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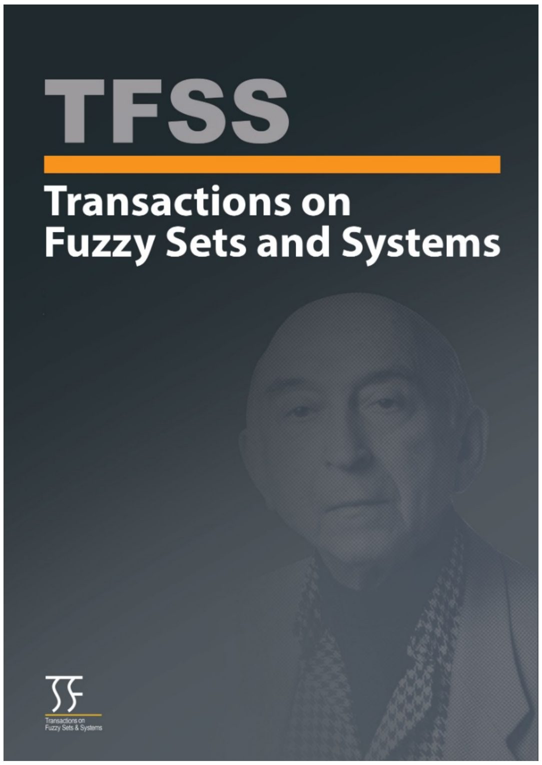

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Anti-Fuzzy Ideal in (m, n)-Near Rings

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Anti-Fuzzy Ideal in (m, n)-Near Rings

Fahime Mohammadi , **Bijan Davvaz***

Abstract. In this article, we introduce and discuss the definitions of fuzzy (m, n)-sub near rings, anti-fuzzy (m, n)-sub near rings, fuzzy ideals, and anti-fuzzy ideals in (m, n)-near rings. We establish several fundamental properties and prove key theorems related to these fuzzy structures. Additionally, illustrative examples are provided to enhance understanding and demonstrate the applicability of the concepts.

AMS Subject Classification 2020: 16Y20; 16Y30; 03A72
Keywords and Phrases: Anti-fuzzy (m, n)-Sbgroup, Anti-fuzzy (m, n)-sub near ring, Anti-fuzzy ideal, Anti-fuzzy prime ideal, Near rings homomorphism.

1 Introduction

A function $\vartheta : R \longrightarrow [0, 1]$ in a set R is called a fuzzy set [1]. Fuzzy subgroups of a group are studied by Rosenfeld in [2], also see [3]. Davvaz and Dudek investigated fuzzy subgroups of an m-ary group. Suppose that Im(ϑ) denote the image set of ϑ and ϑ is a fuzzy set in a set R. Set $\vartheta^{\geq}_{\alpha} = \{x \in R \mid \vartheta(x) \geq \alpha\}$ ($\vartheta^{\leq}_{\alpha} = \{x \in R \mid \vartheta(x) \leq \alpha\}$) where $\alpha \in [0, 1]$ is named a upper (lower) level subset of ϑ.

Abou-Zaid [4] introduced the concept of a fuzzy sub-near ring and studied the fuzzy ideals of a near ring. This concept has been studied by many researchers, see for example [5, 6, 7, 8].

Definition 1.1. [9] *A fuzzy set ϑ of an m-ary group (R, f) is named a fuzzy subgroup of R if*

(1) *for all $q_1, q_2, ..., q_m \in R$, $\vartheta\big(f(q_1, q_2, ..., q_m)\big) \geq \min\{\vartheta(q_1), \vartheta(q_2), ..., \vartheta(q_m)\}$,*

(2) *for every $o_1^m, b \in R$ and $1 \leq i \leq n$ there is $u_i \in R$ so that $f(o_1^{i-1}, u_i, o_{i+1}^m) = b$ and $\vartheta(u_i) \geq \min\{\vartheta(o_1), \vartheta(o_2), ..., \vartheta(o_{i-1}), \vartheta(o_{i+1}), ..., \vartheta(o_n), \vartheta(b)\}$.*

Biswas introduced the notion of an anti-fuzzy subgroup of a group [10]. Then, in [9], Davvaz and Dudek extended this concept to m-ary groups.

Definition 1.2. [9] *A fuzzy set ϑ of an m-ary group (R, f) is named an anti-fuzzy subgroup of R if*

(1) *for every $o_1, o_2, ..., o_m \in R$, $\vartheta\big(f(o_1, o_2, ..., o_m)\big) \leq \max\{\vartheta(o_1), \vartheta(o_2), ..., \vartheta(o_m)\}$,*

(2) *for every $o_1^m, b \in R$ and $1 \leq i \leq n$ there is $u_i \in R$ so that $f(o_1^{i-1}, u_i, o_{i+1}^m) = b$ and $\vartheta(u_i) \leq \max\{\vartheta(o_1), \vartheta(o_2), ..., \vartheta(o_{i-1}), \vartheta(o_{i+1}), ..., \vartheta(o_n), \vartheta(b)\}$.*

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In [11], Mohammadi and Davvaz characterized a new class of n-ary algebras that we call (m,n)-near rings. They investigated the notions of i-R-groups, i-(m,n)-near field, prime ideals, primary ideals, and subtractive ideals of (m,n)-near rings. Then, in [12], they studied fuzzy ideal. See [11] for the definitions of i-(m,n)-near ring and (m,n)-sub near ring.

In this article, we denote i-(m,n)-near ring by (m,n)-near ring.

Definition 1.3. [12] *Suppose that (R,f,g) is an (m,n)-near ring and ϑ is a fuzzy set of R. In this case, ϑ is named a fuzzy (m,n)-sub near ring of R, if*

(1) *for all $o_1, o_2, ..., o_m \in R$,*

$$\vartheta\big(f(o_1, o_2, ..., o_m)\big) \geq \min\{\vartheta(o_1), \vartheta(o_2), ..., \vartheta(o_m)\},$$

(2) *for all $o_1^m, b \in R$ and $1 \leq i \leq n$ there is $l_i \in R$ so that*

$$f(o_1^{i-1}, l_i, o_{i+1}^m) = b,\ \vartheta(l_i) \geq \min\{\vartheta(o_1), \vartheta(o_2), ..., \vartheta(o_{i-1}), \vartheta(o_{i+1}), ..., \vartheta(o_m), \vartheta(b)\},$$

(3) *for all $q_1, q_2, ..., q_n \in R$,*

$$\vartheta\big(g(q_1, q_2, ..., q_n)\big) \geq \min\{\vartheta(q_1), \vartheta(q_2), ..., \vartheta(q_n)\}.$$

2 Anti-fuzzy (m,n)-sub near rings and ideals

In this section, we define the concept of anti-fuzzy (m,n)-sub near rings and ideals of a (m,n)-near ring, and prove a few theorems concerning this concept. We also obtain a necessary and sufficient condition for a fuzzy subset of a (m,n)-near ring to be an anti-fuzzy (m,n)-sub near ring. We also obtain a relation between fuzzy (m,n)-sub near rings and anti-fuzzy (m,n)-sub near rings

