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Inverse network DEA incorporating cost efficiency with fuzzy data: an application to Petrochemical equipment industries

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Abstract
Petrochemical industry effects on economy of the oil-rich country, such as Iran and all around the world. Nowadays decision-making management is an important subject. The successful manager must be ready to set the future goals by evaluating the past performances then consider the resources and the limitations of the present. Data Envelopment Analysis (DEA) is a powerful technique evaluating relative efficiency for homogenies Decision Making Units (DMU). Since the production process usually consists of related sub-processes with uncertain data, efforts are made to generalize the conventional network models with imprecise data especially fuzzy data where the results are more realistic. This article proposed a model with fuzzy arithmetic approach estimating the inputs while the technical and cost efficiency of all DMUs remain constant with perturbed outputs by manager's preferences.

Keywords: Two- stage efficiency DEA, Inverse DEA, Cost efficiency, Fuzzy data

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1. Introduction

The manager's decisions should not be depended only on his priorities; they must study past performances and inefficiency components, resources restriction, then they consider consciously resource allocation and investment for the future planning. Data Envelopment Analysis (DEA) can do all. DEA is a non-parametric method based on linear programming that measures the relative efficiency of decision-making units (DMUs) producing the same outputs by consuming the same inputs. There is a frontier efficiency approximating production function according to axioms. If a DMU is placed on it, called efficient and efficiency's score will be one; otherwise, it is less than one. DEA also suggests a pattern to reach it for doing better. For the first time, a model for evaluating efficiency of units with multiple inputs and one output was proposed by Farrell [1]. Charnes and Cooper [2] extended Farrell's model for multiple inputs and outputs, called CCR model calculating relative efficiency of DMUs. Then BCC model was introduced by Banker, Charnes and Cooper [3] for variable returns to scale (VRS). In addition, with increasing more studies in this field, non-radial models are proposed.

In the real world, DMUs as banks, institutions, schools, hospitals, etc., are two or multiple- stage with series or parallel structure. Conventional DEA models consider them as black box without intermediate measure; only focus first and final data. This approach-imposed defects on the models such as not investigating inefficiency resources in sub-processes. DEA models for evaluating multi-stage processes are called Network Data Envelopment Analysis (NDEA). Two-stage DEA model is the basis of network models. In such models, the output of the first stage is considered as the input of the second stage and called intermediate measures used to produce the outputs of the second stage. The works of Fare, GrossKopf [4] and seiford, Zho [5] emphasized on decomposition of complicated production process. Kao and Hwang [6] presented in 2008 relational two-stage DEA model according to multiplication idea for Taiwan insurance company under constant returns to scale (CRS). After, the additive weighted two-stage DEA model is proposed by Chen et al. [7]. Overall efficiency value of system was arithmetic average of sub-processes efficiency values in variable returns to scale (VRS) circumstance.

Mathematical models will be applied in the world problems, but the data are not usually certain and accurate. Data can be fuzzy in nature. Lotfizadeh [8] first proposed fuzzy set theory in 1956. There is fuzzy number in DEA. DEA models with fuzzy data cannot be solved by the classic mathematical methods. To overcome this challenge, different approaches have been presented and the most widely used of them is α-cut method. The main idea of this approach is transforming the fuzzy models into a pair of parametric problems with upper and lower boundary of efficiency by membership function. Many researches have studied on fuzzy two-stage DEA models. Kao and Liu [9] presented a model for evaluating the efficiency of two-stage system by α-cut approach. Kachouei et al. [10] used arithmetic fuzzy method on fractional efficiency DEA model with fuzzy data two-stage system with alphabetic approach for solving multi-objective resulted model. Saeedi et al. [11, 12] worked on two and three -stages network system in the presence of undesirable and desirable fuzzy outputs by arithmetic fuzzy method.

On the other hand, recent researches are interested in helping managers to overcome problems such as financial resource restriction, optimizing profits, investment, etc. It needs increasing inputs or outputs with no changing in efficiency score. Inverse DEA (InvDEA) involves two types. Firstly, the input vector of one DMU are increased among a group of them, while the current efficiency scores of all DMUs are maintained and the question arises how much the related outputs should be more produced (estimation of possible outputs). In addition, other one, if the outputs of a DMU need to increase to a certain level with the fixed efficiency scores of all DMUs, how much inputs should be prepared for it (estimation of possible inputs). Inverse DEA is an optimization approach for estimating the data measures of a firm that was first introduced by Wei et al. [13, 14], Jahanshahlou et al [15], Hadi Vancheh [16], have many studies in this field. Shiri et al. [17, 18] used inverse DEA in two-stage network to evaluate cost efficiency and proposed a model.

As mentioned above, in this article, with considering the fuzzy data in the real-world problems based on previous studies specially Shiri et al. [17,18] worked on inverse DEA problem incorporating fuzzy data in two-stage process. The fuzzy inputs and outputs are shown with triangular fuzzy numbers. The proposed model is based on fuzzy arithmetic method. The rest of this paper is outlined as follows: in the section 2, fuzzy numbers, basic definitions and fuzzy arithmetic operations are introduced. Then, in section 3, two- stage efficiency DEA models are mentioned and incorporated with fuzzy data. Cost efficiency in two-stage network system is involved with fuzzy number in section 4. Inverse cost efficiency two-stage DEA model is proposed with fuzzy data in section 5. At the end, proposed model is examined by empirical example in section 6. Finally, the conclusion is presented in section 7.

2. Fuzzy Data

2.1. Basic Definitions of the Fuzzy Set and Arithmetic Operations

Definition 2.1: A fuzzy set on a classical set of objects X is defined as the set of ordered pairs $\widetilde{A} = \{(x, \mu_{\tilde{A}}(x)) : x \in X\}$, where $\mu_{\tilde{A}}: X \to [0,1]$ is the membership function [8,19,20].

