A DEA Approach for Optimal Value Efficiency (OVE) Conditions Using Supporting Hyperplanes

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Abstract – This paper presents a control-oriented approach to measuring Optimal Value Efficiency (OVE) in Data Envelopment Analysis (DEA) under negative data conditions, particularly when the data originate from interval scale (IS) variables such as differences in sales or other financial indicators. These IS variables are first decomposed into two ratio scale (RS) components to preserve DEA consistency. The OVE framework, originally introduced by Halme, etc. incorporates the decision maker's preferences into the evaluation process. The method first identifies technically efficient Decision Making Units (DMUs) and supports the decision maker in selecting the Most Preferred Solution (MPS) among them. To approximate the unknown value function's indifference contour at the MPS, a supporting hyper plane is constructed on the production possibility set (PPS). The value function is assumed to be strictly increasing in outputs, decreasing in inputs, and pseudo-concave. We implement this within a radial DEA model under Variable Returns to Scale (VRS), where both input minimization and output maximization are considered. The proposed approach effectively supports efficiency evaluation even with negative data, offering a robust tool for real-world decision environments involving dynamic systems and control strategies. The methodological innovation lies in the technique for approximating the decision maker's unknown and assumed pseudo-concave value function. Instead of relying solely on complex preference elicitation, the method constructs a supporting hyperplane to the Production Possibility Set (PPS) at the identified Most Preferred Solution. This hyperplane serves as a local linear approximation of the value function's indifference contour at the MPS. The weights for this hyperplane—which define the trade-offs between inputs and outputs—are derived not from an arbitrary assignment, but from the optimal dual solution of a specially formulated radial DEA model (specifically, a combined Banker-Charnes-Cooper orientation model) applied to the decomposed dataset. This approach allows for the calculation of OVE scores through a simple geometric projection. For a given DMU, its distance to the approximated indifference contour is measured along a chosen directional vector (e.g., an output-oriented direction). The resulting OVE score quantifies how much a DMU must improve relative to the MPS to reach a point deemed equally desirable according to the approximated preferences embedded in the supporting hyperplane. The proposed method was demonstrated using a numerical example involving 13 DMUs with one input and two outputs, where one output contained negative values. After decomposing the IS variable, technical efficiencies were computed, and an efficient DMU was selected as the MPS. The dual DEA model was solved to obtain the hyperplane weights. Subsequently, OVE scores for all DMUs were calculated. A comparative analysis with established methods by Halme et al. showed that the proposed approach can yield distinct and often more conservative inefficiency estimates, as seen in the example where several DMUs received significantly lower OVE scores (e.g., U6: 0.12 vs. 0.54/0.60; U8: 0.13 vs. 0.53/0.58), highlighting the impact of the specific preference approximation mechanism.

Keywords: Optimal Value Efficiency, Interval Scale Variables, Most Preferred Solution (MPS), Variable Returns to Scale (VRS), Control-Oriented Evaluation

1. Introduction

In many real-world applications, negative data arise naturally, particularly from interval scale (IS) variables such as profits, sales differences, or financial changes. IS data, by definition, lack a true zero point, rendering ratio-based analysis invalid [1]. When IS variables are used as inputs or outputs in DEA, they must first be decomposed into two ratio scale (RS) components to enable accurate evaluation. OVE measures the distance of a DMU from the decision maker's Most Preferred Solution (MPS), defined on the production possibility set (PPS). Since the value function (VF) representing DM preferences is typically unknown, an indifference contour at the MPS is approximated by a supporting hyperplane. This hyperplane is used to assess how far each inefficient DMU is from the preferred frontier. The proposed method utilizes dual forms of radial DEA models, specifically adapted to IS data, to construct the supporting hyperplane without solving additional linear programming problems in the output direction. Our approach builds upon the structure proposed by

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Halme et al. while maintaining computational efficiency and conceptual clarity. It enables consistent evaluation of OVE scores even in the presence of negative data, offering a practical framework for decision-making in dynamic and control-sensitive environments.[2].

Data Envelopment Analysis (DEA), introduce d by [3], is a non-parametric method for assessin g the technical efficiency of Decision Making U nits (DMUs) performing similar tasks. Classical DEA models are value-neutral and do not requir e any input regarding the decision maker's (DM' s) preferences. While this impartiality is consider ed a strength, several extensions of DEA have b een developed to incorporate preference informati on, including the Value Efficiency (VE) framewo rk proposed by [3], and further elaborated by [4, 5,6].

