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Research Article

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Proofs of the Reverses of some Inequalities for-Algebras in the non-Standard case

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Abstract
This paper investigates the reverse forms of certain inequalities within-algebras in the non-standard case. We establish that the inequalities in this context are precisely the reverses of those in the standard case, extending results previously obtained for-algebras. Our study explores relationships between significant operator means, such as the Heinz, Heron, and geometric means, in the framework of JB-algebras. For two positive invertible elements in a unital JB-algebra and for specific values of, we demonstrate novel reverse inequalities, including, refining known results. Additionally, we analyze operator monotonicity, operator convexity, and functional calculus in-algebras, leading to extended inequalities that hold in the non-standard case. Using algebraic and functional properties of JB-algebras, we generalize previous results on operator means, demonstrating their validity beyond associative operator settings. These findings contribute to a deeper understanding of the structure of-algebras and their applications in functional analysis and operator theory. The results further highlight the significance of non-standard settings in refining classical inequalities.

Keywords: algebra, Heinz mean, Heron mean, Geometric mean.

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1. Introduction

Jordan algebras were introduced in 1934 by physicists Pascual Jordan, John von Neumann, and Eugene Wigner while exploring the mathematical foundations of quantum mechanics. Their goal was to generalize the algebraic structures used in quantum theory, extending beyond associative algebras to include more general structures, with an initial focus on finite-dimensional algebras [17]. Von Neumann later extended this work to infinite dimensions [18]. Segal [25] pioneered the examination of Jordan subalgebras of self-adjoint operators on Hilbert spaces, with further advancements made by Effros, Størmer [1], Topping, and others. JB -algebras, introduced by Alfsen, Shultz, and Størmer, emerged as a natural framework for quantum observables and have since found applications in analysis, geometry, and operator theory [12, 26, 27].
The theory of operator means began with Anderson and Duffin [2], who introduced parallel addition for positive matrices in electrical network synthesis. Anderson and Trapp [3] later extended this idea to positive operators. In 1975, Pusz and Woronowicz [23] introduced the geometric mean for positive operators, and Ando [4,6,7], Kubo, Fujii, and others [20,21,22] developed a general framework for operator means. While operator means such as the arithmetic, harmonic, and geometric means have greatly influenced operator theory, their use has been mostly confined to Hilbert spaces, with no exploration of their application in JB -algebras.

Observables in quantum mechanics are represented by self-adjoint operators on a Hilbert space. While these operators are not closed under the usual associative product, they are closed under the Jordan product, making it a suitable framework. For more details, see [14].

A Jordan algebra is a non-associative algebra A over $\mathbb{R}$ that satisfies the following properties for all elements $x, y \in \mathrm{A}$:

$$x \circ y = y \circ x \quad (Commutativity)$$
$$(x^2 \circ y) \circ x = x^2 \circ (y \circ x) \quad (Jordan\ identity)$$

where $\circ$ denotes the Jordan product.

Note that special Jordan algebras are a class of Jordan algebras that can be embedded into associative algebras equipped with the symmetrized product

$$x \circ y = \frac{xy + yx}{2} \qquad (1\text{-}1)$$

In contrast, exceptional Jordan algebras are those that cannot be constructed in this way.

As an example, the set of $n \times n$ self-adjoint matrices over the complex numbers, $H_n(C)$, is a special Jordan algebra.

If a real Jordan algebra A is equipped with a complete norm that satisfying

$$\|A \circ B\| \le \|A\| \|B\|, \qquad A, B \in \mathrm{A}$$

it is referred to as a Jordan Banach algebra.

Definition 1. A JB -algebra is a Jordan Banach algebra $\mathcal{A}$ that satisfies two additional conditions for $A, B \in \mathcal{A}$:

$$\|A^2\| = \|A\|^2$$

$$\|A^2\| \le \|A^2 + B^2\|$$

Complex Hermitian and real symmetric matrices, by Jordan product, are important examples of JB -algebras.

Let A be a unital JB -algebra with a unit element I, For an element $A \in \mathcal{A}$, its Jordan inverse A^{-1} (if it exists) is defined as the unique element satisfying:

$$A \circ A^{-1} = I \qquad and \qquad A^2 \circ A^{-1} = A$$

The spectrum of an element A in a Jordan algebra A consists of all real numbers λ for which the element $A - \lambda I$ does not have a multiplicative inverse within the Jordan algebra $\mathcal{A}$. For a positive semidefinite matrix (whether real symmetric or complex Hermitian) denoted as $A \geq 0$, the spectrum contains only non-negative eigenvalues.

