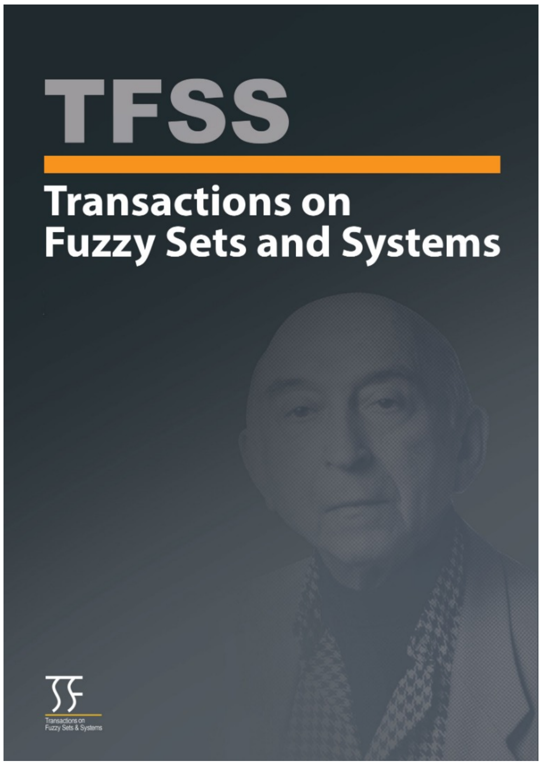
Transactions on Fuzzy Sets and Systems



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Fuzzy Bayesian, E-Bayesian, and Hierarchical Bayesian Estimations of $R = P(X>Y)$ in Weibull Distribution under Type II censored Data

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(Special Section on "Dynamical Systems in Fuzzy Environments".)

Abstract. This study examines Bayesian, E-Bayesian (E-B), and hierarchical Bayesian (H-B) estimation methods for the stress-strength reliability parameter (SSRP) $R = P(X>Y)$, within the Weibull distribution framework under Type II censoring and fuzzy data conditions. Stress and strength random variables are modeled as Weibull distributions with distinct scale parameters but a common shape parameter. Estimations are conducted using the squared error (SE) loss function and Lindleys approximation. Furthermore, a comprehensive simulation study, complemented by real-world data analysis, has been carried out to assess and compare the performance of the proposed estimators. The results from both simulation and empirical analyses demonstrate that the H-B estimator consistently outperforms both the Bayesian and E-B estimators under the squared error (SE) loss function.

AMS Subject Classification 2020: 62F15; 62N02; 62C05
Keywords and Phrases: E-Bayesian estimator, Hierarchical Bayesian estimator, Fuzzy Data, Stress-strength parameter, Type II censoring.

1 Introduction

In dependability, the $R = P(X > Y)$ is a measure of a system's efficiency and is sometimes called the stress strength factor. In this context, X represents the resilience of an element under stress Y. If the applied stress exceeds the variable's strength, the system fails. Estimating R is a critical topic in various scientific disciplines, including lifetime analysis, mechanical reliability of systems, structural engineering, rocket engine wear studies, and the assessment of aircraft components. Numerous researchers have investigated methods for estimating R, particularly in scenarios where X and Y are independent random variables from the same distribution. For instance, estimation techniques for R have been studied in various contexts, including the bivariate exponential distribution (see [1]), the Weibull distribution (see [2]), the multivariate normal case (see [3]), and the Burr Type XII distribution (see [4, 5]). Additional work includes investigations in the generalized exponential distribution (see [6]), its three-parameter variant (see [7]), and through lower record values of the same distribution (see [8]). Various distributions have been explored for estimating the reliability parameter, including the Weibull distribution with progressively censored samples (see [9]), the exponential distribution under progressive Type II censoring [10], and the Burr XII distribution with progressively censored samples

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for initial failures [5]. Additionally, research has investigated the Lindley distribution [11] and its power-transformed counterpart, the power Lindley distribution [12]. These studies emphasize the crucial role of reliability parameter estimation in diverse applications within reliability analysis and related fields. Over the years, various methodologies have been proposed for estimating the parameters of statistical distributions, with Bayesian estimation becoming a widely recognized approach. Selecting an appropriate prior density function across the parameter space is crucial for minimizing errors in Bayesian estimates. However, extending the range of parameter variation within this space may inadvertently increase error rates. Therefore, it is essential to define suitable prior density functions and establish specific conditions for the hyperparameters of the prior distribution. Within this framework, both E-B and H-B estimation methods are considered. The concept of the H-B prior distribution was first introduced in [13], while [14] further developed techniques for constructing such distributions and introduced the E-B and H-B methods. Bayesian and E-B formulations for estimating the reliability parameter in scenarios involving zero-failure data were explored by [15, 16]. Subsequently, [17, 18] derived E-B and H-B estimations for system reliability parameters, with their properties analyzed in detail [16]. The characteristics, reliability, and hazard functions associated with the Weibull distribution were ascertained in [19] using maximum likelihood evaluation in conjunction with Bayesian and E-B techniques. Furthermore, applications of the H-B method to data analysis were demonstrated in [20, 21]. The estimation of under various distributions and sampling schemes has been researched in great detail by numerous authors (see, for example, [22]-[29]). These cumulative efforts highlight the ongoing advancements in parameter estimation techniques across multiple statistical and reliability frameworks. The Weibull distribution is widely recognized as one of the most frequently used models. The Weibull distribution's probability density function (pdf) is provided by

$$f\left(x;\alpha,\eta\right)=\alpha\eta x^{\alpha-1}e^{-\eta x^{\alpha}};x>0 \tag{1}$$

where $\alpha>0$ is the shape parameter, and $\eta>0$ is the scale parameter. The same, the cumulative distribution function (cdf) is given as

$$F\left(x;\alpha,\eta\right)=1-e^{-\eta x^{\alpha}};x>0 \tag{2}$$

The reliability function $R(t)$ and the hazard rate function $h(t)$ at time t are given by

$$R(t)=e^{-\eta t^{\alpha}} \qquad\qquad h\left(t\right)=\alpha\eta t^{\alpha-1}$$

The Weibull distribution is a highly flexible model widely utilized in survival analysis and reliability engineering, particularly well-suited for handling censored data. By adjusting its shape α and scale η parameters, it can represent a wide range of hazard rate behaviors, including monotonically increasing, decreasing, or constant hazard functions. This versatility makes the Weibull distribution an effective tool for modeling failure mechanisms in engineering systems, medical survival data, and risk assessment applications. Its analytically tractable probability density function (PDF) and hazard rate function, combined with its empirical adaptability, have established the Weibull distribution as a cornerstone in Bayesian inference and fuzzy data modeling frameworks. Figures 1 and 2 illustrate the PDF and hazard rate function for various values of the shape parameter α, with the scale parameter fixed at $\eta=3$. As observed, when $\alpha>1$, the hazard rate function increases over time, which characterizes systems prone to early failures. Conversely, for $\alpha<1$, the hazard rate decreases over time, reflecting systems prone to early failures. When $\alpha=1$, the Weibull distribution reduces to the exponential distribution, resulting in a constant hazard rate over time. This implies a memoryless process where the likelihood of failure is independent of the elapsed time. Furthermore, variations in α affect the shape and skewness of the PDF, corresponding to changes in the concentration and dispersion of failure times.

