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A New Approach of Generalized Interval-Valued Neutrosophic Rough Set and Its Application in Medical Diagnosis

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Abstract. This study introduces a framework of Generalized Interval-Valued Neutrosophic Rough Sets (GIVNRSs) to improve the modeling of uncertainty, indeterminacy, and incompleteness in complex data. The framework extends classical rough set theory by incorporating interval-valued neutrosophic elements, providing a richer representation of imprecise information. We establish the fundamental operations of GIVNRSs and explore their theoretical properties in comparison with existing neutrosophic and rough set models. To enhance approximation flexibility, we propose two novel operatorsring sum and ring productwithin the GIVNRS structure. The practical utility of the model is demonstrated through decision-making applications, highlighting its potential to address real-world problems characterized by vagueness and incomplete data.

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1 Introduction

In fields such as medical diagnosis, where data is often uncertain, incomplete, or imprecise, traditional mathematical models may not fully capture the complexity of actual-world data. Medical decision-making, in particular, relies on nuanced judgments under uncertain conditions, with diagnoses dependent on varied symptoms, test results, and expert opinions, all of which may have differing degrees of reliability. To address these challenges, neutrosophic sets and rough set theory provide effective instruments for modeling imprecision and uncertainty in complex data. Neutrosophic sets ([1, 2]) with their ability to incorporate truth, indeterminacy, and falsity values, extend the capabilities of fuzzy [3] and intuitionistic fuzzy sets (IFSs) ([4]-[6]). Interval-valued neutrosophic sets (IVNSs) ([7]-[9]), in particular, provide a flexible way to model uncertainty by allowing each component truth, indeterminacy, and falsity to be represented by intervals rather than fixed values. Meanwhile, rough set theory [10] complements this by using lower and upper approximations to classify objects with vague or incomplete information.

Building on the ideas of fuzzy sets proposed by Zadeh in 1965 [3] and IFSs developed by Atanassov ([4]-[6]), neutrosophic sets were later developed to accommodate a broader range of uncertainty often encountered in real-world situations ([1, 2]). Neutrosophic sets allow for each element to have degrees of truth, indeterminacy,

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and falsity, effectively capturing more complex forms of uncertainty and imprecision than earlier models. While fuzzy sets focus on degrees of membership, and IFSs include both membership and non-membership, neutrosophic sets incorporate an additional layer of indeterminacy, making them well-suited for analyzing situations where information may be incomplete, conflicting, or inconsistent. Further advancing this model, interval neutrosophic sets (INSs) were introduced ([7]-[9]) to provide even greater flexibility by representing each componenttruth, indeterminacy, and falsityas an interval within the range $[0, 1]$. Saha and Broumi in [7] introduced novel operators on IVNSs, enhancing the ability to manage uncertainty and imprecision in MCDM and data analysis Broumi and Smarandache [8] proposed interval-valued neutrosophic soft rough sets, which combine the concepts of interval-valued neutrosophic sets and rough set theory, providing a comprehensive framework for analyzing vague and uncertain information. Broumi et al. [11] presented rough neutrosophic sets, merging neutrosophic and rough set theories to tackle the complexities of ambiguous data. Together, these studies advance the theoretical foundations of neutrosophic logic while offering practical tools for applications in areas such as artificial intelligence and decision-making processes, thereby enhancing the capacity to analyze complex datasets with inherent uncertainty. This interval-based approach allows each aspect to reflect a range of possible values rather than a single precise number, offering a robust framework to address cases with high levels of variability and vagueness. INSs have proven especially valuable in complex decision-making environments, where capturing the full range of possible values for truth, indeterminacy, and falsity offers a more comprehensive understanding of uncertainty.

Neutrosophy has established the groundwork for a broad range of mathematical frameworks that extend beyond traditional and fuzzy set theories, leading to the development of neutrosophic set theory and its related models. This field addresses the limitations of classical and fuzzy sets by introducing additional degrees of freedom to handle uncertainty, indeterminacy, and inconsistency. Neutrosophic sets incorporate three parameterstruth, indeterminacy, and falsityallowing for more nuanced representation of information, especially in complex systems where standard models fall short. By expanding upon these classical foundations, neutrosophic theories provide versatile tools for fields requiring detailed analysis of ambiguous or incomplete information, including decision-making, artificial intelligence, and data science. These developments have allowed researchers to handle uncertainties more flexibly and robustly, making these theories valuable across various applications. For this work, we have studied the following articles: Hai-Long et al. in [12] proposed the hybrid model of generalized interval neutrosophic rough sets in 2018 by combining INSs with rough set theory, thus leveraging two powerful tools for managing complex information. Recently, Mukherjee and Das [13] introduced an extended model, known as generalized interval-valued neutrosophic sets (GIVNSs), building on generalized neutrosophic sets (GNSs), Salama and Alblowi [14]. Mukherjee and Das [15] study the Neutrosophic bipolar vague soft set and its application to decision-making problems. Chang [16] explored the idea of fuzzy topological spaces, contributing to the foundational understanding of fuzzy set theory. This work emphasizes the significance of fuzzy topological spaces in analyzing continuity and convergence within fuzzy environments, which is pivotal for further advancements in fuzzy mathematics and its applications in various fields, including MCDM and data analysis. Salama and Al-Blowi [17] delved into the interplay between neutrosophic sets and neutrosophic topological spaces. This research highlights how neutrosophic topology can provide a more comprehensive understanding of mathematical structures and enhance the analysis of complex systems where traditional set theories may fall short. Broumi et al. in [11] introduced the concept of rough neutrosophic sets. This work extends the classical notions of rough sets by incorporating neutrosophic elements, thereby enabling a more nuanced representation of uncertainty and imprecision in the data. By combining the strengths of rough set theory, characterized by its ability to handle vague and incomplete information, with the flexibility of neutrosophic sets, this research offers valuable insights for applications in various fields, including MCDM processes and information systems. The authors illustrate the potential of rough neutrosophic sets to enhance the modeling of complex scenarios where both roughness and uncertainty coexist, making it a significant contribution to the field of computational mathematics. It is to be noted, by

Hai-Long in [12], "Generalized interval neutrosophic rough sets and its application in multi-attribute decision making", is based on the relational concepts, while this paper is based on a set-theoretic approach.

This paper introduces a new framework that combines the strengths of these approaches, GIVNRSs. By integrating the characteristics of both GIVNSs [13] and rough sets [10], GIVNRSs allow for a refined analysis of uncertain data through the use of interval-valued truth, indeterminacy, and falsity components within a rough set framework. We define fundamental operations including intersection, union, inclusion, and equality for GIVNRSs, providing a comprehensive mathematical foundation. Additionally, we propose novel ring sum and ring product operators tailored to GIVNRS approximations, enhancing the ability to combine and manipulate approximations in MCDM contexts.

Finally, we demonstrate the practical applicability of this approach through an MCDM problem, showcasing its potential in real-world scenarios that demand sophisticated handling of uncertain and imprecise information. Further studies, including those by [16, 17, 11], have informed the development and applications of this model. This research contributes to the growing field of neutrosophic rough set theory, offering valuable insights for applications in medical diagnosis and beyond.

Motivation of the Work

The motivation for this research arises from the limitations of existing mathematical tools in effectively handling uncertainty, indeterminacy, and inconsistency in real-world data. While fuzzy sets, intuitionistic fuzzy sets, and neutrosophic sets have significantly advanced uncertainty modeling, they often fail to capture situations where information is both interval-based and rough in nature. Interval-valued neutrosophic sets offer flexibility by representing truth, indeterminacy, and falsity with ranges, and rough set theory provides lower and upper approximations to deal with incomplete information. However, these two frameworks in isolation are not sufficient to address highly complex decision-making problems, such as those encountered in medical diagnosis, where symptoms are overlapping, test results may be inconclusive, and expert judgments may vary.

To overcome these challenges, we are motivated to propose a generalized interval-valued neutrosophic rough set (GIVNRS) model that synergistically integrates the strengths of interval-valued neutrosophic sets and rough sets. This integration allows for a richer representation of uncertain information, leading to more reliable approximations and robust decision-making frameworks. We aim to establish a comprehensive theoretical foundation for GIVNRSs and demonstrate their practical applicability through medical diagnostic scenarios, thus offering a powerful tool to researchers and practitioners in dealing with uncertain, inconsistent, and incomplete information.