The remainder of this paper is structured as f ollows: Section 2 reviews the treatment of interv al scale variables and background on value effici ency. Section 3 outlines the proposed method for measuring OVE scores. Section 4 provides a nu merical example, and Section 5 concludes with fi nal remarks and suggestions for future research.

2. IS Data and Value Efficiency Analysis

A critical issue with IS variables is the lack of a meaningful zero point, which prohibits ratio -based operations such as division. Therefore, we strongly recommend decomposing each IS varia ble into two RS variables, even if the original va lues appear to be positive. This ensures consisten cy within the DEA framework and enables reliab le estimation of efficiency scores. Negative data values are frequently encountered in DEA applic ations, particularly when variables are defined as the difference between two ratio scale (RS) com ponents. As noted by [7], several commonly use d variables in the DEA literature - such as the ra te of GDP growth per capita, profit, and taxes (e .g., profit = income – cost) - can take negative v alues. These are typically derived from interval s cale (IS) measurements.

Assume that among the total set of inputs an d outputs, a subset is measured on an interval sc ale. Each IS variable is replaced by a pair of RS variables such that their difference recovers the original IS variable. In this structure: -The new input matrix includes: (1) RS input variables deri ved from IS inputs (minuends), and (2) RS variab les originating from IS outputs (subtrahends). -T he new output matrix include: (1) RS output var iables derived from IS inputs (subtrahends), and (2) RS variables derived from IS outputs (minuen ds). In the dual formulation of the DEA model, t he coefficients of these newly introduced RS var iables are set equal. Each new constraint introdu ced in the dual problem corresponds to a new va riable in the primal formulation.

The following section presents the structure o f the combined radial BCC model incorporating t he decomposed IS variables, where each IS input /output is transformed into a pair of RS variables (one input and one output).

$$
\begin{aligned}
&\min \sum_{i=1}^{m+s} v_i x_{io} - \sum_{r=1}^{p+t} \mu_r y_{ro} + u \\
&\text{subject to} \\
&\sum_{i=1}^{m+s} v_i x_{ij} - \sum_{r=1}^{p+t} \mu_r y_{rj} + u \ge 0, \\
&\sum_{r=1}^{p+t} \mu_r y_{ro} + \sum_{i=1}^{m+s} v_i x_{io} = 1 \\
&\mu_r - v_i = 0, \quad i = r = 1,\ldots,t+s, \\
&\mu_r, v_i \ge 0, \quad \forall r, \forall i
\end{aligned}
\tag{1}
$$

The model (1) is the dual of the following ra dial combined BCC primal model.

$$
\begin{aligned}
&\max \quad \sigma \\
&\text{subject to} \\
&\sum_{j=1}^{n} y_{rj}\lambda_j - \sigma y_{ro} - \pi_r \ge y_{ro}, \quad r = 1,\ldots,t+s \\
&\sum_{j=1}^{n} y_{rj}\lambda_j - \sigma y_{ro} \ge y_{ro}, \quad r = t+s+1,\ldots,p+t, \\
&\sum_{j=1}^{n} x_{ij}\lambda_j + \sigma x_{io} - \pi_i \le x_{io}, \quad i = 1,\ldots,t+s \\
&\sum_{j=1}^{n} x_{ij}\lambda_j + \sigma x_{io} \le x_{io}, \quad i = t+s+1,\ldots,m+s \\
&\sum_{j=1}^{n} \lambda_j = 1, \\
&\lambda_j \ge 0, \quad j = 1,\ldots,n
\end{aligned}
\tag{2}
$$

Naturally, in addition to the combined orienta tion model presented above, standard input- or o utput-oriented DEA models can also be considere d. Specifically, setting the combination parameter θ to zero in model (2) yields the output-oriente d formulation. The input-oriented model can be derived analogously. It is important to note that efficient DMUs remain efficient even after deco mposing IS variables. However, the increase in t he number of variables may result in some ineffi cient units becoming efficient. Nonetheless, only the efficiency scores of originally inefficient unit s are affected. For further details, refer to [1], " Dealing with Interval Scale Data in DEA [8].