In any Jordan algebra, the triple product $\{ABC\}$ is defined as follows:

$$\{ABC\} := (A \circ B) \circ C + (B \circ C) \circ A - (A \circ C) \circ B$$

Observe that the triple product $\{ABC\}$ is linear with respect to each factor and satisfies the property $\{ABC\} = \{CBA\}$. In the case of a special Jordan algebra, the triple product is given by $\{ABC\} = \frac{1}{2}(ABC + CBA)$.

Definition 2. Let A be a unital JB -algebra. A quadratic map, denoted as U_A, is defined on A as follows:

$$U_A B = \{ABC\} = 2(A \circ B) \circ A - A^2 \circ B$$

It is clear that U_A is linear in B but nonlinear in A; specifically,

$$U_A(B - C) = \{ABC\} - \{ACA\}$$

Note that when A is special; the product ABA becomes meaningful and is equal to the expression ABA itself (i.e., $\{ABA\} = ABA$). Furthermore, $U_A B = \{ABA\} \geq 0$ where $B \geq 0$.

The following two identities hold in any Jordan algebra, demonstrating that the Jordan triple product $\{ABA\}$ behaves similarly to the associative product ABA, making algebraic calculations more manageable:

$$\{\{ABA\} C \{ABA\}\} = \{A\{B\{ACA\}B\}A\}, \quad (1-2)$$

$$\{BAB\}^2 = \{B\{AB^2A\}B\}. \quad (1-3)$$

Equality (1-2) can be expressed in the following way for the U_A map.

$$U_{\{ABA\}} = U_A U_B U_A \quad (1-4)$$

We will highlight specific properties of U_A that will be frequently used in the following section.

Let $A, B \in \mathrm{A}$ where A is a JB -Banach algebra. Then

- A^{-1} exists if U_A has a bounded inverse. In this case, $U_A^{-1} = U_{A^{-1}}$.
- If A and B have inverses, then $\{ABA\}^{-1} = \{A^{-1}B^{-1}A^{-1}\}$.
- If A and B have inverses and $0 < A < B$, then $B^{-1} < A^{-1}$.

For additional information, readers are referred to.

When studying JB −algebras, it is important to consider specific classes of real-valued functions. An operator monotone (increasing) function g on $\mathbb{R}$ defined on a JB -algebra A such that $A \leq B$ implies $g(A) \leq g(B)$. Furthermore, g is defined as operator convex if for any $0 \leq t \leq 1$, the following condition is satisfied:

$$g\left((1-t)A+tB\right) \leq (1-t)g(A)+tg(B).$$

g is said to be operator concave if $-g$ is operator convex.

In 2021, Wang et al [28]. introduced operator means for $A, B \in \mathrm{A}$ where A is a JB -Banach algebra, with $0 \leq t \leq 1$:

- t -weighted harmonic mean: $A!_t B = \left((1-t)A^{-1} + tB^{-1}\right)^{-1}$;
- t -weighted geometric mean: $A\sharp_t B = \{A^{1/2}\{A^{-1/2}BA^{-1/2}\}^t A^{1/2}\}$;
- t -weighted arithmetic mean: $A\nabla_t B = (1-t)A + tB$.

They also established the following relationships among them in [28].

$$A\sharp_t B = B\sharp_{1-t}A, \qquad (1-5)$$
$$(A\sharp_t B)^{-1} = A^{-1}\sharp_t B^{-1}, \qquad (1-6)$$
$$A!_t B \leq A\sharp_t B \leq A\nabla_t B, \qquad (1-7)$$
$$\alpha A\sharp_t \beta B = (\alpha\sharp_t\beta)(A\sharp_t B) \quad (\alpha > 0, \beta > 0) \qquad (1-8)$$

$\{C(A\sharp_t B)C\} = \{CAC\}\sharp_t\{CBC\}$ for any invertible$C \in \mathrm{A}$. $\quad (1-9)$

In the C∗-algebra of bounded linear operators on a Hilbert space, the theory of operator means is essential for understanding its structure.

Concepts like operator monotonicity and operator convexity, analogous to those in JB -algebras, are central to this theory. For detailed insights, see references [5, 8, 9, 16, 19, 23].