Definition 2.1. *Suppose that (R,f,g) is an (m,n)-near ring and ϑ is a fuzzy set of R. Then ϑ is named the anti-fuzzy (m,n)-sub near ring of R, if*

(1) *for all $o_1, o_2, ..., o_m \in R$,*

$$\vartheta\big(f(o_1, o_2, ..., o_m)\big) \leq \max\{\vartheta(o_1), \vartheta(o_2), ..., \vartheta(o_m)\},$$

(2) *for all $o_1^m, b \in R$ and $1 \leq i \leq n$ there is $q_i \in R$ so that*

$$f(o_1^{i-1}, q_i, o_{i+1}^m) = b,\ \vartheta(q_i) \leq \max\{\vartheta(o_1), \vartheta(o_2), ..., \vartheta(o_{i-1}), \vartheta(o_{i+1}), ..., \vartheta(o_m), \vartheta(b)\},$$

(3) *for all $q_1, q_2, ..., q_n \in R$,*

$$\vartheta\big(g(q_1, q_2, ..., q_n)\big) \leq \max\{\vartheta(q_1), \vartheta(q_2), ..., \vartheta(q_n)\}.$$

Theorem 2.2. *Suppose that (R,f,g) is an (m,n)-near ring and ϑ is a fuzzy subset of R, then ϑ^c is a fuzzy sub near ring of R if and only if ϑ is an anti-fuzzy sub near ring of R.*

Proof. Suppose that ϑ is an anti-fuzzy sub near ring of R. In this case, for all $o_1^m, v_1^n \in R$,

$$\begin{aligned}\vartheta^c\big(f(o_1, o_2, ..., o_m)\big) &= 1 - \vartheta\big(f(o_1, o_2, ..., o_m)\big)\\ &\geq 1 - \max\{\vartheta(o_1), \vartheta(o_2), ..., \vartheta(o_m)\}\\ &= \min\{1 - \vartheta(o_1), 1 - \vartheta(o_2), ..., 1 - \vartheta(o_m)\}\\ &= \min\{\vartheta^c(o_1), \vartheta^c(o_2), ..., \vartheta^c(o_m)\}.\end{aligned}$$

for all $o_1^m, b \in R$ and $1 \leq i \leq n$ there is $l_i \in R$ so that

$$f(o_1^{i-1}, l_i, o_{i+1}^m) = b,\ \vartheta(l_i) \leq \max\{\vartheta(o_1), \vartheta(o_2), ..., \vartheta(o_{i-1}), \vartheta(o_{i+1}), ..., \vartheta(o_m), \vartheta(b)\},$$

so

$$\begin{aligned} \vartheta^c(l_i) &= 1 - \vartheta(l_i) \\ &\geq 1 - \max\{\vartheta(o_1), \vartheta(o_2), ..., \vartheta(o_{i-1}), \vartheta(o_{i+1}), ..., \vartheta(o_m), \vartheta(b)\} \\ &= \min\{1-\vartheta(o_1), 1-\vartheta(o_2), ..., 1-\vartheta(o_{i-1}), 1-\vartheta(o_{i+1}), ..., 1-\vartheta(o_m), 1-\vartheta(b)\} \\ &= \min\{\vartheta^c(o_1), \vartheta^c(o_2), ..., \vartheta^c(o_{i-1}), \vartheta^c(o_{i+1}), ..., \vartheta^c(o_m), \vartheta^c(b)\}. \end{aligned}$$

$$\begin{aligned} \vartheta^c\big(g(q_1, q_2, ..., q_n)\big) &= 1 - \vartheta\big(g(q_1, q_2, ..., q_n)\big) \\ &\geq 1 - \max\{\vartheta(q_1), \vartheta(q_2), ..., \vartheta(q_n)\} \\ &= \min\{1-\vartheta(q_1), 1-\vartheta(q_2), ..., 1-\vartheta(q_n)\} \\ &= \min\{\vartheta^c(q_1), \vartheta^c(q_2), ..., \vartheta^c(q_n)\}. \end{aligned}$$

Therefore ϑ^c is a fuzzy sub near ring of (R, f, g).

Conversely, suppose that ϑ^c is a fuzzy sub near ring of (R, f, g),

$$\begin{aligned} \vartheta\big(f(o_1, o_2, ..., o_m)\big) &= 1 - \vartheta^c\big(f(o_1, o_2, ..., o_m)\big) \\ &\leq 1 - \min\{\vartheta^c(o_1), \vartheta^c(o_2), ..., \vartheta^c(o_m)\} \\ &= 1 - \min\{1-\vartheta(o_1), 1-\vartheta(o_2), ..., 1-\vartheta(o_m)\} \\ &= \max\{\vartheta(o_1), \vartheta(o_2), ..., \vartheta(o_m)\}. \end{aligned}$$

for all $o_1^m, b \in R$ and $1 \leq i \leq n$ there is $l_i \in R$ so that

$$f(o_1^{i-1}, l_i, o_{i+1}^m) = b,\ \vartheta^c(l_i) \geq \min\{\vartheta^c(o_1), \vartheta^c(o_2), ..., \vartheta^c(o_{i-1}), \vartheta^c(o_{i+1}), ..., \vartheta^c(o_m), \vartheta^c(b)\},$$

so

$$\begin{aligned} \vartheta(l_i) &= 1 - \vartheta^c(l_i) \\ &\leq 1 - \min\{\vartheta^c(o_1), \vartheta^c(o_2), ..., \vartheta^c(o_{i-1}), \vartheta^c(o_{i+1}), ..., \vartheta^c(o_m), \vartheta^c(b)\} \\ &= 1 - \min\{1-\vartheta(o_1), ..., 1-\vartheta(o_{i-1}), 1-\vartheta(o_{i+1}), ..., 1-\vartheta(o_m), 1-\vartheta(b)\} \\ &= \max\{\vartheta(o_1), \vartheta(o_2), ..., \vartheta(o_{i-1}), \vartheta(o_{i+1}), ..., \vartheta(o_m), \vartheta(b)\}. \end{aligned}$$

$$\begin{aligned} \vartheta\big(g(v_1, v_2, ..., v_n)\big) &= 1 - \vartheta^c\big(g(v_1, v_2, ..., v_n)\big) \\ &\leq 1 - \min\{\vartheta^c(v_1), \vartheta^c(v_2), ..., \vartheta^c(v_n)\} \\ &= 1 - \min\{1-\vartheta(v_1), 1-\vartheta(v_2), ..., 1-\vartheta(v_n)\} \\ &= \max\{\vartheta(v_1), \vartheta(v_2), ..., \vartheta(v_n)\}. \end{aligned}$$

□

Example 2.3. Similar to Example 2 in [11], it can be easily proven that $(\mathbb{Z}, f, g)$ with $f(l_1, l_2, ..., l_m) = l_1 + l_2 + ... + l_m$ and $g(w_1, w_2, ..., w_n) = w_1$ is an (m,n)-near ring. Now we show that η defined below is an anti-fuzzy sub near ring.

$$\eta(x) = \begin{cases} 0.7 & \text{if } x \in m\mathbb{Z} \\ 0.8 & \text{if } x \notin m\mathbb{Z}. \end{cases}$$

Considering Example 5 in [12],

$$\eta^c = \mu(x) = \begin{cases} 0.3 & \text{if } x \in m\mathbb{Z} \\ 0.2 & \text{if } x \notin m\mathbb{Z}. \end{cases}$$

μ is a fuzzy (m,n)-sub near ring of $(\mathbb{Z}, f, g)$. So using the previous theorem η is an anti-fuzzy (m,n)-sub near ring of $(\mathbb{Z}, f, g)$.