Literature Review

The integration of Rough Set Theory [10] and Generalized Interval-Valued Neutrosophic Sets (GIVNS) [12] into Generalized Interval-Valued Neutrosophic Rough Sets (GIVNRS) represents a novel and robust framework for handling uncertainty, particularly in data characterized by incompleteness, indeterminacy, and inconsistency. A comparison with existing hybrid approaches highlights the unique advantages of GIVNRS. Fuzzy logic, introduced by Zadeh [3] in 1965, effectively handles gradual transitions between membership and non-membership with a well-established theoretical foundation. However, it lacks explicit mechanisms for addressing indeterminacy and fails to simultaneously capture multiple types of uncertainty, such as vagueness and incompleteness. In contrast, GIVNRS incorporates three interval-valued components (truth, indeterminacy, and falsity), allowing it to model a broader spectrum of uncertainty. Additionally, the rough set element provides lower and upper approximations to classify data, overcoming the limitations of fuzzy sets that rely solely on membership functions. Similarly, Intuitionistic Fuzzy Sets (IFS), developed by Atanassov [4, 5], extend fuzzy logic by introducing a degree of non-membership and a hesitation margin, providing a more nuanced representation of uncertainty. Despite this, IFS struggles with inconsistencies in data and relies on hesitation as a derived measure, limiting interpretive flexibility. GIVNRS advances this approach by representing truth, indeterminacy, and falsity as independent interval-valued components, enhancing its capacity to handle uncertainty. The integration with rough sets further allows GIVNRS to address incomplete or inconsistent data effectively, making it a powerful tool for decision-making. Dempster-Shafer Theory (DST) [18] offers strengths in probabilistic reasoning and evidence-based decision-making by defining degrees of belief and plausibility. However, its reliance on precise evidence and computational complexity for large universal sets limits its applicability in contexts with high indeterminacy. GIVNRS, by contrast, does not depend on probabilistic evidence but models uncertainty using neutrosophic logics interval-valued components, making it particularly useful in scenarios where precise probabilities are unavailable. The rough set aspect enables approximate reasoning, providing an advantage over DST, which requires well-defined belief functions. GIVNRS offers significant advantages, including a rich representational framework with interval-valued components for uncertainty, dual handling of indeterminacy and incompleteness through the synergy of rough sets and GIVNS, practical applicability in real-world problems, and flexibility in adapting to incomplete, ambiguous, and conflicting data. By clearly distinguishing these novel aspects and practical benefits, GIVNRS provides a robust foundation for advancing uncertainty modeling in diverse applications.

The structure of this paper is as follows: In Section 2, we present key terminologies and foundational concepts that establish the groundwork for our research regarding neutrosophic sets and their utilizations. Section 3 introduces a new approach to GIVNRS, detailing the integration of GIVNSs and rough sets, along with various operations including union, intersection, inclusion, and equality. In Sections 4 and 5, we explore the application of GIVNR approximation operators in decision-making problems, demonstrating their utility in enhancing decision processes, particularly in medical diagnostics. Finally, Section 6 concludes the paper by summarizing our findings and discussing potential future research directions, including the development of algorithms and the integration of machine learning techniques with GIVNRSs to improve real-world applications.

2 Terminologies

In this part, we outline essential definitions that establish the foundational principles for advancing our research regarding neutrosophic sets and their applications. Let X represent a non-empty set. These preliminary concepts are crucial as they provide the groundwork for exploring the properties, behaviors, and potential uses of neutrosophic sets within MCDM contexts and other applications. By defining these essential elements, we create a basis for further theoretical developments and practical implementations in handling uncertainty, indeterminacy, and incompleteness in data. This framework will facilitate a more structured approach in utilizing neutrosophic sets to address practical challenges where ambiguity is a significant factor.

Definition 2.1. [1, 2] *A Neutrosophic Set is characterized as a structured object denoted by* $A = \{< t, T_A(t), I_A(t), F_A(t) >: t \in X\}$*, where* X *is a non-empty fixed set. In this context,* $T_A(t), I_A(t)$*, and* $F_A(t)$ *represent the degrees of membership, indeterminacy, and nonmembership for each element* $t \in X$*, respectively. The values are constrained within the interval* $]\,0^-, 1^+[$ *for T (truth), I (indeterminacy), and F (falsity), adhering to the condition* $0^- \leq T_A(t) + I_A(t) + F_A(t) \leq 3^+$*. This definition allows for a nuanced representation of uncertainty, accommodating various states of membership.*

Definition 2.2. [14] *A Generalized Neutrosophic Set (GNS) extends the concept of a neutrosophic set, defined similarly as* $A = \{< t, T_A(t), I_A(t), F_A(t) >: t \in X\}$*. The functions* $T_A(t), I_A(t)$*, and* $F_A(t)$ *maintain their roles as degrees of membership, indeterminacy, and nonmembership, respectively. A critical distinction is the condition* $T_A(t) \wedge I_A(t) \wedge F_A(t) \leq 0.5$ *and* $T, I, F \to]0^-, 1^+[$ *and* $0^- \leq T_A(t) + I_A(t) + F_A(t) \leq 3^+$*, which introduces a constraint that enhances the model's applicability and robustness in decision-making scenarios.*

Table 1: GIVN Set

X	$[T_A^l(t), T_A^r(t)]$	$[I_A^l(t), I_A^r(t)]$	$[F_A^l(t), F_A^r(t)]$	$\text{Sup}\, T_A(t) \wedge \text{Sup}\, I_A(t) \wedge \text{Sup}\, F_A(t)$
a	$[0.3, 0.6]$	$[0.2, 0.5]$	$[0.1, 0.3]$	$0.6 \wedge 0.5 \wedge 0.3 = 0.3$
b	$[0.3, 0.5]$	$[0.4, 0.6]$	$[0.2, 0.3]$	$0.5 \wedge 0.6 \wedge 0.3 = 0.3$
c	$[0.2, 0.4]$	$[0.2, 0.5]$	$[0.2, 0.4]$	$0.4 \wedge 0.5 \wedge 0.4 = 0.4$
d	$[0.1, 0.3]$	$[0.2, 0.3]$	$[0.1, 0.5]$	$0.3 \wedge 0.3 \wedge 0.5 = 0.3$

Definition 2.3. [7, 8, 9] *An IVNS is a specialized form defined as* $A = \{< t, T_A(t), I_A(t), F_A(t) >: t \in X\}$, *where each membership function is represented by intervals:* $T_A(x) = [T_A^l(t), T_A^r(t)]\,, I_A(t) = [I_A^l(t), I_A^r(t)]$, *and* $F_A(t) = [F_A^l(t), F_A^r(t)]$. *These intervals indicate the range of values for each function, ensuring that* $T_A(t), I_A(t), F_A(t) \in \text{Int}[0,1]$. *Here,* $\text{Int}([0,1])$ *refers to the set of all closed subintervals within the unit interval* [0, 1]*, providing a more flexible representation of uncertainty.*

Definition 2.4. [9] *The Complement of an IVNS, denoted* A^c*, is defined as* $A^c = \{< t, T_{A^c}(t), I_{A^c}(t), F_{A^c}(t) >: t \in X\}$*, where the mappings are derived as outlined below:*
$T_{A^c}(t) = [F_A^l(t), F_A^r(t)], I_{A^c}(t) = [1 - I_A^r(t), 1 - I_A^l(t)]$, and $F_{A^c}(t) = [T_A^l(t), T_A^r(t)]$.

This definition emphasizes the relationship between the original set and its complement can be expressed through their membership functions.

Definition 2.5. [13] *A Generalized Interval Valued Neutrosophic Set (GIVNS)isarticulated as* $A = \{< t, T_A(t), I_A(t), F_A(t) >: t \in X\}$*, where* $T_A(t) = [T_A^l(t), T_A^r(t)], I_A(t) = [I_A^l(t), I_A^r(t)]$*, and* $F_A(t) = [F_A^l(t), F_A^r(t)]$*. The functions must satisfy* $\text{Sup}\, T_A(t) \wedge \text{Sup}\, I_A(t) \wedge \text{Sup}\, F_A(t) \leq 0.5$*, ensuring that the generalized set remains within certain bounds of uncertainty.*

Let us now consider one example of GIVNS.

Example 2.6. Let us consider $X = \{a, b, c, d\}$ and
$A = \{< t, [T_A^l(t), T_A^r(t)]\,, [I_A^l(t), I_A^r(t)]\,, [F_A^l(t), F_A^r(t)] >: t \in X\}$ given by Table 1. Then A is a GIVNS on X.