The purpose of Value Efficiency Analysis (V EA) is to evaluate the relative efficiency of each unit with respect to the indifference contour of a Decision Maker's (DM's) Value Function (VF), passing through the Most Preferred Solution (MPS). This would be straightforward if the VF were known explicitly. However, in practice, the VF is unknown and must be approximated. VEA integrates the DM's preference information regarding desirable input-output combinations into the DEA framework. The MPS is a (real or virtual) DMU located on the efficient frontier, representing the most desirable values of inputs and outputs. [1] proposed that the DM's unknown VF is **pseudo-concave**, **strictly increasing in outputs**, and **strictly decreasing in inputs**, attaining a **maximum at the MPS.** [1, 9].

To approximate the indifference contour at the MPS, Authors in [3] suggested constructing a **tangent hyperplane** to the contour by defining a region that includes all input-output combinations that are less preferred or equally preferred to the MPS. This region is represented using a **cone** that approximates the set of such vectors. VEA can be implemented through linear programming using standard DEA models. A DMU is considered **value inefficient** with respect to any strictly increasing, pseudo-concave VF (with maximum at the MPS) if the optimal value of the following problem is strictly positive. [1, 10]

$$\begin{aligned} \max \quad & Z=\sigma+\varepsilon(1^T s^+ + 1^T s^-) \\ subject\ to \quad & \\ & Y\lambda-\sigma w^y - s^+ = g^y, \\ & X\lambda+\sigma w^x + s^- = g^x,\ F\lambda+\eta=d, \\ & \lambda_j \ge 0,\ if\ \lambda_j^*=0,\ j=1,\cdots,n \\ & \eta_j \ge 0,\ if\ \eta_j^*=0,\ j=1,\cdots,k \end{aligned} \tag{3}$$

(3)

And, the dual of the model (3) is:

$$\begin{aligned} \min \quad & W=v^T g^x-\mu^T g^y+\rho^T d \\ subject\ to \quad & \\ & -\mu^T y_j+v^T x_j+\rho^T F_j=0, \\ & j\in\{j|\lambda_j^*>0, j=1,\cdots,n\}, \\ & -\mu^T y_j+v^T x_j+\rho^T F_j\ge 0, \\ & j\in\{j|\lambda_j^*=0, j=1,\cdots,n\}, \\ & \mu^T w^y+v^T w^x=1,\quad \mu,v\ge\varepsilon 1 \end{aligned}$$

(4)

$$\begin{aligned} & \rho_j=0,\ if\ \eta_j^*=0,\ j=1,\cdots,k, \\ & \rho_j\ge 0,\ if\ \eta_j^*>0,\ j=1,\cdots,k \end{aligned}$$

Where λ^* and η^* correspond to the MPS: $y^*=Y\lambda^*$, $x^*=X\lambda^*$. The only difference compared with a standard primal DEA model is that some variables of λ,η are allowed to have negative values. This simple modification of the DEA model makes it possible to take into account value judgments in the form of the MPS.

3.Optimal Value Efficiency (OVE) Scores

In this section, we propose an approach to approximate the indifference contour of the unknown Value Function (VF) at the Most Preferred Solution (MPS) using a supporting hyperplane on the Production Possibility Set (PPS). This approximation enables us to evaluate the value efficiency of Decision Making Units (DMUs) in the presence of negative data .

To construct the supporting hyperplane, we utilize the dual formulation of a radial DEA model. The resulting dual weights assigned to input and output variables serve as the components of the normal vector of the supporting hyperplane at the MPS. These weights reflect the marginal rate of substitution between inputs and outputs according to the decision maker's preferences embedded within the mode.

3.1 *Combined model after decomposing IS variables (BCC-Primal-OVE)*

$$\begin{aligned} \max \quad & \sigma+\varepsilon\left(\sum_{i=1}^{m+s} s_i^- + \sum_{r=1}^{p+t} s_r^+\right) \\ subject\ to \quad & \\ & \sum_{j=1}^{n} y_{rj}\lambda_j-\sigma y_{ro}-\pi_r-s_r^+=y_{ro}, \\ & r=1,\ldots,t+s \\ & \sum_{j=1}^{n} y_{rj}\lambda_j-\sigma y_{ro}-s_r^+=y_{ro}, \\ & r=t+s+1,\ldots,p+t \\ & \sum_{j=1}^{n} x_{ij}\lambda_j+\sigma x_{io}-\pi_i+s_i^-=x_{io}, \\ & i=1,\ldots,t+s \end{aligned} \tag{5}$$