Theorem 2.4. *Assume that ϑ is a fuzzy set in an (m,n)-near ring (R,f,g). In this case, ϑ is an anti-fuzzy sub near ring of R if and only if the lower level cut $\vartheta_t^{\leq}$ is an sub near ring of R for each $t \in [\vartheta(0),1]$.*

Proof. Suppose that ϑ is an anti-fuzzy sub near ring of R and $\vartheta_t^{\leq} \neq 0$ so for all $v_1, v_2, ..., v_m \in \vartheta_t^{\leq}$,

$$\vartheta\big(f(v_1, v_2, ..., v_m)\big) \leq \max\{\vartheta(v_1), \vartheta(v_2), ..., \vartheta(v_m)\} \leq t.$$

Then $\vartheta\big(f(v_1, v_2, ..., v_m)\big) \leq t$ so $f(v_1, v_2, ..., v_m) \in \vartheta_t^{\leq}$. Hence $(\vartheta_t^{\leq}, f)$ is a semi group.

For all $v_1^{i-1}, v_{i+1}^m, b \in \vartheta_t^{\leq}$ there is $u_i \in R$ that $f(v_1^{i-1}, u_i, v_{i+1}^m) = b$ and

$$\vartheta(u_i) \leq \max\{\vartheta(v_1), \vartheta(v_2), ..., \vartheta(v_{i-1}), \vartheta(v_{i+1}), ..., \vartheta(v_m)\} \leq t.$$

So $u_i \in \vartheta_t^{\leq}$ this gives that $(\vartheta_t^{\leq}, f)$ is an m-group. For all $l_1, l_2, ..., l_n \in \vartheta_t^{\leq}$ have

$$\vartheta\big(g(l_1, l_2, ..., l_n)\big) \leq \max\{\vartheta(l_1), \vartheta(l_2), ..., \vartheta(l_n)\} \leq t.$$

Thus $\vartheta\big(g(l_1, l_2, ..., l_n)\big) \leq t$ hence $g(l_1, l_2, ..., l_n) \in \vartheta_t^{\leq}$, therefore $(\vartheta_t^{\leq}, g)$ is an n-semi group. Then the level subset $\vartheta_t^{\leq}$, $t \in (0,1]$, is a sub near ring of R.

Now, suppose that the level subset $\vartheta_t^{\leq}$, $t \in (0,1]$, is a sub near ring of R. For all $o_1, o_2, ..., o_m \in R$ let $b = \max\{\vartheta(o_1), \vartheta(o_2), ..., \vartheta(o_m)\}$ so for all $1 \leq i \leq m$, $\vartheta(o_i) \leq b$. As a result $o_1, o_2, ..., o_m \in \vartheta_b^{\leq}$, $f(o_1, o_2, ..., o_m) \in \vartheta_b^{\leq}$ therefore $\vartheta\big(f(o_1, o_2, ..., o_m)\big) \leq b$

$$\vartheta\big(f(o_1, o_2, ..., o_m)\big) \leq \max\{\vartheta(o_1), \vartheta(o_2), ..., \vartheta(o_m)\}.$$

For all $l_1, l_2, ..., l_n \in R$ let $a = \max\{\vartheta(l_1), \vartheta(l_2), ..., \vartheta(l_n)\}$, so for all $1 \leq i \leq n$, $\vartheta(l_i) \leq a$. As a result $l_1, l_2, ..., l_n \in \vartheta_a$ so $g(l_1, l_2, ..., l_n) \in \vartheta_a^{\leq}$ this implies that $\vartheta\big(g(l_1, l_2, ..., l_n)\big) \leq a$ thus

$$\vartheta\big(g(l_1, l_2, ..., l_n)\big) \leq \max\{\vartheta(l_1), \vartheta(l_2), ..., \vartheta(l_n)\}.$$

For all $v_1^{i-1}, v_{i+1}^m, b \in R$ let $d = \max\{\vartheta(v_1), \vartheta(v_2), ..., \vartheta(v_{i-1}), \vartheta(v_{i+1}), ..., \vartheta(v_m), \vartheta(b)\}$, so $v_1^{i-1}, v_{i+1}^m, b \in \vartheta_d^{\leq}$. Hence there is $u_i \in \vartheta_d^{\leq}$ so that $b = f(v_1^{i-1}, u_i, v_{i+1}^m)$,

$$\vartheta(u_i) \leq d = \max\{\vartheta(v_1), \vartheta(v_2), ..., \vartheta(v_{i-1}), \vartheta(v_{i+1}), ..., \vartheta(v_m), \vartheta(b)\}.$$

Therefore ϑ is a fuzzy sub near ring. $\square$

Example 2.5. Similar to Example 2 in [11], it can be easily proven that $(\mathbb{Z}, f, g)$ with $f(l_1, l_2, ..., l_m) = l_1 + l_2 + ... + l_m$ and $g(w_1, w_2, ..., w_n) = w_1$ is an (m,n)-near ring. Now we show that η defined below is an anti-fuzzy sub near ring.

$$\eta(x) = \begin{cases} 0.7 & \text{if } x \in m\mathbb{Z} \\ 0.8 & \text{if } x \notin m\mathbb{Z}. \end{cases}$$

Then, we have $\eta_{0.7}^{\leq} = m\mathbb{Z}$ and $\eta_{0.8}^{\leq} = \mathbb{Z}$ are both (m,n)-sub near rings of $(\mathbb{Z}, f, g)$, so according to previous theorem, η is an anti-fuzzy (m,n)-sub near ring of $(\mathbb{Z}, f, g)$.