Definition 2.7. *The complement of a given GIVNS*
$A = \{< t, [T_A^l(t), T_A^r(t)]\,, [I_A^l(t), I_A^r(t)]\,, [F_A^l(t), F_A^r(t)] > t :\in X\}$ *is defined by*
$A^c = \{< t, [F_A^l(t), F_A^r(t)]\,, [1 - I_A^r(t), 1 - I_A^l(t)]\,, [T_A^l(t), T_A^r(t)] >: t \in X\}$,
where $\text{Sup}\, F_A(t) \wedge\ \text{Sup}\, I_A(t) \wedge \text{Sup}_A(t) \leq 0.5$.
The maximum of a GIVNS is $< [1,1], [0,0], [0,0] >$ *and minimum is* $< [0,0], [1,1], [1,1] >$.
The maximum of GIVNS $< [1,1], [0,0], [0,0] >$ *and minimum of GIVNS* $< [0,0], [1,1], [1,1] >$ *are denoted by* 1_N *and* 0_N *respectively.*

Definition 2.8. *A GIVNSA is considered to be a subset of another GIVNS B, represented* asA $\subseteq$ B, *if* $T_A^l(z) \leq T_B^l(z), T_A^r(z) \leq T_B^r(z), I_A^l(z) \geq I_B^l(z), I_A^r(z) \geq I_B^r(z)$ *and* $F_A^l(z) \geq F_B^l(z), F_A^r(z) \geq F_B^r(z)$ *for all* $z \in X$*. This definition emphasizes the comparative relationships between neutrosophic sets, providing a clear framework for assessing inclusion.*

Definition 2.9. *The Union of two GIVNSs* A *and* B *is represented as* $C = A \cup B$*. The truth, indeterminacy, and degree of non-membership functions for* C *are described as*

$$T_C^l(x) = \sup\left\{T_A^l(x), T_B^l(x)\right\}$$
$$T_C^r(x) = \sup\left\{T_A^r(x), T_B^r(x)\right\}$$
$$I_C^l(x) = \inf\left\{I_A^l(x), I_B^l(x)\right\}$$
$$I_C^r(x) = \inf\left\{I_A^r(x), I_B^r(x)\right\}$$
$$F_C^l(x) = \inf\left\{F_A^l(x), F_B^l(x)\right\}$$
$$F_C^r(x) = \inf\left\{F_A^r(x), F_B^r(x)\right\}, \text{ for all } x \in X.$$

Note: $A \cup B$ is the smallest GIVNS containing both the sets A and B.

Definition 2.10. *The intersection of two GIVNSs A and B is represented as $D = A \cap B$. The truth, indeterminacy, and degree of non-membership functions for D are described as*

$$T_D^l(x) = \inf\left\{T_A^l(x), T_B^l(x)\right\}$$
$$T_D^r(x) = \inf\left\{T_A^r(x), T_B^r(x)\right\}$$
$$I_D^l(x) = \sup\left\{I_A^l(x), I_B^l(x)\right\}$$

$$I_D^r(x) = \sup\left\{I_A^r(x), I_B^r(x)\right\} \text{ and}$$
$$F_D^l(x) = \sup\left\{F_A^l(x), F_B^l(x)\right\}$$
$$F_D^r(x) = \sup\left\{F_A^r(x), F_B^r(x)\right\} \text{ for all } x \in X$$

Note: The intersection A $\cap$ B represents the most extensive GIVNS that is included in both sets A and B.

Definition 2.11. [10] *For a subset $A \subseteq X$, the lower and upper approximations of A concerning the approximation space (X, R) are denoted by $\underline{\mathrm{A}}_{\mathrm{R}} \& \bar{A}_R$, respectively. These are formally defined as follows:*

1. *Lower approximation $\underline{A}_R$: This consists of all elements $x \in X$ for which the equivalence class $[x]_R$ is entirely contained in AAA. Formally, $\underline{A}_R = \{x \in X : [x]_R \subseteq A\}$ or $\cup \{[x]_R : [x]_R \subseteq A\}, x \in X$.*
2. *Upper approximation $\bar{A}_R$: This includes all elements $\mathrm{x} \in \mathrm{X}$ such that the intersection of $[x]_R$ with A is non-empty. That is; $\bar{A}_R = \{\mathrm{x} \in \mathrm{X} : [\mathrm{x}]\mathrm{R} \cap \ \mathrm{A} \neq \phi\}$. or $\cup \{[\mathrm{x}]_{\mathrm{R}} : [\mathrm{x}]_{\mathrm{R}} \cap \ \mathrm{A} \neq \phi\}, x \in X$.*

These approximations serve distinct purposes: the lower approximation $\underline{A}_R$ captures elements that can be confidently classified within A based on available information, while the upper approximation $\bar{A}_R$ identifies elements that could potentially belong to A . When the lower and upper approximations coincide-that is, when $\underline{A}_R = \bar{A}_R$ the set A is termed definable, as it can be precisely described using the equivalence classes induced by R. However, if $\underline{A}_R \neq \bar{A}_R$, A is classified as a rough set, indicating some ambiguity or uncertainty in its boundaries.

3 A New Approach to Generalized Interval-Valued Neutrosophic Rough Set

In this segment, we present a new notion of GIVNRSs by combining GIVNSs and rough sets. We also study various operations, including union, intersection, inclusion, and equality over these sets.

Definition 3.1. *Assume U is a non-empty collection, and let R be a relation of equivalence established on U. A GIVNSA in U is defined as: $A = \{< t, T_A(t), I_A(t), F_A(t) >: t \in U\}$.*
Where, $T_A(t) = \left[T_A^l(t), T_A^r(t)\right], I_A(t) = \left[I_A^l(t), I_A^r(t)\right],$ *and* $F_A(t) = \left[F_A^l(t), F_A^r(t)\right] \mathrm{Int}[0,1]$ *(Int $[0,1]$ is the collection of all sub intervals of $[0,1]$)*
with $\max T_A(t) \wedge \max I_A(t) \wedge \max F_A(t) \leq 0.5$
The lower and upper approximation $\underline{A}_R$ and $\bar{A}_R$ of A in the Pawlak approximation (U, R) are as follows:
$\underline{A}_R =$
$\{< t, [\wedge_{y\in[t]_R}\{T_A^l(t)\}, \wedge_{y\in[t]_R}\{T_A^r(t)\}], [\mathrm{v}_{y\in[t]_R}\left\{I_A^l(t)\right\}, \mathrm{V}_{y\in[t]_R}\{I_A^r(t)\}], [\mathrm{v}_{y\in[t]_R}\left\{F_A^l(t)\right\}, \mathrm{v}_{y\in[t]_R}\left\{F_A^r(t)\right\}] >:$
$t \in U\}$, *and*
$\bar{A}_R =$
$\{< t, [\vee_{y\in[t]_R}\{T_A^l(t)\}, \vee_{y\in[t]_R}\{T_A^r(t)\}], [\wedge_{y\in[t]_R}\left\{I_A^l(t)\right\}, \wedge_{y\in[t]_R}\{I_A^r(t)\}], [\wedge_{y\in[t]_R}\left\{F_A^l(t)\right\}, \wedge_{y\in[t]_R}\left\{F_A^r(t)\right\}] >:$
$t \in U\}$

"$\wedge$" mean min and "$\vee$" mean maximum. The relation R denotes the equivalence relationship associated with the GIVNS A. In this context, $[t]_R$ represents the equivalence class of the element t.

$$\left[\wedge_{y\in[t]_R}\left\{T_A^l(t)\right\}, \wedge_{y\in[t]_R}\left\{T_A^r(t)\right\}\right] \in \mathrm{Int}[0,1]$$
$$\left[\mathrm{v}_{y\in[t]_R}\left\{I_A^l(t)\right\}, \mathrm{v}_{y\in[t]_R}\left\{I_A^r(t)\right\}\right] \in \mathrm{Int}[0,1]$$
$$\left[\mathrm{v}_{y\in[t]_R}\left\{F_A^l(t)\right\}, \mathrm{v}_{y\in[t]_R}\left\{F_A^r(t)\right\}\right] \in \mathrm{Int}[0,1]$$

with $0 \leq \wedge_{y\in[x]_R}\left\{T_A^r(t)\right\} + \mathrm{V}_{y\in[x]_R}\left\{I_A^r(t)\right\} + \mathrm{v}_{y\in[x]_R}\left\{F_A^r(t)\right\} \leq 3$ *and* $0 \leq \left(\wedge_{y\in[t]_R}\left\{T_A^r(t)\right\}\right) \wedge \left(\vee_{y\in[t]_R}\left\{I_A^r(t)\right\}\right) \wedge \left(\vee_{y\in[t]_R}\left\{F_A^r(t)\right\}\right) \leq 0.5$. *Then $\underline{A}_R$ is a GIVNS.*
Similarly, we have

$$\left[\vee_{y\in[t]_R}\left\{T_A^l(t)\right\}, \vee_{y\in[t]_R}\left\{T_A^r(t)\right\}\right] \in \mathrm{int}[0,1]$$
$$\left[\wedge_{y\in[t]_R}\left\{I_A^l(t)\right\}, \wedge_{y\in[t]_R}\left\{I_A^r(t)\right\}\right] \in \mathrm{int}[0,1]$$
$$\left[\wedge_{y\in[t]_R}\left\{F_A^l(t)\right\}, \wedge_{y\in[t]_R}\left\{F_A^r(t)\right\}\right] \in \mathrm{int}[0,1]$$

with $0 \leq \left(\vee_{y\in[t]_R}\left\{T_A^r(t)\right\}\right) + \wedge_{y\in[x]_R}\left\{I_A^r(t)\right\} + \wedge_{y\in[t]_R}\left\{F_A^r(t)\right\} \leq 3$
and $0 \leq \left(\vee_{y\in[t]_R}\left\{T_A^r(t)\right\}\right) \wedge \left(\wedge_{y\in[t]_R}\left\{I_A^r(t)\right\}\right) \wedge \left(\wedge_{y\in[t]_R}\left\{F_A^r(t)\right\}\right) \leq 0.5$. *Then $\bar{A}_R$ is a GIVNS.*
If $\bar{A}_R = \underline{A}_R$, then A is classified as a definable set; otherwise, A is considered a GIVNRS where $A = \left(\bar{A}_R, \underline{A}_R\right)$.