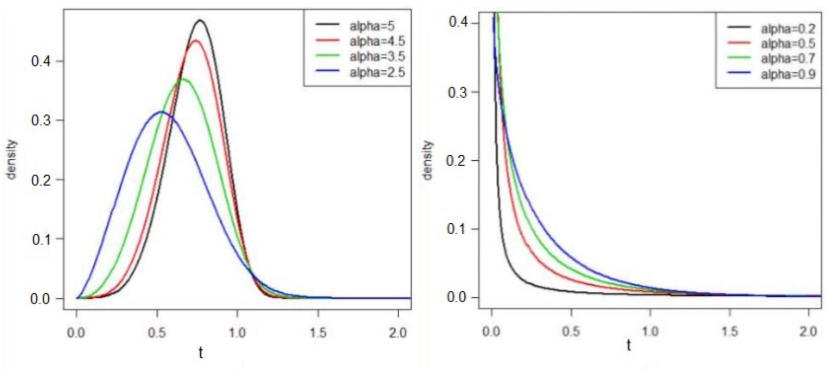

Figure 1: The Probability density function of Weibull distribution for different values of α parameters

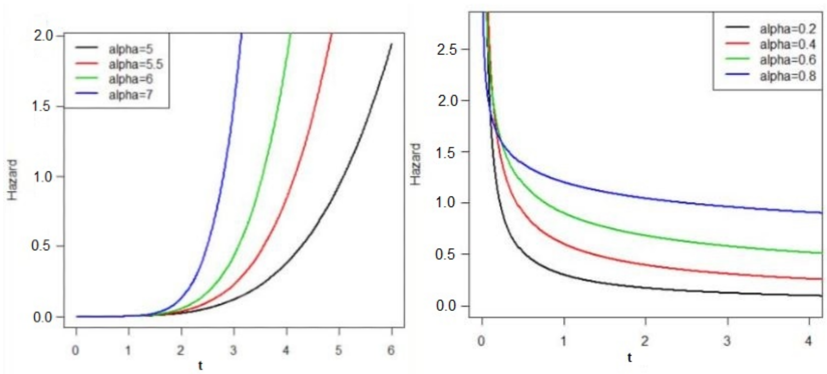

Figure 2: The Hazard rate function of Weibull distribution for different values of α parameters

In this study, the Weibull distribution is referred to as $W(\alpha, \eta)$ for convenience, where $\alpha > 0$ and $\eta > 0$. With the assumption that X and Y are independent random variables with $W(\alpha, \eta_1)$ and $W(\alpha, \eta_2)$ distributions, respectively, the Bayesian, E-B, and H-B estimators of $R = P(X > Y)$ constitute the main focus of this investigation. Additionally, the parameters α,η_1, and η_2 are presumed to be mutually independent.

In lifetime experiments and reliability studies, incomplete information on failure times is a common challenge due to various factors. As a result, researchers frequently encounter censored data. Among the different censoring models, the Type-II censoring scheme is one of the most widely used. In life-testing experiments, suppose n units are subjected to testing. The observed failure times, denoted as $X_{(1)} \leq \cdots \leq X_{(n)}$, represents the order statistics obtained from a random sample of size. However, it is often impractical to continue the experiment until all units fail, as the time required for the final failure may be excessively long [30]. To overcome this issue, the experiment is typically terminated when the r^{th} failure $X_{(r)}$ occurs. This approach defines a Type-II censoring scheme, in which data collection stops after the r^{th} failure. Consequently, only the first r failure times are recorded from a total of n units.

In a Type-II censoring setup, the number of observed failures, r, is predetermined, whereas the actual termination point, $X_{(r)}$, is a random variable. Although this method reduces both testing time and cost, it inevitably results in some loss of information about the underlying parameters. For a detailed discussion on the Type-II censoring scheme, readers may refer to [31, 18, 28].

Type-II censoring is also attractive from a statistical perspective, as it simplifies the likelihood function and facilitates estimation procedures by fixing the number of observed events. Moreover, it provides a balance between data efficiency and practicality, making it highly applicable in engineering and industrial reliability contexts [31]. As discussed in [31], under Type-II censoring, the likelihood function based on the first r

observed failure times from a total of n units are given by

$$L(\alpha,\eta)=\prod_{i=1}^{n} f(x_i;\alpha,\eta)\,[1-F(x_i;\alpha,\eta)]^r$$

which, for the Weibull distribution with parameters α and η, simplifies to

$$\begin{aligned} L(\alpha,\eta) &= \prod_{i=1}^{n} f(x_i;\alpha,\eta)\,[1-F(x_i;\alpha,\eta)]^r \\ &= \alpha^n\eta^n e^{-\eta\sum_{i=1}^{n} x_i^{\alpha}}\left(e^{-\eta\sum_{i=1}^{n} x_i^{\alpha}}\right)^r \\ &= \alpha^n\eta^n e^{-\eta(r+1)\sum x_i^{\alpha}} \end{aligned}$$

E-B and H-B estimation frameworks provide statistically robust alternatives to the classical Bayesian paradigm. The E-B approach utilizes the expected Bayes risk as the decision criterion, resulting in estimators that are optimal on average with respect to the prior distribution. This is particularly advantageous in situations where prior uncertainty is significant. In contrast, H-B models introduce hyperprior structures that enable the modeling of multilevel dependencies and heterogeneity across observational units. These methods enhance inferential accuracy, reduce estimator variance, and offer greater flexibility in representing complex data-generating processes. [32] developed statistical inference techniques for Lomax populations under balanced joint progressive Type-II censoring, thereby enhancing parameter estimation in censored lifetime data. [33, 34] introduced E-B estimation and the corresponding E-MSE under Progressive Type-II Censored Data for certain characteristics of Weibull and compound Rayleigh Distributions.

In statistical sciences, many parameter estimation methods are designed for precise data. However, in practice, it is often infeasible to measure and record exact values for all observations due to various factors. In some cases, data collection is subject to a degree of imprecision caused by unforeseen influences. Consequently, it becomes essential to extend estimation techniques from crisp (exact) values to fuzzy numbers. Several researchers have investigated methods for parameter estimation using imprecise data (see, for instance, [35]-[38]). A previous study [39] derived E-B and H-B calculations for the parameter of the Gompertz Distribution using fuzzy information. More recently, [40]-[42] developed Bayesian, E-B, and H-B estimation methods for the parameters of the Lindley; Rayleigh and Gompertz distributions under Type-II censoring data with fuzzy values, incorporating these methods within the framework of various loss functions.

This study explores the estimation of $R = P\,(X > Y)$ in the context of the Weibull distribution within a Type-II censoring framework. It employs Bayesian, E-B, and H-B techniques to estimate the reliability parameter R while taking into account imprecise data represented as fuzzy integers. The estimation is performed under the squared error loss function, defined as $L\left(\widehat{\theta},\theta\right)=(\widehat{\theta}-\theta)^2$.

This study is structured as follows: Section 2 employs a fuzzy set technique to compute the likelihood function for the Weibull distribution. Section 3 investigates the E-B and H-B estimators of the reliability parameter based on imprecise data, using the squared error loss function. Section 4 describes how Lindley's approximation is used to calculate these estimates. Section 5 presents numerical experiments and Monte Carlo simulations that provide a comprehensive overview of all estimation methods. Additionally, a real-world dataset is analyzed to demonstrate the practical applicability of the proposed methods. Finally, Section 6 summarizes the key results and offers a conclusion.

