org/10.3233/JIFS-232602

[33] Mahapatra R, Allahviranloo T, Kalampakas A, Samanta S. Centrality measure in Quantum graphs and its application. In: Allahviranloo T, Samanta S. (eds.) *Quantum Theory and Fuzzy Systems: Traversing uncertainty in group decision-making and social networks. Studies in Computational Intelligence, vol. 1186.* Springer; 2025. p.307-320. DOI: https://doi.org/10.1007/978-3-031-82058-8_14

[34] Lv Q, Yi Y, Zhang Z. Corona graphs as a model of small-world networks. *Journal of Statistical Mechanics Theory and Experiment*. 2015; 2015(11): 1 - 21. DOI: https://doi.org/10.1088/1742-5468/2015/11/P11024

[35] Mordeson JN, Josy A, Mathew S. Weighted graphs and fuzzy graphs. *Transactions on Fuzzy Sets and Systems*. 2023; 2(1): 195-203. DOI: https://doi.org/10.30495/tfss.2022.1967246.1048

[36] Hong TP, Chen CH, Li YK, Wu MT. Using fuzzy C-means to discover concept-drift patterns for membership functions. *Transactions on Fuzzy Sets and Systems*. 2022; 1(2): 21 - 31. DOI: https://doi.org/10.30495/tfss.2022.1958730.1030

Shaoli Nandi
Department of Mathematics
Raja N. L. Khan Womens College (Autonomous)
Midnapore, India
E-mail: prof.snandi81@gmail.com

Sukumar Mondal
Department of Mathematics
Raja N. L. Khan Womens College (Autonomous)
Midnapore, India
E-mail: sukumarmondal@rnlkwc.ac.in

Sambhu Charan Barman
Department of Mathematics
Shahid Matangini Hazra Government General Degree College for Women
Medinipur, India
E-mail: barman.sambhu@gmail.com