(5)

$$\begin{aligned} & \sum_{j=1}^{n} x_{ij}\lambda_j+\sigma x_{io}+s_i^-=x_{io}, \\ & i=t+s+1,\ldots,m+s \\ & \sum_{j=1}^{n}\lambda_j=1, \\ & s_i^-,s_r^+\ge\varepsilon,\forall i,r,\quad \lambda_j\ge 0,\ \text{If}\ \lambda_j^*=0, j=1,\ldots,n \end{aligned}$$

Where λ^* is corresponds to the MPS after the decomposing IS variables. The dual of the above model is as following:

3.2 (BCC-Dual-OVE)

$$\min \sum_{i=1}^{m+s} v_i x_{io} - \sum_{r=1}^{p+t} \mu_r y_{ro} + u$$

subject to

$$\sum_{i=1}^{m+s} v_i x_{ij} - \sum_{r=1}^{p+t} \mu_r y_{rj} + u = 0 \quad , \quad j \in \{ j \mid \lambda_j^* > 0, j = 1, \ldots, n \}$$

$$\sum_{i=1}^{m+s} v_i x_{ij} - \sum_{r=1}^{p+t} \mu_r y_{rj} + u \geq 0 \quad , \quad j \in \{ j \mid \lambda_j^* = 0, j = 1, \ldots, n \}$$

$$\sum_{r=1}^{p+t} \mu_r y_{ro} + \sum_{i=1}^{m+s} v_i x_{io} = 1$$

(6)

$$\mu_r - v_i = 0, \quad i = r = 1, \ldots, t+s \,, \quad \mu_r, v_i \geq \varepsilon \quad \forall r, \forall i$$

The obtained hyperplane from the model (6) is tangent on PPS at DMU_j that $j \in \{ j \mid \lambda_j^* > 0, j = 1, \ldots, n \}$ and in fact these are the reference DMUs of MPS. Since the MPS is efficient, so set on the efficient frontier and usually the set $\{ j \mid \lambda_j^* > 0, j = 1, \ldots, n \}$ is including only from MPS. Hence, this hyperplane passing among MPS.

The first, we obtain MPS. For its finding to compute the technical efficiency of each DMU (after the decomposing IS variables) and pick out MPS among the efficient DMUs by aid DM. We want to approximate the value θ such that $(x_j, y_j) + \theta (W^x, W^y) = (\bar{x}, \bar{y})$, where $(\bar{x}, \bar{y})$ is the projected point of DMU_j on the indifference contour VF at MPS which we utilizing the supporting hyperplane at MPS instead of it. We use from the model (BCC-Dual_OVE) and suppose that (v^*, μ^*, u^*) is its optimal solution. So the equation of the supporting hyperplane of VF at MPS is as $v^{*T} x - \mu^{*T} y + u^* = 0$. Hence, we have: $v^{*t} (x_j + \theta W^x) - \mu^{*t} (y_j + \theta W^y) + u^* = 0$. In other words, $\theta = -\dfrac{v^{*t} x_j - \mu^{*t} y_j + u^*}{v^{*t} W^* - \mu^{*t} W^y}$. We purpose obtain OVE scores only in output orientation. So as to, we use from the output oriented direction $(W^x, W^y) = (0, y_j)$, that thus we have: $\theta = \dfrac{v^{*t} x_j + u^*}{\mu^{*t} y_j} - 1$. Note that we consider the case when both the two new variable in decomposing the IS variable into two ratio scale variables as objectives and don't consider the case when one of the new variables is non-discretionary by character.

The basic idea of VEA is illustrated in Fig. 1. We assume that the DMUs produce two outputs and all consume the same amount of one input. For evaluating point A, we would like to assess the ratio $\dfrac{OA^4}{OA}$. Because the VF is unknown, so we can assume that the supporting hyperplane at MPS is tangent on the indifference contour and if we use from the supporting hyperplane then obtain the OVE score as the ratio $\dfrac{OA^2}{OA}$ (equal to result of the method of Halme et al.) or $\dfrac{OA^3}{OA}$ (this the OVE score is better from the method of Halme et al.). VE score θ_j for evaluate DMU_j is as

$$\theta_j = \frac{v^{*T} x_j + u^*}{\mu^{*T} y_j} - 1 = \frac{OA^2}{OA} - 1 = \frac{AA^2}{OA} \quad \text{or}$$

$$\theta_j = \frac{OA^3}{OA} - 1 = \frac{AA^3}{OA}.$$

In fact, since may the supposing hyperplane at MPS be as the line ℓ, thus we can said that $\dfrac{OA^2}{OA} \leq \theta_j \leq \dfrac{OA^3}{OA}$.