2 Likelihood Function with Fuzzy Set

Let $X_1,\ldots,X_n$ of size n be i.i.d. lifetimes from a Weibull distribution, which is described by PDF given in Equation (1). Under a Type II censoring scheme, the experiment ends at the rth failure ($r < n$). The first

r failure times are imprecisely observed as fuzzy numbers $\tilde{x}_i$ with membership functions $\mu_{\tilde{x}_i}$, $i = 1, 2, \ldots, r$. Let the maximum value of the means of these fuzzy numbers be $m_{(r)}$. The remaining $n - r$ units, removed at the rth failure, are represented by fuzzy numbers $\tilde{x}_{r+1}, \ldots, \tilde{x}_n$ with membership functions defined as:

$$\mu_{\tilde{x}_j}(x) = \begin{cases} 0 & x \leq m_{(r)} \\ 1 & x > m_{(r)} \end{cases}, \quad j = r+1, \ldots, n.$$

Assume the lifetimes follow a Weibull distribution. If the exact sample $\boldsymbol{x} = (x_1, \ldots, x_n)$ of $\boldsymbol{X}$ were observed, the complete-data likelihood functions could be formulated as:

$$l(\eta, \alpha, \boldsymbol{x}) = \alpha^n \eta^n \left(\prod_{i=1}^{n} {x_i}^{\alpha-1} \right) e^{-\eta \sum_{i=1}^{n} x_i^{\alpha}}$$

Now, consider a situation in which it is impossible to assess or accurately record the measurements of these units of study. Instead, the data are represented as fuzzy numbers. Based on Zadehs definition of the probability of fuzzy events (see [39, 43]), this incomplete information of $\boldsymbol{x}$ can be expressed as a fuzzy subset $\widetilde{\boldsymbol{x}} = (\tilde{x}_1, \ldots, \tilde{x}_n)$, defined by a Borel membership function $\mu_{\widetilde{x}}(\boldsymbol{x}) = \mu_{\tilde{x}_1}(x_1) \times \cdots \times \mu_{\tilde{x}_n}(x_n)$.

Using this approach, the observed-data likelihood function for the parameters η_1 and α is given by:

$$\begin{aligned} L(\alpha, \eta_1, \tilde{x}) &= \int f(x, \eta_1, \alpha)\, \mu_{\tilde{x}}(x)\, dx \\ &= \prod_{j=1}^{n} \int_0^{\infty} \alpha \eta_1 x^{\alpha-1} e^{-\eta_1 x^{\alpha}} \mu_{\tilde{x}_j}(x)\, dx \\ &= \left(\prod_{j=1}^{r} \int_0^{\infty} \alpha \eta_1 x^{\alpha-1} e^{-\eta_1 x^{\alpha}} \mu_{\tilde{x}_j}(x)\, dx \right) \left(\prod_{j=r+1}^{n} \int_0^{\infty} \alpha \eta_1 x^{\alpha-1} e^{-\eta_1 x^{\alpha}} \mu_{\tilde{x}_j}(x)\, dx \right) \\ &= \left(\prod_{j=1}^{r} \int_0^{\infty} \alpha \eta_1 x^{\alpha-1} e^{-\eta_1 x^{\alpha}} \mu_{\tilde{x}_j}(x) dx \right) \\ &\quad \left(\prod_{j=r+1}^{n} \int_0^{m_1(r)} \alpha \eta_1 x^{\alpha-1} e^{-\eta_1 x^{\alpha}} \mu_{\tilde{x}_j}(x)\, dx + \int_{m_1(r)}^{\infty} \alpha \eta_1 x^{\alpha-1} e^{-\eta_1 x^{\alpha}} \mu_{\tilde{x}_j}(x)\, dx \right) \\ &= U_{\eta_1}(\alpha) \left(\prod_{j=r+1}^{n} \int_{m_1(r)}^{\infty} \alpha \eta_1 x^{\eta_1 - 1} e^{-\eta_1 x^{\alpha}} dx \right) \\ &= U_{\eta_1}(\alpha) \left(e^{-\eta_1 m_1^{\alpha}(r)} \right)^{n-r} \\ &= U_{\eta_1}(\alpha)\, e^{-(n-r)\eta_1 m_1^{\alpha}(r)} \end{aligned}$$

Similarly, we obtain, $L(\eta_2, \alpha, \tilde{y}) = U_{\eta_2}(\alpha)\, e^{-(n-r)\eta_2 m_2^{\alpha}(r)}$. Where

$$U_{\eta_1}(\alpha) = \prod_{j=1}^{r} \int_{\circ}^{\infty} \alpha \eta_1 x^{\alpha-1} e^{-\eta_1 x^{\alpha}} \mu_{\tilde{x}_j}(x)\, dx \qquad U_{\eta_2}(\alpha) = \prod_{j=1}^{r} \int_{\circ}^{\infty} \alpha \eta_1 y^{\alpha-1} e^{-\eta_2 y^{\alpha}} \mu_{\tilde{y}_j}(y)\, dy$$

Then, likelihood function obtained as

$$L(\eta_1, \eta_2, \alpha) = U_{\eta_1}(\alpha)\, U_{\eta_2}(\alpha)\, e^{-(n-r)\eta_1 m_1^{\alpha}(r) - (n-r)\eta_2 m_2^{\alpha}(r)} \tag{3}$$

3 E-Bayesian and H-Bayesian estimations of R

The E-B and H-B methods extend the classical Bayesian estimation framework by addressing key limitations related to the specification of prior hyperparameters. Traditional Bayesian estimation is often sensitive to the choice of hyperparameters, which can lead to biased or unstable results. The E-B approach averages Bayesian estimators over a distribution of hyperparameters, thereby reducing this sensitivity and providing more robust estimates. Similarly, H-B methods incorporate an additional layer in the model hierarchy to estimate hyperparameters from the data itself, enhancing flexibility and allowing for more accurate inference [44]. These methods have been shown to improve estimation accuracy and robustness in various statistical models, particularly in reliability and lifetime data analysis [6]. Therefore, their application in this study is motivated by the need for more stable and reliable estimation of the $R = P(X > Y)$ parameter under Type-II censoring and in the presence of fuzzy data.

The Bayesian, E-B, and H-B of $R = P(X > Y)$ are obtained in this section when X and Y are independent random variables with $W(\alpha, \eta_1)$ and $W(\alpha, \eta_2)$ distributions, respectively. Additionally, the parameters α, η_1 and η_2 are assumed to be independent. The E-B and H-B estimates are described as follows, respectively, in accordance with [14].

Definition 3.1. [14] *Assume that b_1 and b_2 are hyperparameters in the prior density function of θ. The joint prior density function of (b_1, b_2) is $\pi(b_1, b_2)$, and let the Bayesian estimate of θ is $\widehat{\theta}_B(b_1, b_2)$ Then, the E-B estimator of θ is given by*

$$\widehat{\theta}_{EB} = E_{\pi(b_1,b_2)}(\widehat{\theta}_B(b_1, b_2)) = \int_{\mathit{\Lambda}_1}\int_{\mathit{\Lambda}_2} \widehat{\theta}_B(a_1, a_2)\,\pi(b_1, b_2)\,db_1 db_2; b_1 \in \mathit{\Lambda}_1, b_2 \in \mathit{\Lambda}_2$$

where $\mathit{\Lambda}_1$ and $\mathit{\Lambda}_2$ are the domains of b_1 and b_2.

Furthermore, the H-B estimator incorporates uncertainty regarding the hyperparameters by treating them as random variables with their own prior distributions. In the hierarchical Bayesian setup, the hyperparameters are modeled as random variables. The final estimator of θ is then obtained by integrating over the joint posterior distribution of θ, resulting in the posterior mean:

$$\hat{\theta}_{HB} = \int \theta \cdot \pi(\theta \mid data)\, d\theta$$

Here, data refers to the observed information under a Type-II censoring scheme, where the recorded values are represented as fuzzy numbers. These observations reflect both imprecision and partial information, forming the basis for the posterior inference of the parameter θ. The hierarchical Bayesian estimator is obtained by integrating over the joint posterior distribution of the parameters and hyperparameters (see [44]).