Definition 2. 2: A triangular fuzzy number $\tilde{A}$ with membership function $\mu_{\tilde{A}}(x)$ is said to be a real number $\mathbb{R}$ and the following rule [8, 19, 20]:

$$\mu_{\tilde{A}}(x) = \begin{cases} \dfrac{x - a^l}{a^m - a^l}, & a^l < x \le a^m, \\ \dfrac{x - a^u}{a^m - a^u}, & a^m \le x < a^u, \\ 0, & otherwise. \end{cases}$$

Definition 2.3: A triangular fuzzy number $\tilde{A} = (a^l, a^m, a^u)$ is considered a non-negative fuzzy number if and only if $a^l \ge 0,\ a^m - a^l \ge 0,\ a^u - a^m \ge 0.$ It is considered positive if and only if $a^l > 0,\ a^m - a^l \ge 0,\ a^u - a^m \ge 0$ [8, 19, 20].

2.2 Arithmetic Operations

Arithmetic operations between two triangular fuzzy numbers are defined on a universal set of real numbers. Given two triangular fuzzy numbers $\tilde{A} = (a^l, a^m, a^u)$ and $\tilde{B} = (b^l, b^m, b^u)$, the operations are as follows [8, 19, 20]:

(i) $\tilde{A} \oplus \tilde{B} = (a^l + b^l, a^m + b^m, a^u + b^u)$,

(ii) $\tilde{A} \ominus \tilde{B} = (a^l - b^l, a^m - b^m, a^u - b^u)$,

(iii) Let $\tilde{A} = (a^l, a^m, a^u)$ and $\tilde{B} = (b^l, b^m, b^u)$ are two non-negative triangular fuzzy numbers, then:

$$\tilde{A} \otimes \tilde{B} = \left(a^l \times b^l, a^m \times b^m, a^u \times b^u\right),$$

(iv) Let $k \in \mathbb{R}$ be a real number then we have:

$$k\tilde{A} = \begin{cases} (ka^l, ka^m, ka^u) & k > 0 \\ (ka^u, ka^m, ka^l) & k < 0' \end{cases}$$

(v) Let $\tilde{A}, \tilde{B} \succ \tilde{0}$ then:

$$\frac{\tilde{A}}{\tilde{B}} = \frac{(a^l, a^m, a^u)}{(b^l, b^m, b^u)} = \left(\frac{a^l}{b^u}, \frac{a^m}{b^m}, \frac{a^u}{b^l}\right).$$

3. The relational efficiency two-stage DEA model with fuzzy data

3.1. The relational efficiency 2-stage network DEA model

Different from Conventional network DEA, black box approach; suppose there are n DMU_j $(j = 1, \dots, n)$, that each DMU includes a series of two sub-processes. At first stage $x_{ij}, (i = 1, \dots, m)$ are consumed as inputs for producing $z_{pj}, (p = 1, \dots, s)$, these are intermediate measure producing $y_{rj}, (r = 1, \dots, s)$ in the second stage as following figure:

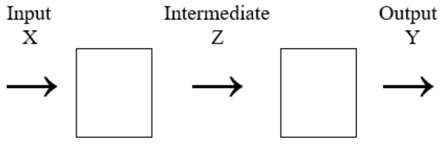

Fig .1. Two-stage system with inputs X, intermediate Z, output Y

Kao and Hang [6] presented the relational efficiency 2-stage DEA model considering intermediate measure in constant returns to scale (CRS):

$$E_o = max \sum_{r=1}^{s} u_r y_{ro} \quad (1)$$

$$s.t. \quad \sum_{i=1}^{m} v_i x_{io} = 1$$

$$\sum_{r=1}^{s} u_r y_{rj} - \sum_{i=1}^{m} v_i x_{ij} \le 0, \; j = 1, \dots, n$$

$$\sum_{p=1}^{q} w_p z_{pj} - \sum_{i=1}^{m} v_i x_{ij} \le 0, j = 1, \dots, n$$

$$\sum_{p=1}^{q} w_p z_{pj} - \sum_{i=1}^{m} v_i x_{ij} \le 0, j = 1, \dots, n$$

$$v_i, w_p, u_r \ge 0, i = 1, \dots, m, p = 1, \dots, q, s = 1, \dots, s.$$

It can be written in input –oriented model:

$$min\ \theta_o \quad (2)$$

$$\text{s.t.} \quad \sum_{j=1}^{n} \lambda_j x_{ij} \le \theta_o x_{io}, \; i = 1, \dots, n$$

$$\sum_{j=1}^{n} (\lambda_j - \mu_j) z_{pj} \ge z_{pj}, p = 1, \dots, q$$

$$\sum_{j=1}^{n} \mu_j y_{rj} \ge y_{ro}, \quad r = 1, \dots, s$$

$$\lambda_j \ge 0, \mu_j \ge 0, \quad j = 1, \dots, n$$

3.2 The efficiency 2-stage network DEA model with fuzzy data

In this section, model (2) will be written with fuzzy numbers. Assume that $\tilde{x}_{ij}, i = (1, \dots, m)$, $\tilde{z}_{pj}, p = (1, \dots, q)$, $\tilde{y}_{rj}, r = (1, \dots, s)$ are inputs, intermediate measures, outputs respectively, so fuzzy efficiency value will be gained as follow:

$$min\tilde{\theta}_o \quad (3)$$

$$\text{s.t.} \sum_{j=1}^{n} \lambda_j \tilde{x}_{ij} \le \tilde{\theta}_o \tilde{x}_{io}, \; i = 1, \dots, n,$$

$$\sum_{j=1}^{n} (\lambda_j - \mu_j) \tilde{z}_{pj} \ge \tilde{z}_{po}, p = 1, \dots, q$$

$$\sum_{j=1}^{n} \mu_j \tilde{y}_{rj} \ge \tilde{y}_{ro}, \quad r = 1, \dots, s$$

$$\lambda_j \ge 0, \mu_j \ge 0, \quad j = 1, \dots, n$$

Given triangle fuzzy number:

$$\tilde{x}_{ij} = (x_{ij}^l, x_{ij}^m, x_{ij}^u), \; \tilde{z}_{ij} = (z_{pj}^l, z_{pj}^m, z_{pj}^u), \; \tilde{y}_{ij} = (y_{ij}^l, y_{ij}^m, y_{ij}^u).$$