The geometric mean for $n \times n$ complex matrices was initially defined for positive matrices and later extended to accretive matrices in [13]. A weighted version was introduced in [24], and the theory was further developed in [11] to include arbitrary operator means, with a detailed analysis of the weighted geometric mean and weight extensions.

Definition 3. Let g and f be real continuous function on a closed interval I with $f > 0$. Assume that $A, B \in \mathrm{A}$ with $Sp(A), Sp(B) \subset I$ and Sp where A be a unital JB -algebra. The no associative perspective function $K_{g\Delta f}(B,A)$ of A and B associated to g and f is defined by

$$K_{g\Delta f}(B,A)=\{f(A)^{\frac{1}{2}}g\left(f(A)^{-\frac{1}{2}}Bf(A)^{-\frac{1}{2}}\}\right)f(A)^{\frac{1}{2}}\}.$$

Lemma 4. [28] *Let* A *be a unital* JB *-algebra. Let* p*,* s *and* f *be real valued continuous functions on a closed interval* I *such that* $f>0$ *and* $p(x)\leq s(x)$*. For* $A,B\in$ A *with* $Sp(A),Sp(B)\subset I$ *and* $\{f(A)^{-\frac{1}{2}}Bf(A)^{-\frac{1}{2}}\}\subset I$, $K_{p\Delta f}(B,A)\leq K_{s\Delta f}(B,A)$.

In 2021, Wang et al [28]. defined the young inequality for JB -algebras. Later, in 2024, Ghazanfari et al [15]. proved the reverse of the young inequality for JB -algebras, as stated in the following Lemma respectivly.

Lemma 5. [15, 28] *Let* A *be a unital* JB *-algebra and* $A,B\in$ A *be positive invertible elements.*

- *For any* $t\in[0,1]$, $A!_t B\leq A\#_t B\leq A\nabla_t B$. (1-10)
- *For any* $t\in(-1,0)\cup(1,2)$, $A\nabla_t B\leq A\#_t B\leq A!_t B$. $(1-11)$

Wang et al. proved [28] the following important equality in the standard case, while Ghazanfari et al. [15] established it in the non-standard case, demonstrating that the equality holds for both scenarios.

Let A be a unital JB -algebra and $A,B\in$ A be positive invertible elements. Then for any nonnegative numbers α and β we have

$$(\alpha A\sharp_t\beta B)=(\alpha\sharp_t\beta)(A\sharp_t B). \qquad (1-12)$$

where $t\in(-1,2)$.

2. Main Results

This section begins by establishing the reverse of the inequalities presented by Wang [28] in (1-10). Additionally, it refines (1-11), which was proven by Ghazanfari [15] for JB -algebras, extending from the standard case to the non-standard case.

In [10] Article, several numerical inequalities for the non-standard case have been proven. Lemma 6 employs one of these inequalities to establish a chain of inequalities for JB -algebras in the non-standard context, as demonstrated in Theorems 7 and 8.

Lemma 6. Let $a,b>0$ and $t\notin\left[\frac{1}{2},1\right]$. Then $ta+(1-t)b+(t-1)\left(\sqrt{a}-\sqrt{b}\right)^2\leq a^t b^{1-t}$.

Theorem 7. Let $A,B\in$ A be positive invertible elements, where A is a unital JB -algebra. Then for any $t\in(1,2)$,

$$\begin{aligned}
A!_t B &\geq \left(A^{-1}\nabla_t B^{-1}+2(t-1)(A^{-1}\nabla B^{-1}-A^{-1}\sharp B^{-1})\right)^{-1} && (2-1)\\
(A^{-1}\sharp_t B^{-1})^{-1} &= A\sharp_t B\geq A\nabla_t B+2(t-1)(A\nabla B-A\sharp B) && (2-2),(2-3)\\
&\geq A\nabla_t B && (2-4)
\end{aligned}$$

Proof. From inequality (1-10), $A\nabla B-A\sharp B\geq 0$. Therefore

$$A\nabla_t B\leq A\nabla_t B+2(t-1)(A\nabla B-A\sharp B).$$

The same as:

$$A^{-1}\nabla_t B^{-1} \leq A^{-1}\nabla_t B^{-1} + 2(t-1)(A^{-1}\nabla B^{-1} - A^{-1}\sharp B^{-1}).$$

By definition, $A!_t B = (A^{-1}\nabla_t B^{-1})^{-1}$. Inverting the inequality above results in

$$A!_t B \geq \left(A^{-1}\nabla_t B^{-1} + 2(t-1)(A^{-1}\nabla B^{-1} - A^{-1}\sharp B^{-1})\right)^{-1}.$$