Definition 3.2. [14] *If the prior density of the parameter θ and hyperparameter ν are represented by $\pi(\theta \mid \nu)$ and $\pi'(\nu)$, respectively, then the hierarchical prior density of θ is given by:*

$$\pi''(\theta) = \int_{\Lambda} \pi(\theta|\nu)\,\pi'(\nu)\,d\nu; \nu \in \Lambda$$

It is also assumed the parameters η_1, η_2 and α have the following prior density functions,

$$\pi_1(\eta_1 \mid a_1, b_1) = \frac{b_1^{a_1}}{\mathit{\Gamma}(a_1)} \eta_1^{a_1-1} e^{-b_1\eta_1}; a_1 > 0, b_1 > 0, \tag{4}$$

$$\pi_2(\eta_2 \mid a_2, b_2) = \frac{b_2^{a_2}}{\mathit{\Gamma}(a_2)} \eta_2^{a_2-1} e^{-b_2\eta_2}; a_2 > 0, b_2 > 0, \tag{5}$$

and

$$\pi_3(\alpha \mid a_3, b_3) = \frac{b_3^{a_3}}{\Gamma(a_3)} \alpha^{a_3-1} e^{-b_3\alpha}; \alpha > 0, a_3 > 0, b_3 > 0, \tag{6}$$

respectively. The selection of the Gamma distribution as the prior distribution for the parameters of the Weibull distribution is based on theoretical foundations and computational convenience. Since the shape (α) and scale (η) parameters of the Weibull distribution take positive values, the Gamma distributiondefined over the positive real line and possessing flexible shape characteristicsis considered a natural and effective prior choice. Importantly, in many cases, the Gamma distribution is conjugate to the Weibull likelihood function; this property allows for analytical tractability and closed-form Bayesian estimators [6, 44]. Such conjugacy significantly reduces computational burden and facilitates the inference process. Additionally, the hyperparameters of the Gamma prior can be adjusted to incorporate prior knowledge or empirical data, thereby enhancing the flexibility and robustness of the Bayesian analysis. According to [14] in equation (4), a_1 and b_1 should be chosen to guarantee that $\pi_1(\eta_1 \mid a_1, b_1)$ is a decreasing function of ν_1. Then,

$$\frac{d\pi_1(\eta_1 \mid a_1, b_1)}{d\eta_1} = \frac{b_1^{a_1} \eta_1^{a_1-2} e^{-b_1\eta_1}}{\Gamma(a_1)}((a_1 - 1) - b_1\eta_1)$$

where $b_1 > 0, 0 < a_1 \leq 1$. In [45], Berger showed that a large b_1 decreases the Bayesian estimation efficiency of η_1. Then, $0 < b_1 < c_1$ and c_1 is a fixed number. The prior distribution of b_1, $\pi_3(b_1)$, is assumed to be a continuous uniform distribution in $(0\ , c_1)$. Also according to (4), if $a_1 = 1$, then we have

$$\pi_1(\eta_1 \mid b_1) = b_1 e^{-b_1\eta_1} \tag{7}$$

where $b_1 > 0$ and $\eta_1 > 0$. Similarly, for b_2 and b_3, we obtain

$$\pi_2(\eta_2 \mid b_2) = b_2 e^{-b_2\eta_2}, \tag{8}$$

$$\pi_3(\alpha \mid b_3) = b_3 e^{-b_3\alpha} \tag{9}$$

respectively, where $b_2 > 0$, $\eta_2 > 0$, $\alpha > 0$, and $\eta_3 > 0$.

Now, when $X \sim W(\alpha, \eta_1)$ and $Y \sim W(\alpha, \eta_2)$ are independently distributed, we get E-B and H-B estimators of $R = P(X > Y)$. To do this, we use the squared error (SE) loss function to calculate the E-B and H-B estimators of the R parameter based on imprecise data and under the Type-II censoring framework. Assuming that $\tilde{z} = (\tilde{x}, \tilde{y})$, we derive from equations (3), (7), (8), and (9)

$$\pi(\eta_1, \eta_2, \alpha \mid \tilde{z}) = \frac{U_{\eta_1}(\alpha) U_{\eta_2}(\alpha) e^{-(C_1(\alpha)\eta_1 + C_2(\alpha)\eta_2 + b_3\alpha)}}{\int_\circ^\infty \int_\circ^\infty \int_\circ^\infty U_{\eta_1}(\alpha) U_{\eta_2}(\alpha) e^{-(C_1(\alpha)\eta_1 + C_2(\alpha)\eta_2 + b_3\alpha)} d\eta_1 d\eta_2 d\alpha}$$

where, $C_1(\alpha) = b_1 + m_1^\alpha(r)$ and $C_2(\alpha) = b_2 + m_2^\alpha(r)\alpha$.

Then, using the Weibull distribution with PDF provided in Equation (1) and the SE loss function, the Bayesian estimator of $R = \frac{\eta_2}{\eta_1 + \eta_2}$ is computed by

$$\hat{R}_{BS}(b_1, b_2) = \frac{\int_\circ^\infty \int_\circ^\infty \int_\circ^\infty \frac{\eta_2}{\eta_1+\eta_2} U_{\eta_1}(\alpha) U_{\eta_2}(\alpha) e^{-(C_1(\alpha)\eta_1 + C_2(\alpha)\eta_2 + b_3\alpha)} d\eta_1 d\eta_2 d\alpha}{\int_\circ^\infty \int_\circ^\infty \int_\circ^\infty U_{\eta_1}(\alpha) U_{\eta_2}(\alpha) e^{-(C_1(\alpha)\eta_1 + C_2(\alpha)\eta_2 + b_3\alpha)} d\eta_1 d\eta_2 d\alpha} \tag{10}$$

4 Lindley's Approximation

We use the Lindley approximation to obtain the Bayesian estimator since Equation (10) cannot be computed analytically [46]. The expectation $E(u(\nu) \mid) z)$ is often given by

$$E(u(\nu) \mid z) = \frac{\int u(\nu) e^{Q(\nu)} d\nu}{\int e^{Q(\nu)} d\nu} \tag{11}$$

In above relation, $Q(\nu) = l(\nu) + \rho(\nu)$, $l(\nu)$, with $l(\nu)$, where $\rho(\nu)$ is the logarithm of the prior density of ν and $l(\nu)$ is the logarithm of the likelihood function. The expectation $E(u(\nu \mid z)$ is then derived as follows:

$$E(u(\nu) \mid z) = \left[u(\nu) + \frac{1}{2}\sum_i \sum_j (u_{ij} + 2u_i\rho_j)\sigma_{ij} + \frac{1}{2}\sum_i \sum_j \sum_k \sum_l L_{ijk}\sigma_{ij}\sigma_{kl}u_l \right]_{\widehat{\nu}} \tag{12}$$