Therefore:

$$min\tilde{\theta} = (\theta_o^l, \theta_o^m \theta_o^u) \quad (4)$$

$$s.t. \left(\sum_{j=1}^{n} \lambda_j x_{ij}^l, \sum_{j=1}^{n} \lambda_j x_{ij}^m, \sum_{j=1}^{n} \lambda_j x_{ij}^u\right) \le (\theta_o^l, \theta_o^m, \theta_o^u)\left(x_i^l, x_i^m, x_i^u\right), i = 1, \dots, m$$

$$\left(\sum_{j=1}^{n} (\lambda_j - \mu_j) z_{pj}^l, \sum_{j=1}^{n} (\lambda_j - \mu_j) z_{pj}^m, \sum_{j=1}^{n} (\lambda_j - \mu_j) z_{pj}^u\right) \ge \tilde{0}, \; p = 1, \dots, q$$

$$\left(\sum_{j=1}^{n} \mu_j y_{rj}^l, \sum_{j=1}^{n} \mu_j y_{rj}^m, \sum_{j=1}^{n} \mu_j y_{rj}^u\right) \ge (y_{ro}^l, y_{ro}^m, y_{ro}^u), \quad r = 1, \dots, s$$

$$\theta_0^l \ge 0, \theta_0^m - \theta_0^l \ge 0, \theta_0^u - \theta_0^m \ge 0,$$

$$\lambda_j \ge 0, \quad \mu_j \ge 0, \quad j = 1, \dots, n.$$

According to arithmetic fuzzy approach, the best value of θ_o^l, θ_o^m and θ_o^u can be computed by following models, in result:

$$\theta_o^{l^*} = min\theta_o^l \quad (5)$$
$$s.t.\ \sum_{j=1}^{n} \lambda_j x_{ij}^l \le \theta_o^l\, x_{io}^u\ ,\ i = 1, \dots, m$$
$$\sum_{j=1}^{n} (\lambda_j - \mu_j) z_{pj}^l \ge 0, \quad p = 1, \dots, q$$
$$\sum_{j=1}^{n} \mu_j y_{rj}^l \ge y_{ro}^l\ , \quad r = 1, \dots, s$$
$$\theta_o^l \ge 0,\ \ \lambda_j \ge 0,\ \mu_j \ge 0\ , j = 1, \dots, n.$$

and

$$\theta_o^{m^*} = min\theta_o^m \quad (6)$$
$$\text{s.t. } \sum_{j=1}^{n} \lambda_j x_{ij}^m \le \theta_o^m\, x_{io}^m\ , i = 1, \dots, m$$
$$\sum_{j=1}^{n} (\lambda_j - \mu_j) z_{pj}^m \ge 0, \quad p = 1, \dots, q$$
$$\sum_{j=1}^{n} \mu_j y_{rj}^m \ge y_{ro}^m\ , \quad r = 1, \dots, s$$
$$\theta_o^m \ge 0,\ \lambda_j \ge o\ \ \mu_j \ge 0.,j = 1, \dots, n.$$

and

$$\theta_o^{u^*} = min\theta_o^u \quad (7)$$
$$s.t.\ \sum_{j=1}^{n} \lambda_j x_{ij}^u \le \theta_o^u\, x_{io}^l\ , i = 1, \dots, m$$
$$\sum_{j=1}^{n} (\lambda_j - \mu_j) z_{pj}^u \ge 0\ , \quad p = 1, \dots, q$$
$$\sum_{j=1}^{n} \mu_j y_{rj}^u \ge y_{ro}^u\ , \quad r = 1, \dots, s$$
$$\theta_o^u \ge 0, \quad \lambda_j \ge o\ \ \mu_j \ge 0, j = 1, \dots, n.$$

There are three optimal objective function values obtained from LP programing, namely models (5), (6), (7), for under evaluated DMU (DMU_O). As result, $(\theta_O^{l^*}, \theta_O^{m^*}, \theta_O^{u^*})$ is a triangle fuzzy number.

Similarly, it will be calculated for all DMUs.

Definition 3.1.

a) If $\tilde{\theta} = (\theta^l, \theta^m, \theta^u) = \tilde{1}$; $\theta^l = \theta^m = \theta^u = \tilde{1}$, DMU with triangle fuzzy data DMU is called strong efficient.

b) If $\theta^l \le 1$ and $\theta^m, \theta^u = 1$, DMU with triangle fuzzy data is called efficient

c) If $\theta^l, \theta^m \le 1$ and $\theta^u = 1$, DMU with triangle fuzzy data is called weak efficient.

d) If $\theta^l \le 1,\ \theta^m \le 1,\ \theta^u \le 1$, DMU with triangle fuzzy data is called inefficient.

4. Cost efficiency 2-stage DEA model

Assume $c \in R^m$ is the cost vector of the first stage inputs actual cost of under observed DMU_o Can be computed by $c^t x_o = \sum_{i=1}^{m} c_i x_{io}$. In order to obtain the cost efficiency score of DMU_o , following model (8) must be solved [1]:

$$c^t x^* = min \quad \sum_{i=1}^{m} c_i x_i \quad (8)$$
$$s.t. \quad \sum_{j=1}^{n} \lambda_j x_{ij} \le x_{io}\ , \quad i = 1, \dots, m,$$
$$\sum_{j=1}^{n} (\lambda_j - \mu_j) z_{pj} \ge z_{po}\ , p = 1, \dots, q$$
$$\sum_{j=1}^{n} \lambda_j y_{rj} \ge y_{ro}, \quad r = 1, \dots, s$$
$$\lambda_j \ge 0, \quad \mu_j \ge 0, \quad j = 1, \dots, n.$$

Suppose (x^*, λ^*) is an optimum solution to model. Then $c^t x^* = \sum_{i=1}^{m} c_i x_i^*$ is the minimum cost obtained for DMU_o, and the cost efficiency is equal to the ratio of the obtained minimum cost to the actual cost [1]:

$$CE_o = \frac{c^t x^*}{c^t x_o} = \frac{\sum_{i=1}^{m} c_i x_i^*}{\sum_{i=1}^{m} c_i x_{io}}. \quad (9)$$

Definition 4.1. If cost efficiency score obtained from formula (9) becomes one, then observed unit (DMU_o) is cost efficient. Otherwise, it is cost inefficient DMU.