Example 3.2. Diagnosis of typhoid fever is challenging when patients exhibit overlapping symptoms such as fever, headache, and fatigue. Additionally, incomplete medical records, varying test accuracies, and inconsistent symptom reports further complicate the process. The GIVNRs framework provides a robust approach to model uncertainty, indeterminacy, and falsity in this scenario, enabling systematic decision-making.

The universe of discourse is defined as $X = \{x_1, x_2, \ldots, x_8\}$, representing 8 patients under evaluation.

Symptoms are assessed by doctors and represented as interval-valued neutrosophic sets, capturing truth (T), indeterminacy (I), and falsity (F) for each patient.

For example, for patient x_5:

- $T(x_5) = [0.2, 0.4]$: Moderate likelihood of typhoid based on initial tests.
- $I(x_5) = [0.4, 0.6]$: Significant indeterminacy due to inconclusive or inconsistent reports.
- $F(x_5) = [0.1, 0.2]$: Weak evidence against typhoid based on the lack of specific markers.

Define Equivalence Classes Symptoms are grouped into equivalence classes based on their observations:

- Fever Class: $\{x_1, x_4\}$: Patients exhibiting sustained high fever.
- Headache Class: $\{x_2, x_3, x_6\}$: Patients reporting consistent headaches.

- Typhoid Class: $\{x_5\}$: Patients reporting consistent typhoid.

Then the equivalence relation R be defined as follows:
$X/R = \{\{x_1, x_4\}, \{x_2, x_3, x_6\}, \{x_5\}\}$.
Let A
$=\{< x_1, [0.1, 0.2], [0.2, 0.3], [0.2, 0.4] >, < x_4, [0.1, 0.3], [0.3, 0.5], [0.3, 0.4] >,$
$< x_5, [0.2, 0.4], [0.4, 0.6], [0.1, 0.2] >, < x_7, [0, 0.1], [0.1, 0.3], [0.2, 0.5] >\}$ be a GIVNS of X .
Then by definition

$$\underline{A}_R = \{\langle x_1, [0.1, 0.2], [0.3, 0.5], [0.3, 0.4] >, < x_4, [0.1, 0.2], [0.3, 0.5], [0.3, 0.4] > ,$$
$$< x_5, [0.2, 0.4], [0.4, 0.6], [0.1, 0.2] >\}$$
$$\bar{A}_R = \{\langle x_1, [0.1, 0.3], [0.2, 0.3], [0.2, 0.4]\rangle, \langle x_4, [0.1, 0.3], [0.2, 0.3], [0.2, 0.4]\rangle,$$
$$\langle x_5, [0.2, 0.4], [0.4, 0.6], [0.1, 0.2]\rangle, \langle x_7, [0, 0.1], [0.1, 0.3], [0.2, 0.5]\rangle,$$
$$< x_8, [0, 0.1], [0.1, 0.3], [0.2, 0.5] >\}$$

For another GIVNS

$$B = \{\langle x_1, [0.1, 0.2], [0.2, 0.3], [0.2, 0.4]\rangle, \langle x_4, [0.1, 0.2], [0.2, 0.3], [0.2, 0.4]\rangle,$$
$$< x_5, [0.2, 0.4], [0.4, 0.6], [0.1, 0.2] >\}$$
$$\underline{B}_R = \{\langle x_1, [0.1, 0.2], [0.2, 0.3], [0.2, 0.4]\rangle, \langle x_4, [0.1, 0.2], [0.2, 0.3], [0.2, 0.4]\rangle,$$
$$< x_5, [0.2, 0.4], [0.4, 0.6], [0.1, 0.2] >\}$$
$$\bar{B}_R = \{\langle x_1, [0.1, 0.2], [0.2, 0.3], [0.2, 0.4]\rangle, \langle x_4, [0.1, 0.2], [0.2, 0.3], [0.2, 0.4]\rangle,$$
$$< x_5, [0.2, 0.4], [0.4, 0.6], [0.1, 0.2] >\}$$

Here $\underline{B}_R = \bar{B}_R$ is a definable GIVNS in the approximation space (X, R).This characteristic implies that every element in B has a complete and consistent representation within each equivalence class, confirming B's certainty in terms of membership relative to R. This example highlights how GIVNRS approximations facilitate the classification of elements based on their degrees of membership, allowing us to manage both the indeterminate and incomplete aspects in decision-making.

Definition 3.3. *If $H = \left(\underline{A}_R, \bar{A}_R\right)$ is a GIVNRS in (X, R) then the complement of H is a GIVNRS denoted by $H^c = \left(\underline{A}^c_R, \bar{A}^c_R\right)$ where $\underline{A}^c_R$ and $\bar{A}^c_R$ are the compliments of GIVNSs $\underline{A}_R$ and $\bar{A}_R$ respectively.*

$$\underline{A}^c_R = \Big\{< t, \Big[\vee_{y\in[t]_R} \Big\{F^l_A(t)\Big\}, \vee_{y\in[t]_R} \{F^r_A(t)\}\Big], \big[\big(1 - \vee_{y\in[t]_R} \{I^r_A(t)\}\big), (1- \vee_{y\in[t]_R} \Big\{I^l_A(t)\Big\}\big)\Big], \Big[\wedge_{y\in[t]_R} \Big\{T^l_A(t)\Big\}, \wedge_{y\in[t]_R} \{T^r_A(t)\}\Big] >: t \in X\Big\}$$

In this representation:

- $\vee_{y\in[t]_R} \left\{F^l_A(t)\right\}$ *and* $\vee_{y\in[t]_R} \{F^r_A(t)\}$ *denote the supremum (maximum) of the lower and upper membership degrees of A over the equivalence class $[t]_R$.*

- $1 - \vee_{y\in[t]_R} \{I^r_A(t)\}$ *and* $1 - \vee_{y\in[t]_R} \left\{I^l_A(t)\right\}$ *are the complements of the supremum (maximum) indeterminacy degrees.*

- $\wedge_{y\in[t]_R} \left\{T^l_A(t)\right\}$ *and* $\wedge_{y\in[t]_R} \{T^r_A(t)\}$ *denote the infimum (minimum) of the truth degrees of A over $[t]_R$.*

$$\bar{A}_R^c = \left\{ < t, \left[\Lambda_{y\in[t]_R} \left\{ F_A^l(t) \right\}, \wedge_{y\in[t]_R} \left\{ F_A^r(t) \right\} \right], \left[\left(1 - \wedge_{y\in[t]_R} \left\{ I_A^r(t) \right\} \right), \left(1 - \wedge_{y\in[t]_R} \left\{ I_A^l(t) \right\} \right) \right], \left[\mathrm{v}_{y\in[t]_R} \left\{ T_A^l(t) \right\}, \mathrm{v}_{y\in[t]_R} \left\{ T_A^r(t) \right\} \right] >: t \in X \right\}$$

Here:

- $\wedge_{y\in[t]_R} \left\{ F_A^l(t) \right\}$ *and* $\wedge_{y\in[t]_R} \left\{ F_A^r(t) \right\}$ *denote the infimum (minimum) of the lower and upper membership degrees of A within* $[t]_R$*.*
- $1 - \wedge_{y\in[t]_R} \left\{ I_A^r(t) \right\})$*, and* $1 - \wedge_{y\in[t]_R} \left\{ I_A^l(t) \right\}$ *are the complements of the infimum (minimum) indeterminacy degrees.*
- $\vee_{y\in[t]_R} \left\{ T_A^l(t) \right\}$ *and* $\vee_{y\in[t]_R} \left\{ T_A^r(t) \right\}$ *indicate the supremum (maximum) of the truth degrees over the equivalence class* $[t]_R$*.*

This complement structure provides an extended framework for assessing membership, indeterminacy, and truth values in the rough approximation of GIVNSs, allowing for more granular analysis of uncertainty within neutrosophic contexts. The combination of upper and lower approximations in both the original and complementary GIVNRS aids in a comprehensive assessment of set characteristics under relational dependencies.

Example 3.4. Here, we extend the analysis from Example 3.2 by exploring the concept of the complement of the GIVNRS approximations. Given the GIVNS A and the equivalence relation R over the universe X, we aim to determine the lower approximation and upper approximation of the complement $\underline{A}_R^c$ and $\bar{A}_R^c$.