In this context, consider a parameter vector represented as $\boldsymbol{\nu} = (\nu_1, \ldots, \nu_m)$, $i, j, k, l = 1, \ldots, m$, $\widehat{\nu}$ is the maximum likelihood estimate (MLE) of $\boldsymbol{\nu}$, $u_i = \frac{\vartheta_u(\nu)}{\vartheta \nu_i}$, $u_{ij} = \frac{\vartheta^2 u(\nu)}{\vartheta \nu_i \vartheta \nu_j}$, $\rho_j = \frac{\vartheta_\rho(\nu)}{\vartheta \nu_j}$, $L_{ijk} = \frac{\vartheta^3 l(\nu)}{\vartheta \nu_i \vartheta \nu_j \vartheta \nu_k}$. Additionally, σ_{ij} corresponds to the $(i,j) - th$ element in the inverse of the Fisher information matrix $\{-L_{ij}\}$, where all components are evaluated at the MLE of the parameters. For the particular scenario involving three parameters where $\boldsymbol{\nu}$ is expressed as $\boldsymbol{\nu} = (\nu_1 = \nu_1, \nu_2 = \nu_2, \nu_3 = \alpha)$, applying these definitions to Equation (12) yields:

$$E(u(\nu) \mid z) = \left[u(\nu) + \sum_{i=1}^{3} u_i a_i + \alpha_4 + a_5 + \frac{1}{2}\left(A\sum_{i=1}^{3} u_i\sigma_{1i} + B\sum_{i=1}^{3} u_i\sigma_{2i} + C\sum_{i=1}^{3} u_i\sigma_{3i} \right) \right]_{(\widehat{\nu}_1, \widehat{\nu}_2, \widehat{\nu}_3)}$$

where,

$$\begin{aligned}
&a_i = \sum_{j=1}^{3} \rho_j\sigma_{ij}, i = 1,2,3,\ a_4 = u_{12}\sigma_{12} + u_{13}\sigma_{13} + u_{23}\sigma_{23},\ a_5 = \sum_{i=1}^{3} u_{ii}\sigma_{ii} \\
&A = \sum_{i=1}^{3} \sigma_{ii}L_{ii1} + 2(\sigma_{12}L_{121} + \sigma_{13}L_{131} + \sigma_{23}L_{231}) \\
&B = \sum_{i=1}^{3} \sigma_{ii}L_{ii2} + 2(\sigma_{12}L_{122} + \sigma_{13}L_{132} + \sigma_{23}L_{232}) \\
&C = \sum_{i=1}^{3} \sigma_{ii}L_{ii3} + 2(\sigma_{12}L_{123} + \sigma_{13}L_{133} + \sigma_{23}L_{233}), D = L_{11}L_{23}^2 - L_{11}L_{22}L_{33} + L_{13}^2L_{22} \\
&\sigma_{11} = \frac{L_{22}L_{33} - L_{13}^2}{D},\ \sigma_{12} = \sigma_{21} = \frac{L_{11}L_{13}}{D},\ \sigma_{13} = \sigma_{31} = \frac{L_{13}L_{22}}{D},\ \sigma_{23} = \sigma_{32} = -\frac{L_{11}L_{23}}{D}, \\
&\sigma_{22} = \frac{L_{11}L_{33} - L_{13}^2}{D},\ \sigma_{33} = \frac{L_{11}L_{22}}{D}
\end{aligned}$$

Also, $\widehat{\nu}_1$, $\widehat{\nu}_2$ and $\widehat{\nu}_3$ are the MLEs of ν_1, ν_2 and ν_3, respectively. In our case, for $(\nu_1,\ \nu_2,\ \nu_3) \equiv (\eta_1, \eta_2,\ \alpha)$,

$u(\eta_1,\ \eta_2,\ \alpha) = \frac{\eta_2}{\eta_1+\eta_2}$, $l(\nu) = \ln L(\eta_1,\ \eta_2,\ \alpha)$ and $\rho(\nu) = -b_1\eta_1 - b_2\eta_2 - b_3\alpha$, we have

$$\rho_i = b_i,\ i = 1, 2, 3,$$

$$u_1 = \frac{-\eta_2}{(\eta_2+\eta_1)^2},\ u_2 = \frac{\eta_1}{(\eta_2+\eta_1)^2},\ u_3 = \circ,\ u_{12} = \frac{\eta_2-\eta_1}{(\eta_2+\eta_1)^2},$$

$$u_{13} = u_{23} = 0,\ L_{11} = -\frac{r}{\eta_1^2} - \sum_{j=1}^{r} \frac{I_3^{\eta_1} I_1^{\eta_1} - (I_2^{\eta_1})^2}{(I_1^{\eta_1})^2},$$

$$L_{22} = -\frac{r}{\eta_2^2} - \sum_{j=1}^{r} \frac{I_3^{\eta_2} I_1^{\eta_2} - (I_2^{\eta_2})^2}{(I_1^{\eta_2})^2} = -\frac{2r}{\alpha^2} - (n-r)\, m_{1(r)}^2 \left(\ln m_{1(r)}\right)^2 - (n-r)\, m_{2(r)}^2 \left(\ln m_{2(r)}\right)^2$$

$$\begin{aligned} L_{33} = &-\frac{2r}{\alpha^2} - (n-r)\, m_{1(r)}^2 \left(\ln m_{1(r)}\right)^2 - (n-r)\, m_{2(r)}^2 \left(\ln m_{2(r)}\right)^2 \\ &+ \sum_{j=1}^{r} \frac{J_1^{\eta_1} I_1^{\eta_1} - 3\eta_1 J_2^{\eta_1} I_1^{\eta_1} + \eta_1^2 J_3^{\eta_1} I_1^{\eta_1} - (H_1^{\eta_1})^2 + 2\eta_1 H_1^{\eta_1} H_2^{\eta_1} - \eta_1^2 (H_2^{\eta_1})^2}{(I_1^{\eta_1})^2} \\ &+ \sum_{j=1}^{r} \frac{J_1^{\eta_2} I_1^{\eta_2} - 3\eta_2 J_2^{\eta_2} I_1^{\eta_2} + \eta_2^2 J_3^{\eta_2} I_1^{\eta_2} - (H_1^{\eta_2})^2 + 2\eta_2 H_1^{\eta_2} H_2^{\eta_2} - \eta_2^2 (H_2^{\eta_2})^2}{(I_1^{\eta_2})^2}, \end{aligned}$$

where

$$J_i^{\eta_1} = \int_\circ^\infty x^{i\alpha-1} (\ln x)^2 e^{-\eta_1 x^\alpha} \mu_{\tilde{x}_j}(x)\, dx, \quad I_i^{\eta_1} = \int_\circ^\infty x^{i\alpha-1} e^{-\eta_1 x^\alpha} \mu_{\tilde{x}_j}(x)\, dx,$$

$$J_i^{\eta_2} = \int_\circ^\infty y^{i\alpha-1} (\ln y)^2 e^{-\eta_2 y^\alpha} \mu_{\tilde{y}_j}(y)\, dy, \quad I_i^{\eta_2} = \int_\circ^\infty y^{i\alpha-1} e^{-\eta_2 y^\alpha} \mu_{\tilde{y}_j}(y)\, dy,$$

$$H_i^{\eta_1} = \int_\circ^\infty x^{i\alpha-1} (\ln x)\, e^{-\eta_1 x^\alpha} \mu_{\tilde{x}_j}(x)\, dx, \quad H_i^{\eta_2} = \int_\circ^\infty y^{i\alpha-1} (\ln y) e^{-\eta_2 y^\alpha} \mu_{\tilde{y}_j}(y)\, dy$$

Similarly, we can obtain L_{111}, L_{222}, $L_{133} = L_{331}$, $L_{233} = L_{332}$, and L_{333}. Hence,

$$a_4 = u_{12}\sigma_{12},\ a_5 = \frac{u_{11}\sigma_{11} + u_{22}\sigma_{22}}{2},$$

$A = \sigma_{11}L_{111} + \sigma_{33}L_{331}, B = \sigma_{22}L_{222} + \sigma_{33}L_{332}$ and $C = 2\sigma_{13}L_{133} + 2\sigma_{23}L_{133} + \sigma_{33}L_{333}$.