4.1. Cost efficiency 2-stage DEA model with fuzzy data

Now let $c \in R^m$ is the cost vector of the fuzzy inputs related to the first stage of the network, the minimum cost of production $DMU_o(c^t \tilde{x})$ is calculated using the following model with triangle fuzzy number:

$$c^t \tilde{x}^* = min\,(\sum_{i=1}^{m} c_i\, x_i^l\ , \sum_{i=1}^{m} c_i\, x_i^m\ , \sum_{i=1}^{m} c_i\, x_i^u)$$
$$s.t. \quad \left(\sum_{j=1}^{n} \lambda_j\, x_{ij}^l\ , \sum_{j=1}^{n} \lambda_j\, x_{ij}^m\ , \sum_{j=1}^{n} \lambda_j\, x_{ij}^u\right) \le \left(x_i^l\ , x_i^m\ , x_i^u\right), \quad , i = 1, \dots, m \quad (10)$$
$$\begin{pmatrix} \sum_{j=1}^{n} (\lambda_j - \mu_j)\, z_{kj}^l\ , \\ \sum_{j=1}^{n} (\lambda_j - \mu_j)\, z_{kj}^m\ , \\ \sum_{j=1}^{n} (\lambda_j - \mu_j)\, z_{kj}^u \end{pmatrix} \ge (0^l, 0^m, 0^u), k = 1, \dots, h$$
$$\left(\sum_{j=1}^{n} \mu_j\, y_{rj}^l\ , \sum_{j=1}^{n} \mu_j\, y_{rj}^m\ , \sum_{j=1}^{n} \mu_j\, y_{rj}^u\right) \ge \left(y_{ro}^l\ , y_{ro}^m\ , y_{ro}^u\right), \quad r = 1, \dots, s$$
$$x_i^l\ \ , x_i^m - x_i^l \ge 0, x_i^u - x_i^m \ge 0, i = 1, \dots, m,$$

$\lambda_j \geq 0\,, \mu_j 0, \qquad j = 1, \dots, n.$

Definition 4.1. Consider $\tilde{x}^* = (\tilde{x}_1^*, \tilde{x}_2^*, \dots, \tilde{x}_m^*)$ the optimum solution of model (9). That the inputs of DMU_o are triangular fuzzy numbers such as $\tilde{x}_{io}^* = (x_{io}^{*l}, x_{io}^{*m}, x_{io}^{*u})$. The ratio of the minimum cost $c^t\tilde{x}^*$ to the observed cost DMU_o, $c^t\tilde{x}_o$, is defined as the cost efficiency of DMU_o. Thus, we have:

$$\widetilde{CE}_o = \frac{c^t\tilde{x}^*}{c^t\tilde{x}_o} = \frac{\sum_{i=1}^m c_i\tilde{x}_i^*}{\sum_{i=1}^m c_i\tilde{x}_{io}} = \tag{11}$$

$$\frac{(\sum_{i=1}^m c_i x_i^{l*}, \sum_{i=1}^m c_i x_i^{m*}, \sum_{i=1}^m c_i x_i^{u*})}{(\sum_{i=1}^m c_i x_{io}^{l}, \sum_{i=1}^m c_i x_{io}^{m}, \sum_{i=1}^m c_i x_{io}^{u})} = \left(\frac{\sum_{i=1}^m c_i x_i^{l*}}{\sum_{i=1}^m c_i x_{io}^{u}}, \frac{\sum_{i=1}^m c_i x_i^{m*}}{\sum_{i=1}^m c_i x_{io}^{m}}, \frac{\sum_{i=1}^m c_i x_i^{u*}}{\sum_{i=1}^m c_i x_{io}^{l}}\right) = (CE_o^l, CE_o^m, CE_o^u).$$

In fact, the best values of $(c^t x^{l^*}, c^t x^{m^*}, c^t x^{u^*})$ will be calculated by suggested subsequent models:

$$
\begin{aligned}
c^t x^{l^*} &= min \ \sum_{i=1}^m c_i\, x_i^l && (12)\\
s.t.\ \ & \sum_{j=1}^n \lambda_j\, x_{ij}^l \leq x_i^l, && i = 1, \dots, m\\
& \sum_{j=1}^n (\lambda_j - \mu_j)\, z_{kj}^u \geq 0, && k = 1, \dots, h\\
& \sum_{j=1}^n \mu_j\, y_{rj}^u \geq y_{ro}^u, && r = 1, \dots, s\\
& x_i^l \geq 0, && i = 1, \dots, m\\
& \lambda_j \ \ \mu_j \geq 0, && j = 1, \dots, n.
\end{aligned}
$$

$$\mathrm{CE_o^l} = \frac{\sum_{i=1}^m c_i x_i^{l*}}{\sum_{i=1}^m c_i x_{io}^{u}}. \tag{13}$$

$$
\begin{aligned}
c^t x^{m*} &= min \ \sum_{i=1}^m c_i\, x_i^m && (14)\\
s.t.\ \ & \sum_{j=1}^n \lambda_j\, x_{ij}^m \leq x_i^m, && i = 1, \dots, m\\
& \sum_{j=1}^n (\lambda_j - \mu_j)\, z_{kj}^m \geq 0^m, && k = 1, \dots, h\\
& \sum_{j=1}^n \mu_j\, y_{rj}^m \geq y_{ro}^m, && r = 1, \dots, s\\
& x_i^m \geq 0 && i = 1, \dots, m\\
& \lambda_j \ , \mu_j \geq 0, && j = 1, \dots, n.
\end{aligned}
$$

$$\mathrm{CE_o^m} = \frac{\sum_{i=1}^m c_i x_i^{m*}}{\sum_{i=1}^m c_i x_{io}^{m}}. \tag{15}$$