The lower approximation of the complement, $\underline{A}_R^c$, is defined as follows:
$\underline{A}_R^c = \{< x_1, [0.3, 0.4], [0.5, 0.7], [0.1, 0.2] >,$
$< x_4, [0.3, 0.4], [0.5, 0.7], [0.1, 0.2] >, < x_5, [0.1, 0.2], [0.4, 0.6], [0.2, 0.4] >: x \in X\}.$

This lower approximation $\underline{A}_R^c$ represents elements within X that do not fully satisfy the membership conditions of A but are part of the complement set with defined intervals for truth, indeterminacy, and falsity. Each interval value here provides an adjusted degree of membership that accounts for the complement's characteristics in R. Similarly, the upper approximation of the complement, $\bar{A}_R^c$, is given by:

$$\begin{aligned}&\bar{A}_R^c = \{\langle x_1, [0.2, 0.4], [0.7, 0.8], [0.1, 0.3] >, < x_4, [0.2, 0.4], [0.7, 0.8], [0.1, 0.3]\rangle, \\ &\langle x_5, [0.1, 0.2], [0.4, 0.6], [0.2, 0.4]\rangle, \langle x_7, [0.2, 0.5], [0.7, 0.9], [0, 0.1]\rangle, \\ &< x_8, [0.2, 0.5], [0.7, 0.9], [0, 0.1] >: x \in X\}\end{aligned}$$

In this upper approximation $\bar{A}_R^c$, the set includes additional elements from the equivalence classes of R that partially meet the criteria of A's complement, encompassing a broader scope of indeterminate and falsity values. This upper approximation reflects potential elements of A^c based on the expanded intervals for each parameter, thereby handling cases where uncertainty and partial indeterminacy play a role.

Together, $\underline{A}_R^c$ and $\bar{A}_R^c$ illustrate how the GIVNRS approach can manage and distinguish between complete and partial membership in a set's complement, providing nuanced approximations that can handle both high and low degrees of membership uncertainty. This dual approximation approach allows for robust decision-making applications where elements' inclusion or exclusion from a set is based on varying degrees of truth, indeterminacy, and falsity across intervals, adapting to the dynamic and often imprecise nature of real-world data.

Theorem 3.5. *Assume A and B be GIVNSs, $\underline{A}$ and $\bar{A}$ are the lower and upper approximation of a GIVNS A with respect to the approximation space (X, R) respectively, $\underline{B}$ and $\bar{B}$ are the lower and upper approximation of a GIVNS B with respect to the approximation space (X, R) respectively. Then the following hold:*

1. $\underline{A} \subseteq A \subseteq \bar{A}$
2. $\overline{A \cup B} = \bar{A} \cup \bar{B}$ *and* $\underline{A \cap B} = \underline{A} \cap \underline{B}$
3. $\overline{A \cap B} = \bar{A} \cap \bar{B}$ *and* $\underline{A \cup B} = \underline{A} \cup \underline{B}$
4. $A \subseteq B \Rightarrow \bar{A} \subseteq \bar{B}$ *and* $\underline{A} \subseteq \underline{B}$
5. $\underline{A}^c = (\bar{A})^c$ *and* $\overline{A^c} = (\underline{A})^c$

Proof.

1. Assume $A = \left\{< t, \left[T_A^l(t), T_A^r(t)\right], \left[I_A^l(t), I_A^r(t)\right], \left[F_A^l(t), F_A^r(t)\right] >: t \in X\right\}$ be a GIVNS satisfying the conditions 3.1 and 3.2.

   Now, from the definition 3.1,

$$T_{\underline{A}}^l(t) = \wedge_{y \in [t]_R} \left\{T_A^l(y)\right\} \leq T_A^l(t) \leq T\frac{l}{A}(t) = \mathrm{v}_{y \in [t]_R} \left\{T_A^l(y)\right\} \text{ for all } t \in X$$
$$T_{\underline{A}}^r(t) = \wedge_{y \in [t]_R} \{T_A^r(y)\} \leq T_A^r(t) \leq T\frac{r}{A}(t) = \mathrm{v}_{y \in [t]_R} \{T_A^r(y)\} \text{ for all } t \in X$$
$$I_{\underline{A}}^l(t) = \vee_{y \in [t]_R} \left\{I_A^l(y)\right\} \geq I_A^l(t) \geq I_A^l(t) = \wedge_{y \in [t]_R} \left\{I_A^l(y)\right\} \text{ for all } t \in X$$

   $I_{\underline{A}}^r(t) = \mathrm{V}_{y \in [t]_R} \{I_A^r(y)\} \geq I_A^r(t) \geq I_{\bar{A}}^r(t) = \wedge_{y \in [t]_R} \{I_A^r(y)\}$ for all $t \in X$
   $F_{\underline{A}}^l(t) = \mathrm{V}_{y \in [t]_R} \{F_A^l(y)\} \geq F_A^l(t) \geq F_{\bar{A}}^l(t) = \wedge_{y \in [t]_R} \{F_A^l(y)\}$ for all $t \in X$
   $F_{\underline{A}}^r(t) = \vee_{y \in [t]_R} \{F_A^r(y)\} \geq F_A^r(t) \geq F_{\bar{A}}^r(t) = \wedge_{y \in [t]_R} \{F_A^r(y)\}$ for all $t \in X$.
   Satisfying the conditions 3.3 to 3.8. Thus $\left(\left[T_{\underline{A}}^l, T_{\underline{A}}^r\right], \left[I_{\underline{A}}^l, I_{\underline{A}}^r\right], \left[F_{\underline{A}}^l, F_{\underline{A}}^r\right]\right)$
   $\subseteq \left(\left[T_A^l, T_A^r\right], \left[I_A^l, I_A^r\right], \left[F_A^l, F_A^r\right]\right) \subseteq \left(\left[T_{\bar{A}}^l, T_{\bar{A}}^r\right], \left[I_{\bar{A}}^l, I_{\bar{A}}^r\right], \left[F_{\bar{A}}^l, F_{\bar{A}}^r\right]\right)$ for all $t \in X$.
   Hence $\underline{A} \subseteq A \subseteq \bar{A}$.

2. Let $A = \left\{< t, \left[T_A^l(t), T_A^r(t)\right], \left[I_A^l(t), I_A^r(t)\right], \left[F_A^l(t), F_A^r(t)\right] >: t \in X\right\}$ satisfying the conditions 3.1 and 3.2.

$$B = \left\{< t, \left[T_B^l(t), T_B^r(t)\right], \left[I_B^l(t), I_B^r(t)\right], \left[F_B^l(t), F_{\mathrm{B}}^r(t)\right] > t :\in X\right\}$$

   with $T_B(t) = \left[T_B^l(t), T_B^r(t)\right], I_A(t) = \left[I_B^l(t), I_B^r(t)\right]$, and $F_B(t) = \left[F_B^l(t), F_B^r(t)\right] \in \mathrm{Int}[0,1]$ andsup $T_B(t) \wedge \sup I_B(t) \wedge \sup F_B(t) \leq 0.5$ be two GIVNSs

$$\overline{A \cup B} = \left\{< t, \left[T\frac{l}{A \cup B}(t), T\frac{r}{A \cup B}(t)\right], \left[I\frac{l}{A \cup B}(t), I\frac{r}{A \cup B}(t)\right], \left[F_{\overline{A \cup B}}(t), F_{\overline{A \cup B}}^r(t)\right] >: t \in X\right\}$$

   $T_{\overline{A \cup B}}(t) = \left[T\frac{l}{A \cup B}(t), T_{\overline{A \cup B}}^r(t)\right], I_{\overline{A \cup B}}(t) = \left[I\frac{l}{A \cup B}(t), I\frac{r}{A \cup B}(t)\right]$, and $F_{\overline{A \cup B}}(t) = \left[F_{\overline{AUB}}^l(t), F_{\overline{AUB}}^r(t)\right] \in$ $\mathrm{Int}[0,1]$
   and $\sup T_{\overline{A \cup B}}(t) \wedge \sup I_{\overline{A \cup B}}(t) \wedge \sup F_{\overline{A \cup B}}(t) \leq 0.5$.

Now,

$$\bar{A} \cup \bar{B} = \left\{ \begin{array}{c} < t, \left[\sup\left(T\frac{l}{A}(t), T\frac{l}{B}(t)\right), \sup\left(T\frac{r}{A}(t), T\frac{r}{B}(t)\right)\right], \left[\inf\left(I\frac{l}{A}(t), I\frac{l}{B}(t)\right), \inf\left(I\frac{r}{A}(t), I\frac{r}{B}(t)\right)\right], \\ \left[\inf\left(F_{\frac{l}{A}}(t), F_{\bar{B}}(t)\right), \inf\left(F_{\frac{r}{A}}(t), \mathrm{F}_{\bar{B}}(t)\right)\right] > t :\in X \end{array} \right\}$$