Definition 2.6. *Suppose that (R,f,g) is an i-(m,n)-near ring and ϑ is a fuzzy set of R. In this case, ϑ is named a fuzzy ideal of R, if*

(1) *for every $o_1, o_2, ..., o_m \in R$,*

$$\vartheta\big(f(o_1, o_2, ..., o_m)\big) \geq \min\{\vartheta(o_1), \vartheta(o_2), ..., \vartheta(o_m)\},$$

(2) *for all* $o_1^m, b \in R$ *and* $1 \leq i \leq n$ *there is* $u_i \in R$ *so that*

$$f(o_1^{i-1}, u_i, o_{i+1}^m) = b,\ \vartheta(u_i) \geq \min\{\vartheta(o_1), \vartheta(o_2), ..., \vartheta(o_{i-1}), \vartheta(o_{i+1}), ..., \vartheta(o_m), \vartheta(b)\},$$

(3) *for all* $u_1^{i-1}, u_{i+1}^m, u \in R$ *and* $1 \leq j \leq m$, *there is* $l \in R$ *that*

$$f(u_1^{j-1}, u, u_{j+1}^m) = f(u_1^{i-1}, l, u_{i+1}^m),\ \vartheta(l) \geq \vartheta(u),$$

(4) *for all* $l_1, l_2, ..., l_n \in R$,

$$\vartheta\big(g(l_1, l_2, ..., l_n)\big) \geq \min\{\vartheta(l_1), \vartheta(l_2), ..., \vartheta(l_n)\},$$

(5) *for all* $u_1^n, u \in R$ *and* $1 \leq i \leq n$,

$$\vartheta\big(g(u_1^{i-1}, u, u_{i+1}^n)\big) \geq \vartheta(u),$$

(6) *for all* $k \neq i$, $1 \leq k \leq m$, $v_1^{i-1}, v_{i+1}^n, o_1^m \in R$ *and* $o_k \in R$ *there is* $l_k \in R$ *so that* $g\big(v_1^{i-1}, f(o_1, o_2, ..., o_m), v_{i+1}^n\big) = f\big(g(v_1^{i-1}, o_1, v_{i+1}^n), g(v_1^{i-1}, o_2, v_{i+1}^n), ..., g(v_1^{i-1}, o_{k-1}, v_{i+1}^n), l_k, g(v_1^{i-1}, o_{k+1}, v_{i+1}^n), ..., g(v_1^{i-1}, o_m, v_{i+1}^n)\big)$ *and* $\vartheta(l_k) \geq \vartheta(o_k)$.

Note that ϑ *is a fuzzy* i*-ideal of* R *if it satisfies* $1, 2, 3, 4, 5$. ϑ *is a fuzzy* j*-ideal,* $j \neq i$, *of* R *if it satisfies* $1, 2, 3, 4, 6$, $1 \leq i, j \leq n$.

Definition 2.7. *Suppose that* (R, f, g) *is an* i*-*(m, n)*-near ring and* ϑ *is a fuzzy set of* R. *In this case,* ϑ *is named an anti-fuzzy ideal of* R, *if*

(1) *for all* $o_1, o_2, ..., o_m \in R$,

$$\vartheta\big(f(o_1, o_2, ..., o_m)\big) \leq \max\{\vartheta(o_1), \vartheta(o_2), ..., \vartheta(o_m)\},$$

(2) *for all* $o_1^m, b \in R$ *and* $1 \leq i \leq n$ *there is* $u_i \in R$ *so that*

$$f(o_1^{i-1}, u_i, o_{i+1}^m) = b,\ \vartheta(u_i) \leq \max\{\vartheta(o_1), \vartheta(o_2), ..., \vartheta(o_{i-1}), \vartheta(o_{i+1}), ..., \vartheta(o_m), \vartheta(b)\},$$

(3) *for all* $u_1^{i-1}, u_{i+1}^m, u \in R$ *and* $1 \leq j \leq m$, *there is* $l \in R$ *that*

$$f(u_1^{j-1}, u, u_{j+1}^m) = f(u_1^{i-1}, l, u_{i+1}^m),\ \vartheta(l) \leq \vartheta(u),$$

(4) *for all* $o_1, o_2, ..., o_n \in R$,

$$\vartheta\big(g(o_1, o_2, ..., o_n)\big) \leq \max\{\vartheta(o_1), \vartheta(o_2), ..., \vartheta(o_n)\},$$

(5) *for all* $u_1^n, u \in R$ *and* $1 \leq i \leq n$,

$$\vartheta\big(g(u_1^{i-1}, u, u_{i+1}^n)\big) \leq \vartheta(u),$$

(6) *for all* $k \neq i$, $1 \leq k \leq m$, $o_1^{i-1}, o_{i+1}^n, v_1^m \in R$ *and* $v_k \in R$ *there is* $l_k \in R$ *so that* $g\big(o_1^{i-1}, f(v_1, v_2, ..., v_m), o_{i+1}^n\big) = f\big(g(o_1^{i-1}, v_1, o_{i+1}^n), g(o_1^{i-1}, v_2, o_{i+1}^n), ..., g(o_1^{i-1}, v_{k-1}, o_{i+1}^n), l_k, g(o_1^{i-1}, v_{k+1}, o_{i+1}^n), ..., g(o_1^{i-1}, v_m, o_{i+1}^n)\big)$ *and* $\vartheta(l_k) \leq \vartheta(v_k)$.

Note that ϑ is an anti-fuzzy i-ideal of R if it satisfies $1,2,3,4,5$. ϑ is an anti-fuzzy j-ideal, $j \neq i$, of R if it satisfies $1,2,3,4,6$, $1 \leq i,j \leq n$.

Theorem 2.8. *Suppose that that ϑ is a fuzzy set in an (m,n)-near ring (R,f,g). Then, ϑ is an anti-fuzzy ideal of R if and only if the lower level cut $\vartheta_t^{\leq}$ is an ideal of R for every $t \in [\vartheta(0),1]$.*

Proof. Suppose that the lower level cut subset $\vartheta_t^{\leq}$, $t \in (0,1]$, is an ideal of R so $\vartheta_t^{\leq}$ is a sub near ring of R. Using the Theorem 2.4 ϑ is an anti-fuzzy sub near ring of R.

(1) Let $x_1^{i-1}, x_{i+1}^m \in R$, $x_i \in R$ and $t = \vartheta(x_i)$. Hence $x_i \in \vartheta_t^{\leq}$ and $\vartheta_t^{\leq}$ is an ideal of R thus for all $1 \leq j \leq m$ there is $a_j \in \vartheta_t^{\leq}$ so that $f(x_1^{i-1}, x_i, x_{i+1}^m) = f(x_1^{j-1}, a_j, x_{j+1}^m)$ and $\vartheta(a_j) \leq t = \vartheta(x_i)$, therefore $\vartheta(a_j) \leq \vartheta(x_i)$.