The Bayesian estimator of R is then given by

$$\begin{aligned} \hat{R}_{BS}(b_1, b_2) = &R + \left(u_1 a_1 + u_2 a_2 + u_{12}\sigma_{12} + \frac{u_{11}\sigma_{11} + u_{22}\sigma_{22}}{2}\right) \\ &+ \frac{1}{2}\left(\sigma_{11}L_{111} + \sigma_{33}L_{333}\right)\left(u_1\sigma_{11} + u_2\sigma_{12}\right) \\ &+ \left(\sigma_{22}L_{222} + \sigma_{33}L_{333}\right)\left(u_1\sigma_{21} + u_2\sigma_{22}\right) \\ &+ \frac{1}{2}\left(2\sigma_{13}L_{133} + 2\sigma_{23}L_{233} + \sigma_{33}L_{333}\right)\left(u_1\sigma_{31} + u_2\sigma_{32}\right). \end{aligned} \tag{13}$$

Keep in mind that every metric is evaluated at $(\widehat{\eta}_1, \widehat{\eta}_2, \widehat{\alpha})$. Based on Equation (13) and Definition 3.1 provide the E-B estimator of R is then determined as follows:

$$\begin{aligned} \hat{R}_{EB} = &R + \rho_3\sigma_{13} + \rho_3\sigma_{23} + u_{12}\sigma_{12} + \frac{u_{11}\sigma_{11} + u_{22}\sigma_{22}}{2} \\ &+ \frac{1}{2}\left(\sigma_{11}L_{111} + \sigma_{33}L_{333}\right)\left(u_1\sigma_{11} + u_2\sigma_{12}\right) \\ &+ \frac{1}{2}\left(2\sigma_{13}L_{133} + 2\sigma_{23}L_{233} + \sigma_{33}L_{333}\right)\left(u_1\sigma_{31} + u_2\sigma_{32}\right) - \frac{c_1}{2}\sigma_{11} - \frac{c_2}{2}\sigma_{12} \end{aligned}$$

The H-B estimator of R is now derived. Based on Definition 3.2, the prior distribution of the hierarchical parameter α is given by

$$\pi\left(\alpha\right)=\frac{1-c_3\alpha e^{-c_3\alpha}-e^{-c_3\alpha}}{c_3\alpha^2}$$

Therefore, the hierarchical posterior joint distribution of η_1, η_2 and α is

$$\pi^*\left(\eta_1,\eta_2,\alpha\mid\tilde{z}\right)=\frac{\alpha^{2r}\left(\eta_1\eta_2\right)^r U_{\eta_1}\left(\alpha\right)U_{\eta_2}\left(\alpha\right)\pi\left(\alpha\right)e^{-(n-r)\left(m_{1(r)}^{\alpha}+m_{2(r)}^{\alpha}\right)}}{\int_\circ^\infty\int_\circ^\infty\int_\circ^\infty\alpha^{2r}\left(\eta_1\eta_2\right)^r U_{\eta_1}\left(\alpha\right)U_{\eta_2}\left(\alpha\right)\pi\left(\alpha\right)e^{-(n-r)\left(m_{1(r)}^{\alpha}+m_{2(r)}^{\alpha}\right)}d\eta_1 d\eta_2 d\alpha}$$

R's H-Bayesian estimator under the SE loss function is thus provided by

$$\hat{R}_{HBS}=\frac{\int_\circ^\infty\int_\circ^\infty\int_\circ^\infty\frac{\eta_2}{\eta_1+\eta_2}\alpha^{2r}\left(\eta_1\eta_2\right)^r U_{\eta_1}\left(\alpha\right)U_{\eta_2}\left(\alpha\right)\pi\left(\alpha\right)e^{-(n-r)\left(m_{1(r)}^{\alpha}+m_{2(r)}^{\alpha}\right)}d\eta_1 d\eta_2 d\alpha}{\int_\circ^\infty\int_\circ^\infty\int_\circ^\infty\alpha^{2r}\left(\eta_1\eta_2\right)^r U_{\eta_1}\left(\alpha\right)U_{\eta_2}\left(\alpha\right)\pi\left(\alpha\right)e^{-(n-r)\left(m_{1(r)}^{\alpha}+m_{2(r)}^{\alpha}\right)}d\eta_1 d\eta_2 d\alpha}$$

Note that R_{HBS} is also calculated using Lindleys approximation, which has the same structure as Equation (13), except that the logarithm of the prior density of ν's logarithm is given by

$$\rho\left(\nu\right)=\ln\left(\frac{1-c_3\alpha e^{-c_3\alpha}-e^{-c_3\alpha}}{c_3\alpha^2}\right)$$

5 Numerical Experiments

To assess and compare the effectiveness of the estimators discussed in the previous sections, this section presents the results of a Monte Carlo simulation and numerical experiments. All calculations and programming were performed using R software.

5.1 Simulation Study

This subsection presents experimental results related to the estimation methods under discussion. The $W(\alpha,\eta_1)$ and $W(\alpha,\eta_2)$ distributions' differences in sample sizes and stress-strength characteristics are the main focus of the investigation. To assess and contrast the effectiveness of Bayesian, E-B, and H-B estimation techniques in predicting the stress-strength variable, a Monte Carlo simulation research is carried out. The assessment is based on key efficiency measures, such as average values (AV) and mean square error (MSE). Furthermore, the study examines the impact of different sample sizes on the performance of these methods. The simulation process follows the steps outlined below and was executed using R software:

Step 1. Generate random samples of varying sizes from the $U(0,1)$ distribution.

Step 2. Consider $X=\left[-\frac{1}{\eta}ln(1-U)\right]^{\frac{1}{\alpha}}$, where U is a continuous random variable uniformly distributed over the interval $[0,1]$. Using this formulation, random samples of varying sizes were generated from two Weibull distributions $W(\alpha,\eta_1)$ and $W(\alpha,\eta_2)$. The parameter sets used for these distributions are $(\alpha,\eta_1,\eta_2)=(1,2,2),(2,2,3)$, and $(2.5,3,2.5)$.

Step 3. Each sample generated from X in Step 2 and subjected to Type-II censoring is treated as fuzzy sample. This is done using the fuzzy framework described by [35, 6], employing the following membership functions:

$$\mu_{\tilde{x}_1}(x) = \begin{cases} 1 & x \le 0.25 \\ \frac{0.5-x}{0.25} & 0.25 \le x \le 0.5 \\ 0 & \text{otherwise} \end{cases} \qquad \mu_{\tilde{x}_2}(x) = \begin{cases} \frac{x-0.25}{0.25} & 0.25 \le x \le 0.5 \\ \frac{0.75-x}{0.25} & 0.5 \le x \le 0.75 \\ 0 & \text{otherwise} \end{cases}$$

$$\mu_{\tilde{x}_3}(x) = \begin{cases} \frac{x-0.5}{0.25} & 0.5 \le x \le 0.75 \\ \frac{1-x}{0.25} & 0.75 \le x \le 1 \\ 0 & \text{otherwise} \end{cases} \qquad \mu_{\tilde{x}_4}(x) = \begin{cases} \frac{x-0.75}{0.25} & 0.75 \le x \le 1 \\ \frac{1.25-x}{0.25} & 1 \le x \le 1.25 \\ 0 & \text{otherwise} \end{cases}$$

$$\mu_{\tilde{x}_5}(x) = \begin{cases} \frac{x-1}{0.25} & 1 \le x \le 1.25 \\ \frac{1.5-x}{0.25} & 1.25 \le x \le 1.5 \\ 0 & \text{otherwise} \end{cases} \qquad \mu_{\tilde{x}_6}(x) = \begin{cases} \frac{x-1.25}{0.25} & 1.25 \le x \le 1.5 \\ \frac{1.75-x}{0.25} & 1.5 \le x \le 1.75 \\ 0 & \text{otherwise} \end{cases}$$

$$\mu_{\tilde{x}_7}(x) = \begin{cases} \frac{x-1.5}{0.25} & 1.5 \le x \le 1.75 \\ \frac{2-x}{0.25} & 1.75 \le x \le 2 \\ 0 & \text{otherwise} \end{cases} \qquad \mu_{\tilde{x}_8}(x) = \begin{cases} \frac{x-1.75}{0.25} & 1.75 \le x \le 2 \\ 1 & x \ge 2 \\ 0 & \text{otherwise} \end{cases}$$

Step 4. This sample is used to compute the Bayesian, E-B, and H-B estimates of R for the parameter settings $c_1 = 3$ and $c_2 = c_3 = 5$.