$$T^l_{\overline{A \cup B}}(t) = \vee \left\{T^l_{A\cup B}(y)\right\} \mid y \in [t]_R$$
$$= \vee \left\{T^l_A(y) \vee T^l_B(y)\right\} \mid y \in [t]_R$$
$$= \left(\vee \left\{T^l_A(y)\right\} \mid y \in [t]_R\right) \vee \left(\vee \left\{T^l_B(y)\right\} \mid y \in [t]_R\right)$$
$$= \left(T^l_A \vee T^l_B\right)(t)$$
$$T\frac{r}{A \cup B}(t) = \vee \left\{T^r_{A\cup B}(y)\right\}\Big| \; y \in [t]_R$$
$$= \vee \left\{T^r_A(y) \vee T^r_B(y)\right\} \mid y \in [t]_R$$
$$= \left(\vee \left\{T^r_A(y)\right\} \mid y \in [t]_R\right) \vee \left(\vee \left\{T^r_B(y)\right\} \mid y \in [t]_R\right)$$
$$= \left(T^r_{\bar{A}} \vee T^r_{\bar{B}}\right)(t)$$
$$I^l_{\overline{A \cup B}}(t) = \wedge \left\{I^l_{A\cup B}(y)\right\} \mid y \in [t]_R$$
$$= \wedge \left\{I^l_A(y) \wedge I^l_B(y)\right\} \mid y \in [t]_R$$
$$= \left(\wedge \left\{I^l_A(y)\right\} \mid y \in [t]_R\right) \wedge \left(\wedge \left\{I^l_B(y)\right\} \mid y \in [t]_R\right)$$
$$= \left(I\frac{l}{A} \wedge I\frac{l}{B}\right)(t)$$
$$I\frac{r}{\overline{A \cup B}}(t) = \wedge \left\{I^r_{A\cup B}(y)\right\}\Big| \; y \in [t]_R$$
$$= \wedge \left\{I^r_A(y) \wedge I^r_B(y)\right\} \mid y \in [t]_R$$
$$= \left(\wedge \left\{I^r_A(y)\right\} \mid y \in [t]_R\right) \wedge \left(\wedge \left\{I^r_B(y)\right\} \mid y \in [t]_R\right)$$
$$= \left(I\frac{r}{A} \wedge I^r_B\right)(t)$$
$$F^l_{\overline{A \cup B}}(t) = \wedge \left\{F^l_{A\cup B}(y)\right\} \mid y \in [t]_R$$
$$= \wedge \left\{F^l_A(y) \wedge F^l_B(y)\right\} \mid y \in [t]_R$$
$$= \left(\wedge \left\{F^l_A(y)\right\} \mid y \in [t]_R\right) \wedge \left(\wedge \left\{F^l_B(y)\right\} \mid y \in [t]_R\right)$$
$$= \left(F^l_{\bar{A}} \wedge F^l_{\bar{B}}\right)(t)$$
$$\mathrm{F}^r_{\overline{A \cup B}}(t) = \wedge \left\{F^r_{A\cup B}(y)\right\} \mid y \in [t]_R$$
$$= \wedge \left\{F^r_A(y) \wedge F^r_B(y)\right\} \mid y \in [t]_R$$
$$= \left(\wedge \left\{F^r_A(y)\right\} \mid y \in [t]_R\right) \wedge \left(\wedge \left\{F^r_B(y)\right\} \mid y \in [t]_R\right)$$
$$= \left(F^r_{\bar{A}} \wedge F^r_{\bar{B}}\right)(t)$$

Hence $\overline{A \cup B} = \bar{A} \cup \bar{B}$ and similarly, $\underline{A \cap B} = \underline{A} \cap \underline{B}$ for all $t \in X$.

3. $\overline{A \cap B} = \bar{A} \cap \bar{B}$ and $\underline{A \cup B} = \underline{A} \cup \underline{B}$.

   The proof for this property follows a similar reasoning as stated in property (2).

4. The proof is obvious.

5. The proof is obvious.

□

4 Application of GIVN Rough Approximation Operators in Decision-Making Problems

In this section, we explore the application of GIVNR approximation operators in aiding decision-making problems, particularly using the ring sum operator $\oplus$ and ring product operator $\otimes$ applied to the rough approximation operators $\bar{A}$ and $\underline{A}$. These operators enhance our capacity to handle uncertainty, imprecision, and indeterminacy in data, which are common challenges in medical and diagnostic decision-making.

4.1 Definition of Operators

Ring Sum Operator: The ring sum operator, represented by P = $\bar{A} \oplus \underline{A}$, is characterized in the following manner:

$$\mathrm{P} = \bar{A} \oplus \underline{A} = \Big\{\langle x, \big[T^l_{\bar{A}}(x) + T^l_{\underline{A}}(x) - T^l_{\bar{A}}(x) \cdot T^l_{\underline{A}}(x), T^r_{\bar{A}}(x) + T^r_{\underline{A}}(x) - T^r_{\bar{A}}(x) \cdot T^r_{\underline{A}}(x)\big], \big[I^l_{\bar{A}}(x) \cdot I^l_{\underline{A}}(x), I^r_{\bar{A}}(x) \cdot I^r_{\underline{A}}(x)\big], \big[F^l_{\bar{A}}(x) \cdot F^l_{\underline{A}}(x), F^r_{\bar{A}}(x) \cdot F^r_{\underline{A}}(x)\big]\rangle : x \in X\Big\}$$

Ring Product Operator: The ring product operator, denoted by G = $\bar{A} \otimes \underline{A}$, is given by:

$$\mathrm{G} = \bar{A} \otimes \underline{A} = \Big\{< x, \Big[T^l_{\bar{A}}(x) \cdot T^l_{\underline{A}}(x), T^r_{\bar{A}}(x) \cdot T^r_{\underline{A}}(x)\Big], \Big[I^l_{\bar{A}}(x) + I^l_{\underline{A}}(x) - I^l_{\bar{A}}(x) \cdot I^l_{\underline{A}}(x), I^r_{\bar{A}}(x) + I^r_{\underline{A}}(x) - I^r_{\bar{A}}(x) \cdot I^r_{\underline{A}}(x)\Big], \Big[F^l_{\bar{A}}(x) + F^l_{\underline{A}}(x) - F^l_{\bar{A}}(x) \cdot F^l_{\underline{A}}(x), F^r_{\bar{A}}(x) + F^r_{\underline{A}}(x) - F^r_{\bar{A}}(x) \cdot F^r_{\underline{A}}(x)\Big] >: x \in X\Big\}$$

4.2 Ring Sum Operator and Ring Product Operator for Example 3.2

Now consider the Example 3.2, then

$$P = \overline{A} \bigoplus \underline{A} = \{< x_1, [0.18, 0.44], [0.06, 0.15], [0.06, 0.16] >, < x_4, [0.18, 0.44], [0.06, 0.15], [0.06, 0.16] >, < x_5, [0.34, 0.64], [0.16, 0.36], [0.01, 0.04] >, < x_7, [0, 0.1], [0.1, 0.3], [0.2, 0.5] >, < x_8, [0, 0.1], [0.1, 0.3], [0.2, 0.5] >: x \in X\} \tag{1}$$

$$G = \overline{A} \bigotimes \underline{A} = \{< x_1, [0.01, 0.06], [0.44, 0.65], [0.44, 0.64] >, < x_4, [0.01, 0.06], [0.44, 0.65], [0.44, 0.64] >, < x_5, [0.04, 0.16], [0.64, 0.86], [0.19, 0.36] >: x \in X\} \tag{2}$$

While the theoretical foundations of GIVNRs are compelling, the lack of concrete case studies or empirical validation limits the practical significance of the framework. Including a real-world case study or empirical results is crucial to demonstrating its effectiveness and applicability. Below is a step-by-step suggestion for structuring and implementing a case study to strengthen the papers impact.

5 Case Studies and Application Examples

5.1 Selection of a Real-World Problem

Choose a practical scenario where uncertainty, incompleteness, and inconsistency in data are significant challenges. For example: Medical diagnosis: Classifying patients with overlapped symptoms of diseases (e.g., typhoid, dengue).

Environmental Monitoring: Assessing river water quality where parameters (e.g., pH, dissolved oxygen) may have inconsistent or incomplete measurements.

Risk Assessment in Finance: Evaluating credit risk for individuals with incomplete financial histories.

5.2 Case Study Example: Medical Diagnosis

Consider Example 3.2, where we computed the GIVNR approximation operators $\overline{A}$ and $\underline{A}$ for the optimal normal decision objects A. Let $X = \{x_1, x_2, x_3, , x_8\}$ be the set of eight patients under the consideration of a decision-maker. Let R be an equivalence relation on X defined as follows:
$X/R = \{\{x_1, x_4\}, \{x_2, x_3, x_6\}, \{x_5\}, \{x_7, x_8\}\}$ The classifications of patients based on their conditions are as follows: $[x_1]_R = [x_4]_R = x_1, x_4$(Suffering from fever).
$[x_2]_R = [x_3]_R = [x_6]_R = \{x_2, x_3, x_6\}$(Suffering from headache).
$[x_5]_R = x_5$(Suffering from dry cough).
$[x_7]_R = [x_8]_R = \{x_7, x_8\}$(Suffering from tiredness).
Assume that the doctor wants to identify patients with typhoid using a GIVNS:

$$A = \{< x_1, [0.1, 0.2], [0.2, 0.3], [0.2, 0.4] >, < x_4, [0.1, 0.3], [0.3, 0.5], [0.3, 0.4] >,\\ < x_5, [0.2, 0.4], [0.4, 0.6], [0.1, 0.2] >, < x_7, [0, 0.1], [0.1, 0.3], [0.2, 0.5] >\}.$$

According to expert criteria, if the maximum membership value is ≥ 0.60 and the maximum non-membership value is ≤ 0.40 in $\overline{A} \bigoplus \underline{A}$, then from Equation 1, patient x_5 is confirmed to be suffering from typhoid. In this example, patient x_5 is undoubtedly affected by typhoid, while patients x_1 and x_4 should be under observation. From Equation 2, we observe that the optimal decision remains with patient x_5 since the maximum membership value is 0.16 (the maximum among others) and the maximum non-membership value is 0.36 (the minimum among others). Hence, patient x_5 is still considered to be suffering from typhoid. This case study demonstrates the practical application of GIVN rough approximation operators to a decision-making scenario involving medical diagnosis. By defining a GIVNSA, the decision-maker (in this case, a doctor) can assess whether a patient might have typhoid based on specific criteria for membership values and non-membership values. The use of ring sum and ring product operators allows for a nuanced evaluation of each patient based on fuzzy membership criteria, as outlined in Equations 1 and 2.