(2) Let $u_1^n, u \in R$ there is $b \in (0,1]$ that $\vartheta(u) = b$. $\vartheta_b^{\leq}$ is an ideal of R so
$g(u_1^{i-1}, \vartheta_b, u_{i+1}^n) \subseteq \vartheta_b^{\leq}$, hence $g(u_1^{i-1}, u, u_{i+1}^n) \in \vartheta_b^{\leq}$, thus $\vartheta\big(g(u_1^{i-1}, u, u_{i+1}^n)\big) \leq b = \vartheta(x)$. Therefore for all $u_1, u_2, ..., u_n, x \in R$ and $1 \leq i \leq n$,

$$\vartheta\big(g(u_1^{i-1}, u, u_{i+1}^n)\big) \leq \vartheta(u).$$

(3) Let $k \neq i$, $1 \leq k \leq n$, $s_1^{i-1}, s_{i+1}^n, o_1^m, \in R$ and $d \in R$ there is $b \in (0,1]$
that $\vartheta(d) = b$. $\vartheta_b^{\leq}$ is an ideal of R so there is $l \in \vartheta_b^{\leq}$ that
$g\big(s_1^{i-1}, f(o_1^{k-1}, d, o_{k+1}^m), s_{i+1}^n\big)$
$= f\big(g(s_1^{i-1}, o_1, s_{i+1}^n), ..., g(s_1^{i-1}, o_{k-1}, s_{i+1}^n), l, g(s_1^{i-1}, o_{k+1}, s_{i+1}^n), ..., g(s_1^{i-1}, o_n, s_{i+1}^n)\big)$.
$l \in \vartheta_b^{\leq}$ so $\vartheta(l) \leq b = \vartheta(d)$.
All conditions in the Definition 2.7 are met. So ϑ is an anti-fuzzy ideal.
Conversely, assume that ϑ is an anti-fuzzy ideal of R so ϑ is an anti-fuzzy sub near ring of R.
Using the Theorem 2.10, $\vartheta_t^{\leq}$ is a sub near ring of R.
(1) Let $x_1^{i-1}, x_{i+1}^m \in R$, $x_i \in \vartheta_t^{\leq}$ and ϑ is an ideal of R so for all $1 \leq j \leq m$, there is
$a_j \in R$ so that $f(x_1^{i-1}, x_i, x_{i+1}^m) = f(x_1^{j-1}, a_j, x_{j+1}^m)$ and $\vartheta(a_j) \leq \vartheta(x_i) = t$, hence $a_j \in \vartheta_t^{\leq}$.
(2) Let $v_1, v_2, ..., v_n \in R$ and $x \in \vartheta_t^{\leq}$ so $\vartheta\big(g(v_1^{i-1}, x, v_{i+1}^n)\big) \leq \vartheta(x)$, thus
$\vartheta\big(g(v_1^{i-1}, x, v_{i+1}^n)\big) \leq t$, hence $g(v_1^{i-1}, x, v_{i+1}^n) \in \vartheta_t^{\leq}$ so $g(v_1^{i-1}, \vartheta_t^{\leq}, v_{i+1}^n) \subseteq \vartheta_t^{\leq}$.
(3) Let $k \neq i$, $1 \leq k \leq n$, $v_1^{i-1}, v_{i+1}^n, o_1^m \in R, o_k \in \vartheta_t^{\leq}$ there is $l_k \in R$ so that
$g\big(v_1^{i-1}, f(o_1, o_2, ..., o_m), v_{i+1}^n\big)$
$= f\big(g(v_1^{i-1}, o_1, v_{i+1}^n), ..., g(v_1^{i-1}, o_{k-1}, v_{i+1}^n), l_k, g(v_1^{i-1}, o_{k+1}, v_{i+1}^n), ..., g(v_1^{i-1}, o_m, v_{i+1}^n)\big)$
and $\vartheta(l_k) \leq \vartheta(o_k) \leq t$, so $l_k \in \vartheta_t^{\leq}$.
Now, all conditions in the definition of ideal hold, so $\vartheta_t^{\leq}$ is an ideal. □

Example 2.9. In Example 2.5, level subsets $\eta_{0.7}^{\leq} = m\mathbb{Z}$ and $\eta_{0.8}^{\leq} = \mathbb{Z}$ are ideals of $(\mathbb{Z}, f, g)$ so according to previous theorem, η is an anti-fuzzy ideal of $(\mathbb{Z}, f, g)$.

Theorem 2.10. *Suppose that (R,f,g) is an (m,n)-near ring and ϑ is a fuzzy sub near ring of R. In this case, the fuzzy subset ϑ^c, $t \in (0,1]$, is a fuzzy ideal of R if and only if ϑ is an anti-fuzzy ideal.*

Proof. Suppose that ϑ is an anti-fuzzy ideal of (R,f,g). Thus ϑ^c is a fuzzy sub near ring of R and for all $l_1^{i-1}, l_{i+1}^m, l \in R$ and $1 \leq j \leq m$, there is $a \in R$ that

$$f(l_1^{j-1}, l, l_{j+1}^m) = f(l_1^{i-1}, a, l_{i+1}^m),\ \vartheta(a) \leq \vartheta(l),$$

so $\vartheta^c(a) = 1 - \vartheta(a) \geq 1 - \vartheta(l) = \vartheta^c(l)$, for all $l_1^n, l \in R$ and $1 \leq i \leq n$,

$$\vartheta\big(g(l_1^{i-1}, l, l_{i+1}^n)\big) \leq \vartheta(l),$$

then $\vartheta^c\big(g(l_1^{i-1},l,l_{i+1}^n)\big)=1-\vartheta\big(g(l_1^{i-1},l,l_{i+1}^n)\big)\geq 1-\vartheta(l)=\vartheta^c(l)$

for all $k\neq i$, $1\leq k\leq m$, $z_1^{i-1},z_{i+1}^n,l_1^m\in R$ and $l_k\in R$ there is $x_k\in R$ so that $g\big(z_1^{i-1},f(l_1,l_2,...,l_m),z_{i+1}^n\big)=f\big(g(z_1^{i-1},l_1,z_{i+1}^n),g(z_1^{i-1},l_2,z_{i+1}^n),...,g(z_1^{i-1},l_{k-1},z_{i+1}^n),x_k,g(z_1^{i-1},l_{k+1},z_{i+1}^n),...,g(z_1^{i-1},l_m,z_{i+1}^n)\big)$ and $\vartheta(x_k)\leq\vartheta(l_k)$.

Thus $\vartheta^c(x_k)=1-\vartheta(x_k)\geq 1-\vartheta(l_k)=\vartheta^c(l_k)$, therefore ϑ^c is a fuzzy ideal of (R,f,g)

Conversely, assume that ϑ^c is a fuzzy ideal of (R,f,g), thus ϑ is an anti-fuzzy sub near ring of R and for all $l_1^{i-1},l_{i+1}^m,l\in R$ and $1\leq j\leq m$, there is $a\in R$ that

$$f(l_1^{j-1},l,l_{j+1}^m)=f(l_1^{i-1},a,l_{i+1}^m),\ \vartheta^c(a)\geq\vartheta^c(l),$$

so $\vartheta(a)=1-\vartheta^c(a)\leq 1-\vartheta^c(l)=\vartheta(l)$. For every $l_1^n,x\in R$ and $1\leq i\leq n$,

$$\vartheta^c\big(g(l_1^{i-1},l,l_{i+1}^n)\big)\geq\vartheta^c(l),$$

hence $\vartheta\big(g(l_1^{i-1},l,l_{i+1}^n)\big)=1-\vartheta^c\big(g(l_1^{i-1},l,l_{i+1}^n)\big)\leq 1-\vartheta^c(l)=\vartheta(l)$,

for all $k\neq i$, $1\leq k\leq m$, $z_1^{i-1},z_{i+1}^n,l_1^m\in R$ and $l_k\in R$ there is $x_k\in R$ so that $g\big(z_1^{i-1},f(l_1,l_2,...,l_m),z_{i+1}^n\big)=f\big(g(z_1^{i-1},l_1,z_{i+1}^n),g(z_1^{i-1},l_2,z_{i+1}^n),...,g(z_1^{i-1},l_{k-1},z_{i+1}^n),x_k,g(z_1^{i-1},l_{k+1},z_{i+1}^n),...,g(z_1^{i-1},l_m,z_{i+1}^n)\big)$ and $\vartheta^c(x_k)\geq\vartheta^c(l_k)$.

then $\vartheta(x_k)=1-\vartheta^c(x_k)\leq 1-\vartheta^c(l_k)=\vartheta(l_k)$.