Using Lemma 6, for any $x > 0$ and $t \in (1,2)$,

$$tx + (1-t) + 2(t-1)\left(\frac{x+1}{2} - \sqrt{x}\right) \leq x^t.$$

Applying Lemma 4, to inequality (2-5) with $f(y) = y$ yields

$$A\nabla_t B + 2(t-1)(A\nabla B - A\sharp B) \leq A\sharp_t B.$$

The same as:

$$A^{-1}\nabla_t B^{-1} + 2(t-1)(A^{-1}\nabla B^{-1} - A^{-1}\sharp B^{-1}) \leq A^{-1}\sharp_t B^{-1}$$

By inverting the inequality above,

$$\left(A^{-1}\nabla_t B^{-1} + 2(t-1)(A^{-1}\nabla B^{-1} - A^{-1}\sharp B^{-1})\right)^{-1} \geq (A^{-1}\sharp_t B^{-1})^{-1} = A\sharp_t B.$$

Theorem 8. Let $A, B \in A$ be positive invertible elements, where A is a unital JB -algebra. Then for any $t \in (-1,0)$,

$$\begin{aligned} A!_t B \quad &\geq \left(A^{-1}\nabla_t B^{-1} - 2t(A^{-1}\nabla B^{-1} - A^{-1}\sharp B^{-1})\right)^{-1} && (2-6) \\ (A^{-1}\sharp_t B^{-1})^{-1} = \mathrm{A}\sharp_t \mathrm{B} &\geq A\nabla_t B - 2t(A\nabla B - A\sharp B) && (2-7),(2-8) \\ &\geq \mathrm{A}\nabla_t \mathrm{B} && (2-9) \end{aligned}$$

Proof. Similar to Proposition 7 by Theorem [young4], we have $A\nabla B - A\sharp B \geq 0$. Considering $0 < -t < 1$, it follows that

$$A\nabla_t B \leq A\nabla_t B - 2t(A\nabla B - A\sharp B).$$

Likewise

$$A^{-1}\nabla_t B^{-1} \leq A^{-1}\nabla_t B^{-1} - 2t(A^{-1}\nabla B^{-1} - A^{-1}\sharp B^{-1}).$$

Consequently

$$A!_t B = (A^{-1}\nabla_t B^{-1})^{-1} \geq \left(A^{-1}\nabla_t B^{-1} - 2t(A^{-1}\nabla B^{-1} - A^{-1}\sharp B^{-1})\right)^{-1}.$$

Now by Lemma 6, for any $x > 0$ and $1 - t \in (1,2)$,

$$tx + (1-t) - 2t\left(\frac{x+1}{2} - \sqrt{x}\right) \leq x^t.$$

Applying Lemma 4, to inequality (2-10) with $f(y) = y$ yields

$$A\nabla_t B - 2t(A\nabla B - A\sharp B) \leq A\sharp_t B$$

and

$$A^{-1}\nabla_t B^{-1} - 2t(A^{-1}\nabla B^{-1} - A^{-1}\sharp B^{-1}) \leq A^{-1}\sharp_t B^{-1}.$$

So

$$\left(A^{-1}\nabla_t B^{-1} - 2t(A^{-1}\nabla B^{-1} - A^{-1}\sharp B^{-1})\right)^{-1} \geq (A^{-1}\sharp_t B^{-1})^{-1} = A\sharp_t B.$$

Ghazanfari et al [15]. defined the Heinz and Heron means for two positive invertible elements $A, B \in \mathrm{A}$ in a unital JB -algebra A and $t \in [0,1]$ as follows:

- Hienz mean:

$$H_t(A,B) = \left\{A^{\frac{1}{2}}\left(\frac{\{A^{-\frac{1}{2}}BA^{-\frac{1}{2}}\}^t + \{A^{-\frac{1}{2}}BA^{-\frac{1}{2}}\}^{1-t}}{2}\right)A^{\frac{1}{2}}\right\}$$

- Heron mean:

$$F_t(A,B) = \left\{A^{\frac{1}{2}}\left((1-t)\{A^{-\frac{1}{2}}BA^{-\frac{1}{2}}\}^{\frac{1}{2}} + t\left(\frac{1 + \{A^{-\frac{1}{2}}BA^{-\frac{1}{2}}\}}{2}\right)\right)A^{\frac{1}{2}}\right\}$$

- Logarithmic mean:

$$L_t(A,B) = \int_0^1 A^{1/2}\left(A^{-1/2}BA^{-1/2}\right)^t A^{1/2}\}dt$$

.