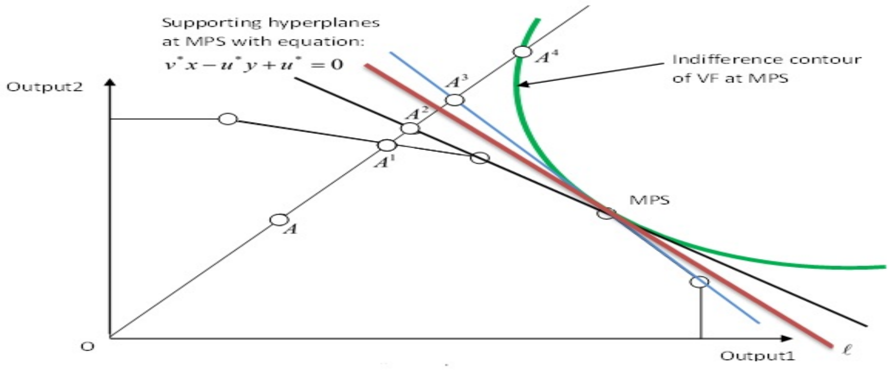

Fig.1 Illustration of VEA

4.Illustrative Example

In this section, we demonstrate the implementation of the proposed approach using the dataset provided in Table 1. The analysis involves evaluating the Optimal Value Efficiency (OVE) of 13 Decision Making Units (DMUs), each of which consumes a single input to produce two outputs. It is important to note that some of the DMUs include negative output values. This occurs as a result of one of the output variables being measured on an interval scale (IS), which, after decomposition, yields both positive and negative values. The example highlights how the model accommodates negative data while computing OVE scores through the supporting hyperplane approximation.

Table1. Units And their input/output variable values.

Units	U1	U2	U3	U4	U5	U6	U7	U8	U9	U10	U11	U12	U13
$O1$	58	48	45	35	34	25	25	25	16	15	14	13	4
$O2$	-16	-17	-6	5	4	-12	3	-14	2	-4	1	1	3
y	38	32	33	36	35	19	31	26	30	31	21	19	7
x	54	49	39	31	31	31	28	40	28	35	20	18	4
I	50	48	49	49	48	50	47	47	45	48	47	35	19

We need that decomposing the IS output variable $O2$ into two RS variables which $O2$ generated by difference two the RS values. For this example we have: $p = 2$, $m = 1$, $t = 0$, $s = 1$.

Table 2. The new variables values after decomposing $O2$.

Units	U1	U2	U3	U4	U5	U6	U7	U8	U9	U10	U11	U12	U13
x_1	54	49	39	31	31	31	28	40	28	35	20	18	4
y_1	38	32	33	36	35	19	31	26	30	31	21	19	7
x_2	50	48	49	49	48	50	47	47	45	48	47	35	19
y_2	58	48	45	35	34	25	25	25	16	15	14	13	4

To compute the technical efficiency for each DMU with the new variables values after decomposing, utilizing from the data of table 2. Each units U1, U5, and is efficient. We picking the unite A as MPS and $\lambda_A^* = 1$. Both variables y_1, x_1 are objectives. U13 The variable x_1 can be viewed either as output or input. We considered that as input. For finding output/input weights, we use from model (BCC-Dual-OVE). Thus we have:

$$\min \quad 54v_1 + 50v_2 - 38\mu_1 - 58\mu_2 + u$$
$$subject\ \ to \quad 54v_1 + 50v_2 - 38\mu_1 - 58\mu_2 + u = 0, \quad 49v_1 + 48v_2 - 32\mu_1 - 48\mu_2 + u \geq 0$$
$$\ldots \tag{8}$$
$$54v_1 + 50v_2 + 38\mu_1 + 58\mu_2 = 1, \quad -v_1 + \mu_1 = 0$$

The obtained weights are as $v_1^* = 0.004987$, $v_2^* = 0.005275$, $\mu_1^* = 0.004987$, $\mu_2^* = 0.004783$, $u^* = -0.006613$.