Therefore ϑ is an anti-fuzzy ideal of R. □

Example 2.11. In Example 2.9, η is an anti-fuzzy ideal of $(\mathbb{Z},f,g)$.

$$\eta^c(x)=\begin{cases}0.3 & \text{if } x\in m\mathbb{Z}\\ 0.2 & \text{if } x\notin m\mathbb{Z}.\end{cases}$$

So by previous theorem η^c is a fuzzy ideal of $(\mathbb{Z},f,g)$.

If ϑ is a fuzzy set of (m,n)-near ring (R,f,g), then $\vartheta_t^{\leq}=(\vartheta^c)_{1-t}^{\geq}$ for every $t\in[0,1]$.

Theorem 2.12. *Suppose that P is an ideal of an (m,n)-near ring (R,h,k). In this case, for each $t\in[0,1]$, there is an anti-fuzzy ideal ϑ of R such that $\vartheta_t^{\leq}=P$.*

Proof. Suppose that $\vartheta:R\longrightarrow[0,1]$ is a fuzzy set defined by:

$$\vartheta(o)=\begin{cases}t & \text{if } o\in I\\ 1 & \text{if } o\notin I\end{cases}$$

where t is a fixed number in $(0,1)$, then clearly $\vartheta_t^{\leq}=I$.

(1) Let $v_1,v_2,...,v_m\in R$, if there is $1\leq i\leq m$ that $v_i\notin I$ then

$$\max\{\vartheta(v_1),\vartheta(v_2),...,\vartheta(v_m)\}=1,$$
$$\vartheta\big(f(v_1,v_2,...,v_m)\big)\leq\max\{\vartheta(v_1),\vartheta(v_2),...,\vartheta(v_m)\}.$$

Otherwise $v_1,v_2,...,v_m\in I$ and I is an ideal of R, so $f(v_1,v_2,...,v_m)\in I$ hence

$$\vartheta\big(f(v_1,v_2,...,v_m)\big)=t,$$
$$\vartheta\big(f(v_1,v_2,...,v_m)\big)\leq\max\{\vartheta(v_1),\vartheta(v_2),...,\vartheta(v_m)\}.$$

(2) Let $v_1^m,b\in R$, because R is an (m,n)-near ring so for all $1\leq i\leq m$ there is $x_i\in R$ so that $f(v_1^{i-1},x_i,v_{i+1}^m)=b$. If there is $1\leq j\leq m$ that $\vartheta_j\notin I$ or $b\notin I$ then

$$\max\{\vartheta(v_1), \vartheta(v_2), ..., \vartheta(v_{i-1}), \vartheta(v_{i+1}), ..., \vartheta(v_m), \vartheta(b)\} = 1,$$
$$\vartheta(x_i) \leq \max\{\vartheta(v_1), \vartheta(v_2), ..., \vartheta(v_{i-1}), \vartheta(v_{i+1}), ..., \vartheta(v_m), \vartheta(b)\}.$$

Otherwise $v_1^m, b \in I$ and I is an ideal of R so $x_i \in I$ hence $\vartheta(x_i) = t$,

$$\vartheta(x_i) \leq \max\{\vartheta(v_1), \vartheta(v_2), ..., \vartheta(v_{i-1}), \vartheta(v_{i+1}), ..., \vartheta(v_m), \vartheta(b)\}.$$

(3) Let $l_1^{i-1}, l_{i+1}^m \in R$, $l \in R$. If $l \in I$, because I is an ideal of R, then for all $1 \leq j \leq m$ there is $a \in I$ so that

$$f(l_1^{i-1}, l, l_{i+1}^m) = f(l_1^{i-1}, a, l_{i+1}^m),\ a \in I$$

so $t = \vartheta(a) \leq \vartheta(l)$. If $l \notin I$, $b = f(l_1^{i-1}, l, l_{i+1}^m)$ and $l_1 = y_1$, $l_2 = y_2$, ..., $l_{i-1} = y_{i-1}, l_{i+1} = y_i, ..., l_m = y_{m-1}$ then by definition near ring for $y_1^{m-1}, b \in R$ there is $a \in R$ so that $b = f(y_1^{j-1}, a, y_j^{m-1})$, hence

$$f(l_1^{i-1}, l, l_{i+1}^m) = f(y_1^{j-1}, a, y_j^{m-1}),\ \vartheta(a) \leq 1 = \vartheta(l).$$

(4) Let $l_1, l_2, ..., l_n \in R$. If there is $1 \leq i \leq n$ that $l_i \notin I$ then

$$\max\{\vartheta(l_1), \vartheta(l_2), ..., \vartheta(l_n)\} = 1,$$
$$\vartheta\big(g(l_1, l_2, ..., l_n)\big) \leq \max\{\vartheta(l_1), \vartheta(l_2), ..., \vartheta(l_n)\}.$$

Otherwise $l_1, l_2, ..., l_n \in I$ and I is an ideal of R so $g(l_1, l_2, ..., l_n) \in I$ hence

$$\vartheta\big(g(l_1, l_2, ..., l_n)\big) = t,$$
$$\vartheta\big(g(l_1, l_2, ..., l_n)\big) \geq \max\{\vartheta(l_1), \vartheta(l_2), ..., \vartheta(l_n)\}.$$

(5) Let $l_1^n, l \in R$ and $1 \leq i \leq n$,
If $l \in I$ then $g(l_1^{i-1}, l, l_{i+1}^n) \in I$ so $\vartheta(l) = \vartheta\big(g(l_1^{i-1}, l, l_{i+1}^n)\big) = t$. If $l \notin I$ then $1 = \vartheta(l) \geq \vartheta\big(g(l_1^{i-1}, l, l_{i+1}^n)\big)$.
(6) Let $k \neq i$, $1 \leq k \leq n$, $a_1^{i-1}, a_{i+1}^n, l_1^m \in R$ and $l_k \in I$ there is $x_k \in I$ so that
$g\big(a_1^{i-1}, f(l_1, l_2, ..., l_m), a_{i+1}^n\big)$
$= f\big(g(a_1^{i-1}, l_1, a_{i+1}^n), ..., g(a_1^{i-1}, l_{k-1}, a_{i+1}^n), x_k, g(a_1^{i-1}, l_{k+1}, a_{i+1}^n), ..., g(a_1^{i-1}, l_m, a_{i+1}^n)\big)$. So $\vartheta(x_k) \leq \vartheta(l_k)$. □