$$
\begin{aligned}
c^t x^{u*} &= min \ \sum_{i=1}^m c_i\, x_i^u && (16)\\
s.t.\ \ & \sum_{j=1}^n \lambda_j\, x_{ij}^u \leq x_i^u, && i = 1, \dots, m\\
& \sum_{j=1}^n (\lambda_j - \mu_j)\, z_{kj}^l \geq 0, && k = 1, \dots, h\\
& \sum_{j=1}^n \mu_j\, y_{rj}^l \geq y_{ro}^l, && r = 1, \dots, s\\
& x_i^u \geq 0 && i = 1, \dots, m\\
& \lambda_j \ \ \mu_j \geq 0, && j = 1, \dots, n.
\end{aligned}
$$

$$CE_o^u = \frac{\sum_{i=1}^m c_i x_i^{u*}}{\sum_{i=1}^m c_i x_{io}^{l}}. \tag{17}$$

5. Inverse Cost Efficiency of Fuzzy Network Systems

5.1. Inverse DEA:

Imagine that the output vector of DMU_o is perturbed from the current level y_o to $\beta_o = (y_o + \Delta y_o)$ with fixed efficiency value of DMU_o , in conventional inverse DEA, the input vector $\alpha_o = x_o + \Delta x_o$ will be determined via following model that it was introduced by wei et al. [17,18,13]:

$$
\begin{aligned}
\min \ \ & (\alpha_1, \alpha_2, \dots, \alpha_m) && (18)\\
s.t.\ \ & \sum_{j=1}^n \lambda_j x_{ij} \leq \theta_o^* \alpha_{io}, && i = 1, \dots, m\\
& \sum_{j=1}^n \lambda_j y_{rj} \geq \beta_{ro}, && r = 1, \dots, s\\
& \lambda_j \geq 0, && j = 1, \dots, n.
\end{aligned}
$$

Suppose (λ, α) is a feasible solution of model. If there is not a practical solution $(\bar{\lambda}, \bar{\alpha})$such that$\bar{\alpha}_i \leq \alpha_i$ $(\forall i)$, then (λ, α) is a weakly-efficient solution of model (18).

There is a simple technique for solving multi-objective model that called the weighted additive method, objective function becomes as follows:

$$w_1\alpha_1 + w_2\alpha_2 + \dots + w_m\alpha_m = \sum_{i=1}^m w_i\alpha_i, \tag{19}$$

Where w_i indicates the importance (weight) of the i-th input and the value is known, usually $\sum_{i=1}^m w_i = 1$.

5.2. Inverse two-stage cost efficiency with fuzzy data

Given that the outputs of DMU_o change from the level $\tilde{y}_o = (y_o^l, y_o^m, y_o^u)$ to the new level as$\tilde{y}_o + \Delta\tilde{y}_o = (y_o^l +$

$\Delta y_o^l\,, y_o^m+\Delta y_o^m\,, y_o^u+\Delta y_o^u)$. To ascertain the minimal expense of generating a new level of output, we posit the subsequent models:

Solve cost efficiency with new outputs as follows:

$$c^t\tilde{\bar{x}}^* = \left(\sum_{i=1}^m c_i\,\bar{x}_i^{l*}, \sum_{i=1}^m c_i\,\bar{x}_i^{m*}, \sum_{i=1}^m c_i\,\bar{x}_i^{u*}\right) = min\left(\sum_{i=1}^m c_i\,\bar{x}_i^{l}, \sum_{i=1}^m c_i\,\bar{x}_i^{m}, \sum_{i=1}^m c_i\,\bar{x}_i^{u}\right) \quad (20)$$
$$s.t.\left(\sum_{j=1}^n \lambda_j\,x_{ij}^l, \sum_{j=1}^n \lambda_j\,x_{ij}^m, \sum_{j=1}^n \lambda_j\,x_{ij}^u\right) \le \left(\bar{x}_i^l, \bar{x}_i^m, \bar{x}_i^u\right), \quad i=1,\dots,m$$
$$\left(\sum_{j=1}^n(\lambda_j-\mu_j)\,z_{kj}^l, \sum_{j=1}^n(\lambda_j-\mu_j)\,z_{kj}^m, \sum_{j=1}^n(\lambda_j-\mu_j)\,z_{kj}^u\right) \ge \tilde{0},\ k=1,\dots,h$$
$$\left(\sum_{j=1}^n\mu_j\,y_{rj}^l, \sum_{j=1}^n\mu_j\,y_{rj}^m, \sum_{j=1}^n\mu_j\,y_{rj}^u\right) \ge (y_{ro}^l+\Delta y_{ro}^l, y_{ro}^m+\Delta y_{ro}^m, y_{ro}^u+\Delta y_{ro}^u), r=1,\dots,s$$
$$\bar{x}_i^l \ge 0, \bar{x}_i^m-\bar{x}_i^l \ge 0, \bar{x}_i^u-\bar{x}_i^m \ge 0, i=1,\dots,m$$
$$\lambda_j,\ \mu_j \ge 0, \qquad j=1,\dots,n.$$

Technical efficiency $\widetilde{\theta}_o^* = \left(\theta_o^{*l}, \theta_o^{*m}, \theta_o^{*u}\right)$ and the cost efficiency $CE_o=(CE_o^l, CE_o^m, CE_o^u)$ related to perturbed DMU_o must be remained unchanged. For the equivalence of the cost efficiency of DMU_o before and after the perturbation, $\frac{c^t\tilde{\bar{x}}^*}{c^t\tilde{\alpha}} = \widetilde{CE}_o$ should be satisfied:

$$\frac{\sum_{i=1}^m c_i\tilde{\bar{x}}_i^*}{\sum_{i=1}^m c_i\tilde{\alpha}_i} = \frac{\sum_{i=1}^m c_i\tilde{x}_i^*}{\sum_{i=1}^m c_i\tilde{x}_{io}} \Rightarrow \quad (21)$$
$$\left(\frac{\sum_{i=1}^m c_i\bar{x}_i^{l*}}{\sum_{i=1}^m c_i\alpha_i^u}, \frac{\sum_{i=1}^m c_i\bar{x}_i^{m*}}{\sum_{i=1}^m c_i\alpha_i^m}, \frac{\sum_{i=1}^m c_i\bar{x}_i^{u*}}{\sum_{i=1}^m c_i\alpha_i^l}\right) = \left(\frac{\sum_{i=1}^m c_i x_i^{l*}}{\sum_{i=1}^m c_i x_{io}^u}, \frac{\sum_{i=1}^m c_i x_i^{m*}}{\sum_{i=1}^m c_i x_{io}^m}, \frac{\sum_{i=1}^m c_i x_i^{u*}}{\sum_{i=1}^m c_i x_{io}^l}\right).$$