After 1000 iterations of Steps 1 through 3, average values and MSEs were computed. R software is used for all calculations. Tables 1 through 3 present the simulation results as well as the actual values of $R = P(X > Y)$. Furthermore, Figure 3 shows the estimates of the R's parameter for each approach for various parameter values α, η_1 and η_2. The results show that, under the SE loss function, the H-B estimating approach outperforms the others. In particular, the H-B estimator consistently displays fewer MSEs than the other estimators across all sample sizes and with the fixed value of $r = 8$ for $(\alpha, \eta_1, \eta_2) = (1, 2, 2)$. This implies that, under the SE loss function, the H-B estimator provides better accuracy in predicting the R's parameter, especially for these parameter levels.

Table 1: Average values and mean squared errors for different estimators of R with $\alpha = 1,\ \eta_1 = 2,\ \eta_2 = 2$

$R_{real} = 0.5$	$\hat{R}_{BS}$		$\hat{R}_{EBS}$		$\hat{R}_{HBS}$	
n	AV	MSE	AV	MSE	AV	MSE
10	.6012	.1254	.6914	.1564	.5876	.1012
20	.6004	.1117	.6754	.1412	.5567	.0987
30	.5876	.1097	.6254	.1256	.5214	.0897
40	.5565	.0997	.5889	.1118	.5191	.0712
50	.5209	.0876	.5343	.1088	.4997	.0549

5.2 Analysis of real data

This subsection provides a detailed analysis of the strength data presented in [47]. The dataset includes strength measurements in gigapascals (GPa) for both impregnated 1000-carbon fiber tows and individual carbon fibers. Tensile tests were conducted on the single fibers, using gauge lengths of 20 mm for Data Set 1 and 10 mm for Data Set 2. These datasets have been previously used in other studies, including those documented in sources [25].

Table 2: Average values and mean squared errors for different estimators of R with $\alpha = 2,\ \eta_1 = 2,\ \eta_2 = 3$

$R_{real} = 0.6$	$\hat{R}_{BS}$		$\hat{R}_{EBS}$		$\hat{R}_{HBS}$	
n	AV	MSE	AV	MSE	AV	MSE
10	.7151	.2215	.7654	.3217	.7091	.2077
20	.7113	.2176	.7566	.3141	.6897	.1987
30	.7015	.2011	.7519	.3092	.6679	.1707
40	.6800	.1917	.7349	.2897	.6512	.1560
50	.6991	.1888	.7188	.2675	.6226	.1278

Table 3: Average values and mean squared errors for different estimators of R with $\alpha = 2.5,\ \eta_1 = 3,\ \eta_2 = 2.5$

$R_{real} = 0.454$	$\hat{R}_{BS}$		$\hat{R}_{EBS}$		$\hat{R}_{HBS}$	
n	AV	MSE	AV	MSE	AV	MSE
10	.5267	.3560	.5613	.4532	.4672	.3203
20	.5097	.3211	.5322	.4232	.4570	.3067
30	.4965	.3176	.4566	.2896	.5247	.3476
40	.4657	.2987	.5054	.3248	.4559	.2553
50	.4548	.2589	.4678	.3095	.4548	.2415

In our analysis, the lifetimes of the fibers is modeled as triangular fuzzy numbers. The data are summarized in Tables 4 and 5, where the actual measurements are treated as the central points of the fuzzy numbers. To model variability, we generate two random samples u_1 and u_2 from a uniform distribution $U(0.03, 0.05)$, with U representing a uniform distribution. Using these random samples, we determine the left and right ambiguities of the triangular fuzzy numbers by incorporating them into the real data. Essentially, the imprecision in the measurements is expressed using fuzzy numbers defined as $\tilde{x}_i = (x_i, x_i u_1, x_i u_2)$. The KolmogorovSmirnov (KS) and AndersonDarling (AD) tests were conducted to evaluate the distribution of the actual tensile strength and resistance data. The results are presented in Table 6. Both the KS and AD test results indicate that the Weibull distribution provides an acceptable and appropriate fit for the data.

Based on real data and under the SE Loss function with $c_1 = 2$ and $c_2 = 3$, the Bayesian, E-B, and H-B estimators of the reliability parameter $R = P(X > Y)$ are 0.6895, 0.5679, and 0.7865, respectively. Since R represents system reliability, with higher values indicating better performance, the H-B estimator which provides the largest estimate among the three methods can be considered the most appropriate estimator for R in this study. It is observed that the results obtained from the real data are consistent with the findings of the simulation study.

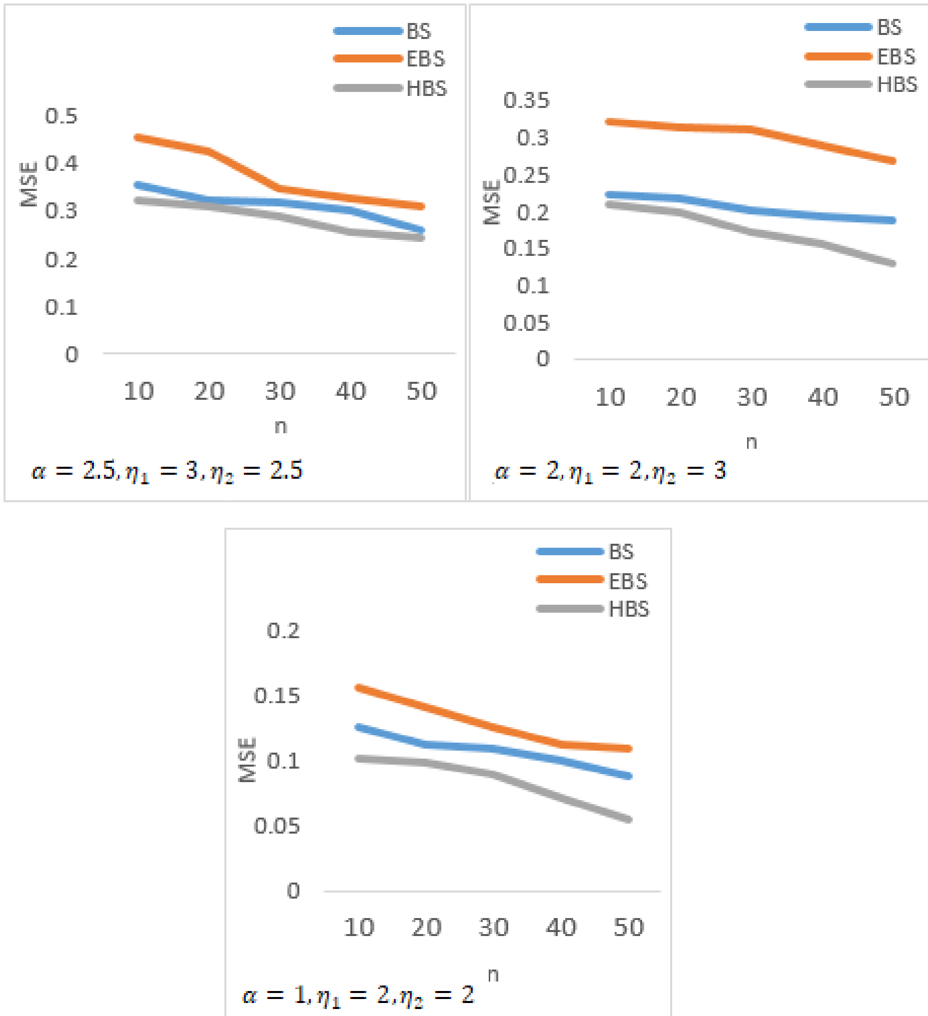

Figure 3: Chart of the Bayesian estimation (BS), E-B estimation (EBS) and H-B estimation (HBS) of R.