The following inequality between the Heinz and Heron means was proven by Zhao et al [29]. for any positive invertible operators A, B on Hilbert space H and $0 \leq t \leq 1$. Later, Ghazanfari et al [15]. extended this result to any positive invertible operators A, B in a unital JB -algebra A with the same condition:

$$H_t(A,B) \leq F_{\alpha(t)}(A,B), \qquad \text{(2-11)}$$

where $\alpha(t) = (1-2t)^2$.

Lemma 9 is another inequality from, and by using it, we have obtained a useful result that establishes a relationship between the Heinz and Heron means in the non-standard case.

Lemma 9. [10] *Let* $a, b > 0$ *and* $t \notin [0,1]$. *Then* $a + b \leq a^t b^{1-t} + b^t a^{1-t}$.

Theorem 10. Let A, B be positive invertible elements in a unital JB -algebra A . Then for any $t \notin [0,1]$, $F_{(1-2t)}(A,B) \leq H_t(A,B)$.

Proof. According to Lemma 9,

$$x^{1-t} \geq 2tx^{\frac{1}{2}} + (1-2t)x$$

for all $x > 0$ and $t \notin [0,1]$. By functional calculus at $\{A^{-\frac{1}{2}}BA^{-\frac{1}{2}}\}$ for JB -algebras [1, proposition 1.21], from (2-13) we get

$$\{A^{-\frac{1}{2}}BA^{-\frac{1}{2}}\}^{1-t} \geq 2t\{A^{-\frac{1}{2}}BA^{-\frac{1}{2}}\}^{\frac{1}{2}} + (1-2t)\{A^{-\frac{1}{2}}BA^{-\frac{1}{2}}\}.$$

Since U_A is a linear mapping, so

$$U_{A^{\frac{1}{2}}}\left(\{A^{-\frac{1}{2}}BA^{-\frac{1}{2}}\}^{1-t}\right) \geq U_{A^{\frac{1}{2}}}\left(2t\{A^{-\frac{1}{2}}BA^{-\frac{1}{2}}\}^{\frac{1}{2}} + (1-2t)\{A^{-\frac{1}{2}}BA^{-\frac{1}{2}}\}\right)$$

from inequality (2-15), we have

$$A\sharp_{1-t}B \geq 2t(A\sharp B) + (1-2t)B$$

therefore according to equation (1-5), we obtained

$$A\sharp_t B = B\sharp_{1-t}A \geq 2t(B\sharp A) + (1-2t)A$$

finally

$$F_{(1-2t)}(A,B) = 2t(A\sharp B) + (1-2t)(A\nabla B) \leq \frac{A\sharp_t B + A\sharp_{1-t}B}{2} = H_t(A,B).$$

In the next theorem, we examine the relationship between the logarithmic mean and the Heron mean in a specific case.

Theorem 11. Let A, B be positive invertible elements in a unital JB -algebra A . Then for any $t \in [0,1]$, $L_t(A,B) = \int_0^1 A\sharp_t B dt \leq F_{\frac{1}{3}}(A,B)$.

Proof. By Inequality (2-11) we have

$$\begin{aligned} L_t(A,B) &= \int_0^1 A^{\frac{1}{2}}\left(A^{-\frac{1}{2}}BA^{-\frac{1}{2}}\right)^t A^{\frac{1}{2}}\}dt = \int_0^1 H_t\,(A,B)dt \\ &\leq \int_0^1 F_{\alpha(t)}\,(A,B)dt = \int_0^1 4(t-t^2)\,(A\sharp B) + \left(1-4(t-t^2)\right)\left(\frac{A+B}{2}\right)dt \\ &\quad \frac{2}{3}(A\sharp B) + \frac{1}{3}(\frac{A+B}{2}) = F_{\frac{1}{3}}(A,B). \end{aligned}$$

It is clear that if we have $t_1 \leq t_2$ then $F_{t_1}(A,B) \leq F_{t_2}(A,B)$ therefore:

Corollary 12. Let A, B be positive invertible elements in a unital JB -algebra A . Then for any $t \in [0,1]$ $L_t(A,B) \leq F_\beta(A,B)$, where $\frac{1}{3} \leq \beta \leq 1$.

Statements & Declarations

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