So:

$$\min\ ((\alpha_1^l, \alpha_2^l, \dots, \alpha_m^l), (\alpha_1^m, \alpha_2^m, \dots, \alpha_m^m), (\alpha_1^u, \alpha_2^u, \dots, \alpha_m^u)) \quad (22)$$
$$s.t.\ \left(\sum_{j=1}^n \lambda_j\,x_{ij}^l, \sum_{j=1}^n \lambda_j\,x_{ij}^m, \sum_{j=1}^n \lambda_j\,x_{ij}^u\right) \le (\theta_O^{l^*}, \theta_O^{m^*}, \theta_O^{u^*})((\alpha_1^l, \alpha_2^l, \dots, \alpha_m^l), (\alpha_1^m, \alpha_2^m, \dots, \alpha_m^m), (\alpha_1^u, \alpha_2^u, \dots, \alpha_m^u)),\ i=1,\dots,m$$
$$\left(\sum_{j=1}^n(\lambda_j-\mu_j)\,z_{kj}^l, \sum_{j=1}^n(\lambda_j-\mu_j)\,z_{kj}^m, \sum_{j=1}^n(\lambda_j-\mu_j)\,z_{kj}^u\right) \ge \tilde{0}, \quad k=1,\dots,h$$
$$\left(\sum_{j=1}^n\mu_j\,y_{rj}^l, \sum_{j=1}^n\mu_j\,y_{rj}^m, \sum_{j=1}^n\mu_j\,y_{rj}^u\right) \ge \left(y_{ro}^l+\Delta y_{ro}^l, y_{ro}^m+\Delta y_{ro}^m, y_{ro}^u+\Delta y_{ro}^u\right), r=1,\dots,s$$
$$\left(\frac{\sum_{i=1}^m c_i\bar{x}_i^{*l}}{\sum_{i=1}^m c_i\alpha_i^u}, \frac{\sum_{i=1}^m c_i\bar{x}_i^{*m}}{\sum_{i=1}^m c_i\alpha_i^m}, \frac{\sum_{i=1}^m c_i\bar{x}_i^{*u}}{\sum_{i=1}^m c_i\alpha_i^l}\right) = \left(\frac{\sum_{i=1}^m c_i x_i^{*l}}{\sum_{i=1}^m c_i x_{io}^u}, \frac{\sum_{i=1}^m c_i x_i^{*m}}{\sum_{i=1}^m c_i x_{io}^m}, \frac{\sum_{i=1}^m c_i x_i^{*u}}{\sum_{i=1}^m c_i x_{io}^l}\right)$$
$$\tilde{\alpha}_i \ge \tilde{0}, \qquad i=1,\dots,m$$
$$\lambda_j, \quad \mu_j \ge 0, \qquad j=1,\dots,n.$$

According to fuzzy arithmetic method, models should be solved as follows:

$$min\ \ (\alpha_1^l, \alpha_2^l, \dots, \alpha_m^l) \quad (23)$$
$$s.t. \sum_{j=1}^n \lambda_j x_{ij}^u \le \theta^{*u}{}_o\alpha_i^l, \quad i=1,2,\dots,m$$
$$\sum_{j=1}^n(\lambda_j-\mu_j)\,z_{kj}^u \ \ge 0^u, \quad k=1,\dots,h$$
$$\sum_{j=1}^{nn}\mu_j\,y_{rj}^u \ge y_{ro}^u+\Delta y_{ro}^u, r=1,\dots,s$$
$$\frac{\sum_{i=1}^m c_i x_i^{u*}}{\sum_{i=1}^m c_i x_{io}^l} = \frac{\sum_{i=1}^m c_i\bar{x}_i^{u*}}{\sum_{i=1}^m c_i\alpha^l{}_i},$$
$$\alpha_i^l \ge\ 0^l, \qquad i=1,\dots,m$$
$$\lambda_j,\ \mu_j \ge 0, \qquad j=1,\dots,n.$$

$$min\ \ (\alpha_1^m, \alpha_2^m, \dots, \alpha_m^m) \quad (24)$$
$$s.t.\ \sum_{j=1}^n \lambda_j x_{ij}^m \le \theta^{*m}{}_o\alpha_i^m,\ i=1,\dots,m$$
$$\sum_{j=1}^n(\lambda_j-\mu_j)\,z_{kj}^m \ \ge 0^m, \quad k=1,\dots,h$$
$$\sum_{j=1}^{nn}\mu_j\,y_{rj}^m \ge y_{ro}^m+\Delta y_{ro}^m, \quad r=1,\dots,s$$
$$\frac{\sum_{i=1}^m c_i x_i^{m*}}{\sum_{i=1}^m c_i x_{io}^m} = \frac{\sum_{i=1}^m c_i\bar{x}_i^{m*}}{\sum_{i=1}^m c_i\alpha^m{}_i},$$
$$\alpha_i^m \ge\ 0^m, \qquad i=1,\dots,m$$
$$\lambda_j, \quad \mu_j \ge 0, \qquad j=1,\dots,n.$$

$$min\ \ (\alpha_1^u,\ \alpha_2^u, \dots,\ \alpha_m^u) \quad (25)$$
$$s.t.\ \sum_{j=1}^n \lambda_j x_{ij}^l \le \theta^{*l}{}_o\alpha_i^u,\ i=1,\dots,m$$
$$\sum_{j=1}^n(\lambda_j-\mu_j)\,z_{kj}^l \ \ge 0^l, \quad k=1,\dots,h$$
$$\sum_{j=1}^{nn}\mu_j\,y_{rj}^l \ge y_{ro}^l+\Delta y_{ro}^l, \quad r=1,\dots,s$$
$$\frac{\sum_{i=1}^m c_i x_i^{l^*}}{\sum_{i=1}^m c_i x_{io}^u} = \frac{\sum_{i=1}^m c_i\bar{x}_i^{l*}}{\sum_{i=1}^m c_i\alpha_i^u}$$
$$\alpha_i^u \ge\ 0^u, \qquad i=1,\dots,m$$
$$\lambda_j, \quad \mu_j \ge 0, \qquad j=1,\dots,n$$