Table 6: Results of K-S and A-D tests on actual tensile strength and resistance data

Test	**Test Statistic**	**p-value**
KolmogorovSmirnov (KS)	0.142	0.983
AndersonDarling (AD)	0.432	0.814

6 Conclusion

This work aimed to estimate the stressstrength parameter, $R = P(X > Y)$, using several approaches. Here, X and Y represent independent Weibull random variables characterized by distinct scale parameters while sharing a common shape parameter. The study developed Bayesian, E-B, and H-B estimators for R, incorporating imprecise prior information and a Type-II censoring scheme. Both Monte Carlo simulations and real-world data were used to evaluate the performance of the proposed estimators. The results indicate that increasing the sample size (n) and the number of completely observed (uncensored) samples (r) reduces the mean squared error (MSE) of the Bayesian estimators, reflecting improved estimation accuracy. Furthermore,

Table 4: Data Set 1 (gauge lengths of 20 mm)

(0.8829, 1.312, 1.998)	(0.8848, 1.314, 2.001)	(1.049, 1.479, 2.182)	(1.228, 1.552, 2.262)
(1.271, 1.700, 2.425)	(1.374, 1.803, 2.602)	(1.432, 1.861, 2.606)	(1.435, 1.865, 2.694)
(1.515, 1.944, 2.709)	(1.528, 1.958, 2.718)	(1.536, 1.966, 2.761)	(1.576, 1.997, 2.778)
(1.591, 2.006, 2.784)	(1.597, 2.021, 2.815)	(1.625, 2.027, 2.824)	(1.633, 2.055, 2.863)
(1.668, 2.063, 2.909)	(1.710, 2.098, 2.952)	(1.749, 2.140, 3.001)	(1.749, 2.179, 3.019)
(1.794, 2.224, 3.033)	(1.810, 2.240, 3.052)	(1.823, 2.253, 3.056)	(1.840, 2.270, 3.086)
(1.842, 2.272, 3.086)	(1.844, 2.274, 3.086)	(1.871, 2.301, 3.150)	(1.871, 2.301, 3.175)
(1.929, 2.359, 3.175)	(1.952, 2.382, 3.223)	(1.952, 2.382, 3.232)	(1.996, 2.426, 3.233)
(2.004, 2.434, 3.281)	(2.005, 2.435, 3.281)	(2.048, 2.478, 3.294)	(2.060, 2.490, 3.294)
(2.081, 2.511, 3.317)	(2.084, 2.514, 3.320)	(2.105, 2.535, 3.343)	(2.124, 2.554, 3.364)
(2.136, 2.566, 3.377)	(2.140, 2.570, 3.382)	(2.126, 2.586, 3.399)	(2.199, 2.629, 3.447)
(2.203, 2.633, 3.451)	(2.212, 2.642, 3.461)	(2.218, 2.648, 3.468)	(2.254, 2.684. 3.507)
(2.267, 2.697, 3.521)	(2.296, 2.726, 3.553)	(2.340, 2.770, 3.602)	(2.343, 2.773, 3.605)
(2.370, 2.800, 3.635)	(2.379, 2.809, 3.645)	(2.388, 2.818, 3.655)	(2.391, 2.821, 3.658)
(2.418, 2.848, 3.688)	(2.450, 2.880, 3.723)	(2.524, 2.954, 3.804)	(2.582, 3.012, 3.868)
(2.637, 3.067, 3.928)	(2.654, 3.084, 3.947)	(2.660, 3.090, 3.954)	(2.666, 3.096, 3.960)
(2.968, 3.128, 3.996)	(2.803, 3.233, 4.111)	(3.003, 3.433, 4.331)	(3.155, 3.585, 4.498)
(3.155, 3.585, 4.498)			

Table 5: Data Set 2 (gauge lengths of 10 mm)

(1.255, 1.901, 2.556)	(1.486, 2.132, 2.786)	(1.557, 2.203, 2.858)	(1.582, 2.228, 2.883)
(1.611, 2.257, 2.912)	(1.704, 2.350, 2.912)	(1.704, 2.361, 3.005)	(1.750, 2.396, 3.016)
(1.751, 2.397, 3.016)	(1.799, 2.445, 3.052)	(1.808, 2.454, 3.100)	(1.799, 2.474, 3.109)
(1.808, 2.518, 3.129)	(1.828, 2.522, 3.173)	(1.872, 2.525, 3.177)	(1.886, 2.532, 3.180)
(1.929, 2.575, 3.187)	(1.968, 2.614, 3.230)	(1.970, 2.616, 3.269)	(1.972, 2.618, 3.271)
(1.978, 2.624, 3.273)	(2.013, 2.659, 3.279)	(2.029, 2.675, 3.314)	(2.092, 2.738, 3.330)
(2.092, 2.740, 3.393)	(2.094, 2.856, 3.395)	(2.210, 2.917, 3.511)	(2.271, 2.928, 3.372)
(2.282, 2.937, 3.583)	(2.291, 2.937, 3.592)	(2.291, 2.977, 3.592)	(2.331, 2.996, 3.632)
(2.350, 3.030, 3.651)	(2.384, 3.125, 3.985)	(2.479, 3.139, 3.780)	(2.493, 3.145, 3.794)
(2.499, 3.220, 3.800)	(2.574, 3.223, 3.875)	(2.577, 3.235, 3.878)	(2.589, 3.243, 3.890)
(2.597, 3.264, 3.898)	(2.618, 3.272, 3.919)	(2.626, 3.294, 3.924)	(2.648, 3.332, 3.949)
(2.686, 3.346, 3.987)	(2.700, 3.377, 4.001)	(2.731, 3.408, 4.032)	(2.762, 3.435, 4.063)
(2.789, 3.493, 4.090)	(2.847, 3.501, 4.148)	(2.855, 3.537, 4.156)	(2.891, 3.554, 4.192)
(2.908, 3.562, 4.209)	(2.916, 3.628, 4.217)	(2.982, 3.852, 4.283)	(3.206, 3.871, 4.507)
(3.225, 3.886, 4.526)	(3.240, 3.971, 4.541)	(3.325, 4.024, 4.626)	(3.378, 4.027, 4.679)
(3.381, 4.225, 4.682)	(3.579, 4.395, 4.880)	(3.749, 5.020, 5.050)	

the results reveal that the H-B estimator achieves significantly lower MSE and smaller average values (AV) compared to the Bayesian and E-B estimators. Across various scenarios, the H-B estimator consistently demonstrates superior efficiency relative to the other two estimators.

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