Step-by-Step Analysis Using GIVN Operators

1. Setting up Patient Classifications: The eight patients, labelled x_1 to x_8, were first grouped based on their symptoms (fever, headache, dry cough, tiredness). Using an equivalence relation R, these patients were classified into subsets that represent different health conditions, which helps in understanding possible correlations between symptoms and diseases.

2. Defining the GIVNS for Typhoid: The GIVNSA was defined with specific membership values T_A, non-membership values I_A, and hesitation values F_A for each patient. For instance, patient x_5 has a membership range $[0.2, 0.4]$, a non-membership range $[0.4, 0.6]$, and a hesitation range $[0.1, 0.2]$. These values reflect an expert's assessment of the likelihood that each patient could have typhoid, with higher values indicating a higher probability.

3. Applying Ring Sum and Ring Product Operators: Using the ring sum and ring product operators: The ring sum operator $\overline{A} \bigoplus \underline{A}$ combines membership values from both approximations to yield a refined probability estimate for each patient, highlighting cases where membership is high and non-membership is low. The ring product operator $\overline{A} \bigoplus \underline{A}$ focuses on the intersection, identifying patients with overlapping symptoms and minimal uncertainty, which is essential for conditions where certain symptoms overlap across diagnoses.

4. Decision Criteria for Typhoid Diagnosis: Based on the expert-defined thresholds, patients with a maximum membership value of at least 0.60 and a non-membership value of 0.40 or lower are flagged as likely cases of typhoid. Here, patient x_5 met the criteria in both the ring sum and ring product operations, confirming a high probability of typhoid.

5. Further Observations: Patients x_1 and x_4, who share similar symptoms, show membership values below the threshold for a definitive diagnosis but may still warrant observation due to elevated but inconclusive values in their ring sum and product operations. This secondary classification allows the decision-maker to prioritize monitoring without immediately labelling them as confirmed cases.

Broader Implications and Utility of GIVN Rough Approximations in Decision-Making

The case study emphasizes the advantages of GIVN rough approximation operators in managing uncertainty and ambiguity in decision-making problems, especially those involving incomplete or overlapping information. The GIVN framework, through its ring sum and ring product operations, allows decision-makers to:

1. Handle Incomplete Data: By considering both lower and upper approximations, the model accounts for situations where full data is unavailable or only partially accurate.

2. Prioritize Critical Cases: The operators make it easier to identify high-priority cases (e.g., patient x_5 in this study), enabling resource allocation towards individuals with the highest likelihood of needing immediate attention.

3. Flexibility across Contexts: This methodology is adaptable to various fields, such as medical diagnostics, environmental assessments, and risk analysis, wherever decisions must be made despite incomplete or fuzzy data.

4. Structured Decision Path: By formalizing the process, GIVN operators provide a systematic approach that minimizes subjective judgment errors, leading to more consistent and reliable decisions.

In this example, GIVN rough approximation operators served as powerful tools for refining diagnostic decisions under uncertainty. For patient x_5, the operators validated the preliminary diagnosis of typhoid, while other patients received conditional assessments. This approach demonstrates the value of GIVN rough approximations in yielding clear, actionable insights, making it an effective methodology for complex decision-making scenarios where traditional binary classifications fall short.

5.3 Case Study Example: Water Quality Assessment

Scenario

A water management authority monitors the quality of a river using parameters such as pH, biochemical oxygen demand (BOD), and dissolved oxygen (DO). Measurements are uncertain due to sampling errors, seasonal variations, and equipment inconsistencies.

Objective

Classify water quality into categories (e.g., Good, Moderate, Poor) using Generalized Interval-Valued Neutrosophic Rough Sets (GIVNRs) to handle uncertainty and inconsistency in the data.

Data Collection and Representation

Water quality data is collected from different monitoring stations. For each parameter, truth, indeterminacy, and falsity values are represented as intervals based on expert opinions or historical data. The data is shown in Table 2.

Table 2: Water Quality Data from Monitoring Stations

Station	pH (T, I, F)	BOD (T, I, F)	DO (T, I, F)
S1	[0.6, 0.8], [0.1, 0.2], [0.0, 0.1]	[0.5, 0.7], [0.2, 0.3], [0.0, 0.2]	[0.7, 0.9], [0.0, 0.1], [0.0, 0.1]
S2	[0.3, 0.5], [0.2, 0.3], [0.3, 0.4]	[0.2, 0.4], [0.3, 0.4], [0.3, 0.5]	[0.5, 0.6], [0.2, 0.3], [0.2, 0.3]
S3	[0.7, 0.9], [0.0, 0.1], [0.0, 0.2]	[0.6, 0.8], [0.1, 0.2], [0.0, 0.1]	[0.8, 1.0], [0.0, 0.1], [0.0, 0.1]

GIVNR-Based Decision-Making Framework

Step 1: Define Equivalence Classes

Based on expert-defined thresholds for water quality parameters, the equivalence classes are defined as:

- **Good Quality:** $\text{pH} \in [6.5, 8.5], \text{BOD} \leq 3, \text{DO} \geq 6$.
- **Moderate Quality:** $\text{pH} \in [5.5, 6.5]$ or $[8.5, 9.5], \text{BOD} \in (3, 6], \text{DO} \in [4, 6)$.
- **Poor Quality:** $\text{pH} < 5.5$ or $> 9.5, \text{BOD} > 6, \text{DO} < 4$.

Step 2: Compute Rough Approximations

Using interval-valued truth, indeterminacy, and falsity, calculate the:

- **Lower Approximation (A_R):** Stations that clearly belong to a specific class.
- **Upper Approximation ($\bar{A}_R$):** Stations that may belong to a specific class due to overlapping intervals.

Step 3: Apply Ring Operators

Apply the ring sum ($\oplus$) and ring product ($\otimes$) operators:

- **Ring Sum ($\oplus$):** Highlights stations with high certainty of membership.
- **Ring Product ($\otimes$):** Identifies stations with minimal uncertainty.

Output Classification

- **Station S1:** Good Quality.
- **Station S2:** Moderate Quality (due to overlapping intervals in pH and BOD).
- **Station S3:** Good Quality.

Validation

Compare GIVNR-based classifications with traditional methods such as fuzzy sets or crisp thresholds:

- **Accuracy:** GIVNRs correctly classify S2 as borderline, whereas traditional methods may misclassify it as Good or Poor due to strict thresholds.
- **Uncertainty Handling:** GIVNRs explicitly quantify indeterminacy, improving decision-making for ambiguous cases like S2.

Advantages Highlighted

1. **Flexibility:** GIVNRs adapt to incomplete and uncertain data, unlike traditional methods.
2. **Refined Classification:** The dual approximations and ring operators allow nuanced decisions.
3. **Broader Applicability:** GIVNRs demonstrate utility in domains like medical diagnostics and financial risk assessment.

The case study framework can be adapted to other fields, such as:
Finance: Evaluate loan applications with incomplete credit histories.
Incorporating such case studies would significantly enhance the papers impact. It not only demonstrates the practical utility of GIVNRs but also provides a clear comparison with existing methods, showcasing its advantages in handling uncertainty and indeterminacy.

Example 5.1. (Counterexample to Theorem 3.5 (2)) Consider an approximation space (U, R) where $U = \{a, b\}$ and R is the universal relation. Thus, $[a]_R = [b]_R = U$. Define two generalized interval-valued neutrosophic sets (GIVNSs) A and B on U (with $I = F = 0$ everywhere, so only T values matter):

$$A : T(a) = 0.4,\ T(b) = 0 \qquad\qquad B : T(a) = 0,\ T(b) = 0.4.$$

Using the definitions of lower and upper approximations:

$$A_R(x) = \min_{y \in [x]_R} T_A(y) \qquad\qquad \overline{A}_R(x) = \max_{y \in [x]_R} T_A(y),$$

and similarly for B.

- For A_R: $\min\{0.4, 0\} = 0$ at every point.
- For B_R: $\min\{0, 0.4\} = 0$ at every point.
- Hence, $A_R \cup B_R$ has $T = 0$ everywhere.
- For $\overline{A}_R$: $\max\{0.4, 0\} = 0.4$ at every point.
- For $\overline{B}_R$: $\max\{0, 0.4\} = 0.4$ at every point.
- Hence, $\overline{A}_R \cup \overline{B}_R$ has $T = 0.4$ everywhere.