Thus:

Proposed Algorithm:

Step 1) Compute the fuzzy technical efficiency of the DMUs using model (4).

Steps 2) calculate the fuzzy cost efficiency value by model (10) and formula (11).

Steps 3) obtain the minimum cost of changed DMU (DMU_O) by solving model (20).

Step 4) estimate the minimum input levels of the first stage after manipulating outputs of stage 2, with constant technical and cost efficiency scores, model (22).

6. Empirical study

In this section, a numerical example is presented to examine the proposed model. Bolt's production workshop and their applications are studied. As the data are not certain, there are presented as triangle fuzzy numbers collected in different periods in a year. Bolts have special importance role in the petrochemical equipment industries. These are vital elements, which can be used in the connection of pipes and flanges and tanks in the oil and gas production refineries. Therefore, they must have high resistance against high pressure, temperature and chemicals; it should be noted that the smallest failure in the production processing could lead to irreparable risks of life and properties. Experts examine them in accredited laboratories. Then expert engineers and technicians send the final products with suitable and safe packaging to the workshops for equipping and setting up refineries, especially in the southern part of Iran country. In this research, 20 petrochemical industrial connection companies and related production workshops are considered as two-stage system. In the first process, rebar with different sizes and workers are inputs that produce stud bolts. Then, Bolts are used to equip units of refineries such as connections of flanges or pipes in the second stage. Data set is exhibited as triangle fuzzy number in table 1 and2.

Table 1. First Sub-process, Data set

	Input			Output (Intermediate measure)	
	Input 1 (By branch number)	Input 2 (By branch number)	Input 3 (By number)	Output 1 (By number)	Output 2 (By number)
	Rebar Model: Mo40 Size 16	Rebar Model: Mo40 Size20	worker	Stud bolt Material: ASTM A193 Gr: B7 Size: 5/8*95 mm	Stud bolt Material: ASTM A320 Gr: L7 Size: 3/4*100 mm
	(42,45,46)	(78,80,91)	(8,9,13)	(2100,2500,2800)	(4520,4700,5100)
2	(47,50,58)	(60,77,80)	(9,10,14)	(2800,3000,3500)	(3750,4010,4600)
3	(49,52,55)	(63,68,75)	(8,10,15)	(2720,3010,3050)	(3500,3900,4100)
4	(46,48,53)	(71,75,79)	(8,10,12)	(2640,3000,3010)	(3900,4100,4420)
5	(52,55,59)	(78,80,85)	(9,12,15)	(2920,3100,3500)	(3980,4600,4710)
6	(30,40,47)	(80,82,91)	(8,11,13)	(2100,2410,2600)	(4100,4780,4900)
7	(38,45,46)	(70,81,83)	(4,5,6)	(2910,3100,3540)	(4910,5100,5800)
8	(50,52,65)	(81,82,90)	(13,14,15)	(2120,2800,2920)	(2800,3000,3100)
9	(37,39,41)	(65,70,73)	(3,4,5)	(2340,2460,2600)	(3840,4200,4500)
10	(35,40,50)	(77,79,90)	(9,10,15)	(2090,2500,2900)	(4600,4720,5150)
11	(37,41,49)	(82,83,91)	(9,11,15)	(2100,2200,2800)	(4610,4630,5100)
12	(49,53,55)	(62,69,75)	(8,9,11)	(2710,3020,3100)	(3600,3950,4150)
13	(30,32,33)	(70,71,72)	(5,6,7)	(2600,2610,2700)	(4750,4800,4810)
14	(46,51,54)	(63,70,75)	(9,10,12)	(2712,3120,3200)	(3600,3960,4250)
15	(53,54,59)	(78,81,84)	(10,13,15)	(2930,3200,3500)	(3990,4700,4810)
16	(31,41,48)	(80,83,90)	(9,10,14)	(2100,2500,2620)	(4120,4800,4900)

17	(48,51,58)	(60,72,82)	(8,11,13)	(2900,3010,3510)	(3720,4012,4610)
18	(37,40,42)	(63,70,72)	(4,5,6)	(2320,2460,2610)	(3800,4320,4350)
19	(46,53,59)	(61,80,83)	(9,11,13)	(2810,3015,3525)	(3740,4000,4615)
20	(38,39,40)	(65,70,73)	(3,5,6)	(2340,2510,2598)	(3820,4310,4520)

Table 2. Second Sub-process, Data set.