Table 3. Obtained OVE scores and comparison the proposed method with the Halme et al methods.

Units	*U1*	*U2*	*U3*	*U4*	*U5*	*U6*	*U7*
The model BCC combined of Halme et al.	0.00	0.11	0.02	0.00	0.00	0.54	0.17
the model BCC output oriented of Halme et al.	0.00	0.14	0.09	0.13	0.00	0.6	0.14
the OVE method	0.00	0.08	0.01	0.00	0.00	0.12	0.05
Units	U8	U9	U10	U11	U12	U13	
The model BCC combined of Halme et al.	0.53	0.37	0.59	0.64	0.32	0.00	
the model BCC output oriented of Halme et al.	0.58	0.28	o.49	0.50	0.26	0.00	
the OVE method	0.13	0.03	0.06	0.19	0.16	0.00	

5.Conclusion

The paper proposes using scalarization-based methods for multi-objective optimization to produce a range of Pareto optimal solutions. Decision-makers can then select the best solution based on their preferences. Our paper introduces a new technique for estimating desired efficiency. We accomplish this by utilizing the supporting hyperplane at MPS and assuming that it is tangent to the unknown preferred function. The weights of different solutions on the Pareto front are adaptively determined by using information from the previously obtained solutions' positions. We also assume that we can find common weights by solving a dual model for the decomposed IS variables. A novel optimal model based on deviation is proposed to determine objective weights. These weights will help us calculate the OVE scores more accurately. Though we can use methods that utilize the original IS variables without decomposing the data, finding the most preferred weights will improve the accuracy of the OVE scores. There is also the option of using CCR models instead of BCC models, which will change the PPS and the supporting hyperplane, resulting in a different measure of OVE. While using optimal weights of each decision-maker unit instead of common weights may take us away from the desired PE for each unit, the decision-maker's preferences can help achieve real performance. Finally, this process can also be used to obtain cost efficiency, provided that the cost function is unknown. Also, in this paper the main contribution of this paper is introducing a particular way to estimate value efficiency by the supporting hyperplane at MPS and we assume that this is tangent hyperplane of (unknown) value function at MPS. In other words, by solving a dual model for the obtained data of decomposing of the IS variables found weights for input/output variables. Then the optimal value efficiency scores, produce by simple calculations. We can use from common weights for output/input variables after decomposing IS variables and by those weights obtain value efficiency scores. We can use methods that utilizing the original IS variables without decomposing data like Zionts-Wallenius method in MOLP and so on.

Reference

[1] Halme M., Joro J., Korhonen P., Salo S., Wallenius J., A value efficiency approach to incorporating preference information in data envelopment analysis, Management Science, 45(1999) 103-115.

[2] Gerami J., Mozafari M.R., Wanke P.F., Improving information reliability of non-radial value efficiency analysis: An additive shacks based measure approach, European Journal Operational Research, 298(3) (2022) 967-978.

[3] Charnes, W.W. Cooper, E. Rhoders, Measuring the efficiency of decision making units, 2 (1978) 429-444.

[4] Korhonen P., Syrjanen M.J., on the interpretation of value efficiency, Journal Productivity Analysis, 24(2005) 197-201.

[5] Zohrabandan M., Using Zionts-Wallenius method to improve estimate of value efficiency in DEA, Applied Mathematical Modeling, 35(2011) 3769-3776.

[6] Joro T., Korhonen P., Zionts S., An interactive approach to improve estimates of value efficiency in data envelopment analysis, European Journal Operational Research, 149(2003) 688-699.

[7] Pastor, j. (1996) Translation invariance in Data Envelopment Analysis: A generalization, Anal. Oper. Res, 66, 93-102.

[8] S. Stanek, D. Kuchta, (2020), Increasing earned value analysis efficiency for IT projects, Journal of Decision Systems, 29(1):1-9.

[9] Hosseinzadeh Lotfi, F., Ebrahimnejad A., Vaez-Ghasemi M., Moghaddas Z. (2020). Data envelopment analysis with R, Springer International Publishing.

[10] Jahanshahloo, G R Zohrebandian, M Alinezhad A Abbasian H Abbasian Naghneh, S Kiani Mavi, R (2010). Finding common weights based on the DM's preference information. Journal of the Operational Research Society, 1796-1800.