Thus,

$$A_R \cup B_R \neq \overline{A}_R \cup \overline{B}_R.$$

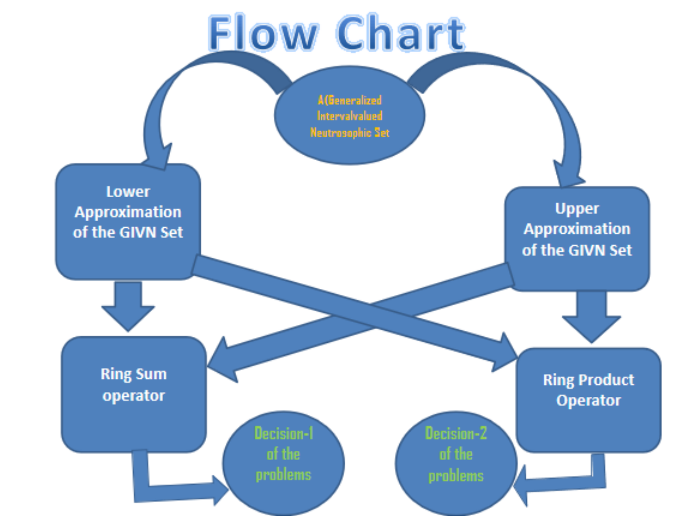

chart1.png chart1.bb

Figure 1: Decision making process for GIVN set

5.4 Limitations

Although this study demonstrates the applicability of GIVNRSs through medical diagnosis and water quality assessment case studies, some limitations remain. First, the analysis is restricted to illustrative datasets; larger and more complex real-world datasets are required to validate the robustness and scalability of the framework. Second, while the case studies show conceptual effectiveness, algorithmic implementations and computational efficiency have not been systematically analyzed. Third, comparative experiments with other established models (e.g., fuzzy rough sets, intuitionistic fuzzy rough sets, or probabilistic approaches) are missing, which would better highlight the relative advantages of GIVNRSs. Finally, the framework has been tested only in static settings; its extension to dynamic and evolving environments remains an open direction for future work.

6 Conclusion

In this paper, we introduce the concept of GIVNRSs, a framework that integrates rough set theory with GIVNS theory to overcome the limitations of traditional approaches in handling uncertainty. While GIVNS theory is mainly concerned with indeterminacy and inconsistency, rough set theory addresses incompleteness. Since both theories ultimately deal with imprecision in data, their combination enables GIVNRSs to manage both indeterminacy and incompleteness simultaneously. This dual capability enhances their effectiveness in diverse domains such as decision-making, data analysis, and information systems, where reliable information is often limited.

We also establish several fundamental properties of GIVNRSs, demonstrating their utility and applicability. Beyond theoretical contributions, this new framework opens opportunities for developing algorithms and methodologies tailored to practical applications. By bridging the gap between rough set theory and neutrosophic logic, GIVNRSs provide a more nuanced tool for addressing complex decision-making problems in uncertain and vague environments.

In conclusion, GIVNRSs represent a significant advancement in the study of uncertainty and imprecision, offering researchers and practitioners a robust approach to decision-making under uncertain conditions. We encourage further exploration and application of this framework to fully realize its potential in solving real-world challenges.

Future Research Directions

The GIVNR framework provides a robust approach to managing uncertainty, incompleteness, and indeterminacy across domains. Future research can focus on integrating GIVNRs with machine learning to handle noisy and incomplete data, improving predictive models in healthcare, finance, and other critical sectors. In optimization, GIVNRs can address uncertain multi-objective problems, such as supply chain management and renewable energy planning. Decision-making frameworks, particularly MCDM and group decision-making, benefit from their ability to model diverse opinions and dynamic preferences.

Applications in big data and the Internet of Things highlight their versatility, enabling data fusion in smart cities and real-time monitoring in healthcare. Hybrid GIVNRML models and algorithmic improvements can further enhance scalability and efficiency. From a theoretical perspective, extensions to multidimensional uncertainty and integration with probabilistic models or DempsterShafer theory can deepen their foundations.

Cross-domain opportunities include environmental management, social sciences, and Industry 4.0, where uncertainty is inherent. Key challenges persist in scalability, benchmark development, and the practical implementation of software tools. Addressing these directions will expand the impact of GIVNRs, making them a powerful framework for solving complex real-world problems.

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References

[1] Smarandache F. Neutrosophy and Neutrosophic logic. In: *The First International Conference on Neutrosophy, Neutrosophic Logic, Neutrosophic Set, Neutrosophic Probability and Statistics, 1-3 Dec 2001, Albuquerque, USA*. Albuquerque: University of New Mexico; 2002. DOI: https://digitalrepository.unm.edu/cgi/viewcontent.cgi?article=1269&context=math_fsp

[2] Smarandache F. *A Unifying Field in Logics. Neutrosophy: Neutrosophic Probability, Set and Logic.* Rehoboth: American Research Press; 1999.

[3] Zadeh LA. Fuzzy sets. *Information and Control.* 1965; 8(3): 338-353. DOI: https://doi.org/10.1016/S0019-9958(65)90241-X

[4] Atanassov KT. Intuitionistic fuzzy sets. In: Sgurev V. (eds.) VII ITKRS Session, Sofia, 20-23 June 1983. *central Sci.-Techn. Library of the Bulgarian Academy of Sciences.* Sofia, Bulgaria: Bulgarian Academy of Sciences; 1984. DOI: https://www.biomed.bas.bg/bioautomation/2016/vol_20.s1/files/20.s1_02.pdf

[5] Atanassov KT. Intuitionistic fuzzy sets. *Fuzzy Sets and Systems.* 1986; 20(1): 87-96. DOI: https://doi.org/10.1016/S0165-0114(86)80034-3

[6] Atanassov KT. Review and new results on intuitionistic fuzzy sets. *Mathematical Foundations of Artificial Intelligence Seminar.* preprint IM-MFAIS-1-88, Sofia, 1988. Reprinted: International Journal Bioautomation, 2016, 20(S1), S7-S16. DOI: https://www.biomed.bas.bg/bioautomation/2016/vol_20.s1/files/20.s1_03.pdf

[7] Saha A, Broumi S. New operators on interval valued Neutrosophic sets. *Neutrosophic Sets and Systems.* 2019; 28(1): 128-137. DOI: https://doi.org/10.5281/zenodo.3382525

[8] Broumi S, Smarandache F. Interval-valued Neutrosophic soft rough sets. *International Journal of Computational Mathematics.* 2015; 2015(1): 232919. DOI: https://doi.org/10.1155/2015/232919

[9] Broumi S, Smarandache F. Interval Neutrosophic rough sets. *Neutrosophic Sets and Systems.* 2015; 7: 23-31. DOI: https://doi.org/10.5281/zenodo.571414

[10] Pawlak Z. Rough sets. *International Journal of Computer and Information Sciences.* 1982; 11: 341-356. DOI: https://doi.org/10.1007/BF01001956

[11] Broumi S, Smarandache F, Dhar M. Rough Neutrosophic sets. *Italian Journal of Pure and Applied Mathematics.* 2014; 32: 493-502. DOI: https://doi.org/10.6084/M9.FIGSHARE.1502608

[12] Yang HL, Bao YL, Guo ZL. Generalized interval Neutrosophic rough sets and its application in multi-attribute decision making. *Filomat.* 2018; 32(1): 11-33. DOI: https://doi.org/10.2298/FIL1801011Y

[13] Mukherjee A, Das R. Generalized interval-valued Neutrosophic set and its application. *Transcations of Fuzzy Sets and Systems.* 2025; 4(2): 1-13. DOI: https://doi.org/10.71602/tfss.2025.1130042

[14] Salama AA, Alblowi SA. Generalized Neutrosophic set and generalized Neutrosophic topological spaces. *Computer Science and Engineering.* 2012; 2(7): 129-132. DOI: https://doi.org/10.5923/j.computer.20120207.01

[15] Mukherjee A, Das R. Neutrosophic bipolar vague soft set and its application to decision making problems. *Neutrosophic Sets and Systems.* 2020; 32: 409-424. DOI: https://doi.org/10.5281/zenodo.3723198

[16] Chang CL. Fuzzy topological spaces. *Journal of Mathematical Analysis and Applications.* 1968; 24(1): 182-190. DOI: https://doi.org/10.1016/0022-247X(68)90057-7

[17] Salama AA, Alblowi SA. Neutrosophic set and Netrosophic topological spaces. *IOSR Journal of Mathematics (IOSR-JM).* 2012; 3(4): 31-35. DOI: https://doi.org/10.9790/5728-0343135

[18] Dempster AP. Upper and lower probabilities induced by a multivalued mapping. In: *The Annals of Mathematical Statistics.* 1967; 38(2): 325-339. DOI: https://link.springer.com/chapter/10.1007/978-3-540-44792-4_3

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