	Input (Intermediate measure)		Final output
	Stud bolt (By number)	Stud bolt Material: ASTM A320 Gr: L7 Size: 3/4*100 mm (By number)	Equipped connections (By meter)
1	(2100,2500,2800)	(4520,4700,5100)	(410,430,470)
2	(2800,3000,3500)	(3750,4010,4600)	(480,483,520)
3	(2720,3010,3050)	(3500,3900,4100)	(470,510,530)
4	(2640,3000,3010)	(3900,4100,4420)	(380,400,450)
5	(2920,3100,3500)	(3980,4600,4710)	(480,520,540)
6	(2100,2410,2600)	(4100,4780,4900)	(310,320,350)
7	(2910,3100,3540)	(4910,5100,5800)	(580,600,640)
8	(2120,2800,2920)	(2800,3000,3100)	(350,370,380)
9	(2340,2460,2600)	(3840,4200,4500)	(600,610,620)
10	(2100,2500,2900)	(4600,4720,5150)	(400,420,460)
11	(2100,2200,2800)	(4610,4630,5100)	(300,310,340)
12	(2710,3020,3100)	(3600,3950,4150)	(471,500,538)
13	(2310,2560,2700)	(3810,4310,4600)	(700,730,780)
14	(2712,3120,3200)	(3600,3960,4250)	(500,510,545)
15	(2930,3200,3500)	(3990,4700,4810)	(470,530,541)
16	(2100,2500,2620)	(4120,4800,4900)	(310,330,360)
17	(2900,3010,3510)	(3720,4012,4610)	(480,489,520)
18	(2320,2460,2610)	(3800,4320,4350)	(600,612,625)
19	(2810,3015,3525)	(3740,4000,4615)	(475,503,527)
20	(2340,2510,2598)	(3820,4310,4520)	(610,620,631)

Table 3. Technical Efficiency

DMU	θ^{*u}_{input}	θ^{*u}_{input}	θ^{*u}_{input}
1	.31	.39	.40
2	.40	.45	.57
3	.42	.54	.55
4	.33	.39	.42
5	.38	.45	.47
6	.24	.28	.37
7	.48	.57	.76
8	.26	.3	.33
9	.56	.71	.98
10	.3	.39	.42
11	.22	.22	.27
12	.42	.52	.57
13	.71	.75	.86
14	.45	.53	.57
15	.38	.45	.48
16	.24	.29	.37
17	.39	.49	.57
18	.56	.66	.75
19	.39	.45	.57
20	.57	.67	1

Table 4. Cost Efficiency

DMU	CE^{*l}	CE^{*m}	CE^{*u}
1	.32	.46	.51
2	.35	.50	.64
3	.36	.56	.63
4	.32	.43	.49
5	.34	.49	.55
6	.24	.33	.41
7	.55	.72	.90
8	.23	.34	.36
9	.61	.86	1
10	.3	.46	.51
11	.22	.32	.37
12	.4	.55	.63
13	.76	1	1
14	.4	.55	.66
15	.34	.49	.52
16	.24	.35	.4
17	.35	.51	.66
18	.59	.83	.1
19	.35	.49	.63
20	.60	.84	1

As seen in table 3, efficiency value in upper bound is equal to one for DMU 20 (workshop); so, it is weak efficient DMU according to Definition 3.1.c. Other DMUs (workshops) are inefficient because of $\theta^l \leq 1,\ \theta^m \leq 1,\ \theta^u \leq 1$.

Assume that $(c_1, c_2 c_3) = (1.6,1.4,7)$ is the cost inputs vector. Cost efficiency scores are computed by model (12-17). There are results in table 4.

Managers or engineers are interested in improving the output levels based on their priorities or future goals set; sometimes they emphasize that technical and cost efficiency scores are remained as before. It means that output levels will be increased with constant efficiency value. Now the question arises that how much inputs are required. In fact, estimation of inputs is done by inverse DEA. Proposed algorithm and proposed inverse two- stage network DEA model with fuzzy data (22) computed easily input vector.

Suppose:

Table 5. increasing output vector

Perturbed DMU (Increased output)	$(\theta^{l^*}, \theta^{m^*}, \theta^u)$	$(CE^{l^*}, CE^{m^8}, CE^{u^*})$	$(\beta^l, \beta^m, \beta^u)$
DMU7	(.48,.57,.76)	(.55,.72,.90)	(590,720,900)
DMU9	(.56,.71,.98)	(.61,.86,1)	(620,730,900)
DMU13	(.71,.75,.86)	(.76,1,1)	(710,850,950)
DMU19	(.39,.45,.57)	(.35,.49,.63)	(480,504,800)
DMU20	(.57,.67,1)	(.60,.84,1)	(612,800,900)

Result:

Table 6. Estimation input vector after increasing output vector

DMU	$(\alpha_1^l, \alpha_2^l,, \alpha_3^l,)$	$(\alpha_1^m, \alpha_2^m,, \alpha_3^m,)$	$(\alpha_1^u, \alpha_2^u,, \alpha_3^u,)$
DMU7	(41,75,5)	(43,97,8)	(62,114,10)
DMU9	(35,64,5)	(37,82,7)	(55,103,9)
DMU13	(40,73,5)	(41,82,6)	(50,92,7)
DMU19	(42,77,5)	(43,97,8)	(63,116,19)
DMU20	(35,64,5)	(41,92,8)	(54,99,10)

Table 6 shows that if output level DMU7 increase to (590,720,900), it will be provided (41,75,5), (43,97,8), (62,114,10) for inputs bounds, respectively. It helps managers decide better in supplying materials and workers in an important industry specially petrochemicals. It gives an opportunity to have a more realistic plan for independence. Steps of production process will be organized before doing with better result and less defects.

7. Conclusion

Since in reality organizations, both governmental and non-governmental, such as hospitals, schools, banks, etc., contain of two or more stage processes; network DEA models can evaluate efficiency of them. Classic network DEA models consider the system as black box without seeking inefficiency sources in sub-process. In recent decades, network DEA models are presented that involving intermediate measures. On the other hand, in the experimental research, the data is not always precise and certain. It can be fuzzy and shown by triangle fuzzy number, so network DEA models incorporating fuzzy data were introduced. There are different approaches for solving them.

In this research, network technical efficiency DEA model with triangle fuzzy data is proposed; fuzzy arithmetic approach solve it. Three linear programming be solved easily to attain technical efficiency value bounds for under observed DEA. According to this method, cost efficiency DEA model is also proposed, the best value of $(c^t x^{l^*}, c^t x^{m^*}, c^t x^{u^*})$ are obtained. Then inverse network DEA model with fuzzy triangle inputs and outputs is introduced. Technical and cost efficiency scores are maintained, output vector is perturbed and input changes can be estimated by proposed model. It is an option that helps managers evaluate past performance beside attention to future planning to make decisions.

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