Example 2.13. Similar to Example 2 in [11], it can be easily proven that $(\mathbb{Z}, f, g)$ with $f(l_1, l_2, ..., l_m) = l_1 + l_2 + ... + l_m$ and $g(w_1, w_2, ..., w_n) = w_1$ is an (m, n)-near ring and for all $p \in \mathbb{Z}$, $p\mathbb{Z}$ is an ideal of $\mathbb{Z}$. Assume that t is a fixed number in $(0, 1)$. We let

$$\vartheta(o) = \begin{cases} t & \text{if } o \in p\mathbb{Z} \\ 1 & \text{if } o \notin p\mathbb{Z} \end{cases}$$

so $\vartheta_t^{\leq} = I$ and by Theorem 2.8, ϑ is an anti-fuzzy ideal of $\mathbb{Z}$.

Definition 2.14. *Assume that Z is a set and W is a subset of Z. In this case, we have*

$$\vartheta_W(z) = \begin{cases} 0 & \text{if } z \in W \\ 1 & \text{if } z \notin W \end{cases}$$

is said to be anti-characteristic function of the set W in Z.

Theorem 2.15. *Suppose that I is a non-empty subset of an (m, n)-near ring (R, f, g) and ϑ is a fuzzy set in R such that ϑ is into $\{0, 1\}$ and ϑ is the anti-characteristic function of I. In this case, ϑ is an anti-fuzzy ideal of R if and only if I is an ideal of R.*

Proof. Suppose that ϑ is an anti-fuzzy ideal of R.

(1) Let $o_1, o_2, ..., o_m \in I$, then $\vartheta(o_1) = \vartheta(o_2) = ... = \vartheta(o_m) = 0$, thus
$\vartheta\big(f(o_1, o_2, ..., o_m)\big) \leq \max\{\vartheta(o_1), \vartheta(o_2), ..., \vartheta(o_m)\} = 0$ and so $f(o_1, o_2, ..., o_m) \in I$.

(2) Let $a_1^m, b \in I$ there is $x_i \in R$, $1 \leq i \leq m$, so that $f(a_1^{i-1}, x_i, a_{i+1}^m) = b$ and $\vartheta(x_i) \leq \max\{\vartheta(a_1), \vartheta(a_2), ..., \vartheta(a_{i-1}), \vartheta(a_{i+1}), ..., \vartheta(a_m), \vartheta(b)\} = 0$, so $x_i \in I$.

(3) ϑ is a fuzzy ideal of R so for all $x_1^{i-1}, x_{i+1}^m \in R$, $x \in I$ and $1 \leq j \leq m$, there is $a \in R$ so that $f(x_1^{j-1}, a, x_{j+1}^m) = f(x_1^{i-1}, x, x_{i+1}^m)$ and $\vartheta(a) \leq \vartheta(x)$, hence $a \in I$.

Thus for all $x_1^{i-1}, x_{i+1}^m \in R$ and $x \in I$ there is $a \in I$ so that

$$f(x_1^{j-1}, a, x_{j+1}^m) = f(x_1^{i-1}, x, x_{i+1}^m) \text{ and } \vartheta(a) \leq \vartheta(x).$$

(4) Let $l_1, l_2, ..., l_n \in I$, then $\vartheta(l_1) = \vartheta(l_2) = ... = \vartheta(l_n) = 0$, thus
$\vartheta\big(g(l_1, l_2, ..., l_n)\big) \leq \max\{\vartheta(l_1), \vartheta(l_2), ..., \vartheta(l_n)\} = 0$ and so $g(l_1, l_2, ..., l_n) \in I$.

(5) For every $z_1^n, z \in I$ and $1 \leq i \leq n$, $0 = \vartheta(z) \geq \vartheta\big(g(z_1^{i-1}, z, z_{i+1}^n)\big)$, so

$$g(z_1^{i-1}, z, z_{i+1}^n) \in I.$$

(6) For every $k \neq i$, $1 \leq k \leq m$, $z_1^{i-1}, z_{i+1}^n, l_1^m \in R$ and $l_k \in I$ there is $x_k \in R$ so that
$g\big(z_1^{i-1}, f(l_1, l_2, ..., l_m), z_{i+1}^n\big)$
$= f\big(g(z_1^{i-1}, l_1, z_{i+1}^n), ..., g(z_1^{i-1}, l_{k-1}, z_{i+1}^n), x_k, g(z_1^{i-1}, l_{k+1}, z_{i+1}^n), ..., g(z_1^{i-1}, l_m, z_{i+1}^n)\big)$
and $\vartheta(x_k) \leq \vartheta(l_k)$, thus $x_k \in I$.

Conversely, suppose that I is an ideal of R.

(1) Suppose that $q_1, q_2, ..., q_m \in R$, if there is $1 \leq i \leq m$ that $q_i \notin I$ then

$$\max\{\vartheta(q_1), \vartheta(q_2), ..., \vartheta(q_m)\} = 1,\ \vartheta\big(f(q_1, q_2, ..., q_m)\big) \leq \max\{\vartheta(q_1), \vartheta(q_2), ..., \vartheta(q_m)\}.$$

Otherwise $q_1, q_2, ..., q_m \in I$ and I is an ideal of R so $f(q_1, q_2, ..., q_m) \in I$ hence

$$\vartheta\big(f(q_1, q_2, ..., q_m)\big) = 0,\ \vartheta\big(f(q_1, q_2, ..., q_m)\big) \leq \max\{\vartheta(q_1), \vartheta(q_2), ..., \vartheta(q_m)\}.$$

(2) Let $o_1^m, b \in R$ because R is an (m, n)-near ring so for all $1 \leq i \leq n$ there is $x_i \in R$, $f(o_1^{i-1}, x_i, o_{i+1}^m) = b$. If there is $1 \leq i \leq m$ that $o_i \notin I$ or $b \notin I$ then

$$\max\{\vartheta(o_1), \vartheta(o_2), ..., \vartheta(o_{i-1}), \vartheta(o_{i+1}), ..., \vartheta(o_m), \vartheta(b)\} = 1,$$
$$\vartheta(x_i) \leq \max\{\vartheta(o_1), \vartheta(o_2), ..., \vartheta(o_{i-1}), \vartheta(o_{i+1}), ..., \vartheta(o_m), \vartheta(b)\}.$$

Otherwise $o_1^m, b \in I$ and I is an ideal of R so $x_i \in I$ hence $\vartheta(x_i) = 0$ and

$$\vartheta(x_i) \leq \max\{\vartheta(o_1), \vartheta(o_2), ..., \vartheta(o_{i-1}), \vartheta(o_{i+1}), ..., \vartheta(o_m), \vartheta(b)\}.$$

(3) Let $o_1^{i-1}, o_{i+1}^m, o \in R$, if $o \in I$, because I is an ideal of R, then for all $1 \leq j \leq m$ there is $a \in I$ so that $f(o_1^{i-1}, o, o_{i+1}^m) = f(o_1^{i-1}, a, o_{i+1}^m)$, $a \in I$ so $0 = \vartheta(a) \leq \vartheta(o)$.

If $o \notin I$, Let $b = f(o_1^{i-1}, o, o_{i+1}^m)$ and $o_1 = y_1$, $o_2 = y_2$, ..., $o_{i-1} = y_{i-1}$, $o_{i+1} = y_i$, ..., $o_m = y_{m-1}$ then By definition near ring for $y_1^{m-1}, b \in R$ there is $a \in R$ so that
$b = f(y_1^{j-1}, a, y_j^{m-1})$, hence

$$f(o_1^{i-1}, o, o_{i+1}^m) = f(y_1^{j-1}, a, y_j^{m-1}),\ \vartheta(a) \leq 1 = \vartheta(o).$$

(4) Let $l_1, l_2, ..., l_n \in R$. If there is $1 \leq i \leq n$ that $l_i \notin I$ then
$\max\{\vartheta(l_1), \vartheta(l_2), ..., \vartheta(l_n)\} = 1$ and $\vartheta\big(g(l_1, l_2, ..., l_n)\big) \leq \max\{\vartheta(l_1), \vartheta(l_2), ..., \vartheta(l_n)\}$.
Otherwise $l_1, l_2, ..., l_n \in I$ and I is an ideal of R so $g(l_1, l_2, ..., l_n) \in I$ hence

$$\vartheta\big(g(l_1, l_2, ..., l_n)\big) = 0,\ \vartheta\big(g(l_1, l_2, ..., l_n)\big) \le \max\{\vartheta(l_1), \vartheta(l_2), ..., \vartheta(l_n)\}.$$

(5) Let $x_1^n, o \in R$ and $1 \le i \le n$, If $o \in I$ then $g(x_1^{i-1}, o, x_{i+1}^n) \in I$. So, $\vartheta(o) = \vartheta\big(g(x_1^{i-1}, o, x_{i+1}^n)\big) = 0$. If $o \notin I$ then $1 = \vartheta(o) \ge \vartheta\big(g(x_1^{i-1}, o, x_{i+1}^n)\big)$.

(6) Let $k \ne i$, $1 \le k \le n$, $k_1^{i-1}, k_{i+1}^n, z_1^m \in R$ and $z_k \in I$ there is $l_k \in I$ so that
$g\big(k_1^{i-1}, f(z_1, z_2, ..., z_m), k_{i+1}^n\big) =$
$f\big(g(k_1^{i-1}, z_1, k_{i+1}^n), ..., g(k_1^{i-1}, z_{k-1}, k_{i+1}^n), l_k, g(k_1^{i-1}, z_{k+1}, k_{i+1}^n), ..., g(k_1^{i-1}, z_m, k_{i+1}^n)\big)$,
$\vartheta(l_k) \le \vartheta(z_k)$. □

Example 2.16. If we let in Example 2.13, $t = 0$, then

$$\vartheta(o) = \begin{cases} 0 & \text{if } o \in I \\ 1 & \text{if } o \notin I \end{cases}$$

so $\vartheta_0^{\le} = I$ and by Theorem 2.8, ϑ is an anti-fuzzy ideal of $\mathbb{Z}$.

For a family of fuzzy sets $\{\vartheta_i |\ i \in \delta\}$ in an (m, n)-near ring R, the union $\bigvee_{i\in\delta} \vartheta_i$ of $\{\vartheta_i \mid i \in \delta\}$ is defined by

$$(\bigvee_{i\in\delta} \vartheta_i)(x) = \sup\{\vartheta_i(x) \mid i \in \delta\}$$

for all $x \in R$.

Theorem 2.17. *If $\{\vartheta_i \mid i \in \delta\}$ is a family of anti-fuzzy ideals of an (m, n)-near ring (R, f, g), then $\bigvee_{i\in\delta} \vartheta_i$ is an anti-fuzzy ideal of R.*

Proof. Suppose that $\{\vartheta_i \mid i \in \delta\}$ is a family of anti-fuzzy ideals of an (m, n)-near ring (R, f, g)

(1) for all $o_1, o_2, ..., o_m \in R$,

$$\begin{aligned}\bigvee_{i\in\delta} \vartheta_i\big(f(o_1, o_2, ..., o_m)\big) &= \sup\{\vartheta_i\big(f(o_1, o_2, ..., o_m)\big) \mid i \in \delta\} \\ &\le \sup\{\max\{\vartheta_i(o_1), \vartheta_i(o_2), ..., \vartheta_i(o_m)\} \mid i \in \delta\} \\ &= \max\{\sup\{\vartheta_i(o_1) \mid i \in \delta\}, ..., \sup\{\vartheta_i(o_m) \mid i \in \delta\}\} \\ &= \max\{\bigvee_{i\in\delta} \vartheta_i(o_1), \bigvee_{i\in\delta} \vartheta_i(o_2), ..., \bigvee_{i\in\delta} \vartheta_i(o_m)\}.\end{aligned}$$

(2) for all $a_1^m, b \in R$ and $1 \le i \le n$ there is $o_i \in R$ so that
$f(a_1^{i-1}, o_i, a_{i+1}^m) = b$,

$$\begin{aligned}\bigvee_{i\in\delta} \vartheta_i(o_i) &= \sup\{\vartheta_i(o_i) \mid i \in \delta\} \\ &\le \sup\{\max\{\vartheta_i(a_1), \vartheta_i(a_2), ..., \vartheta_i(a_{i-1}), \vartheta_i(a_{i+1}), ..., \vartheta_i(a_m), \vartheta_i(b)\} \mid i \in \delta\} \\ &= \max\{\sup\{\vartheta_i(a_1) \mid i \in \delta\}, \sup\{\vartheta_i(a_2) \mid i \in \delta\}, ..., \sup\{\vartheta_i(a_{i-1}) \mid i \in \delta\}, \\ &\quad \sup\{\vartheta_i(a_{i+1}) \mid i \in \delta\}, ..., \sup\{\vartheta_i(a_m) \mid i \in \delta\}, \sup\{\vartheta_i(b) \mid i \in \delta\}\} \\ &= \max\{\bigvee_{i\in\delta} \vartheta_i(a_1), ..., \bigvee_{i\in\delta} \vartheta_i(a_{i-1}), \bigvee_{i\in\delta} \vartheta_i(a_{i+1}), ..., \bigvee_{i\in\delta} \vartheta_i(a_m), \bigvee_{i\in\delta} \vartheta_i(b), \}.\end{aligned}$$

(3) for all $x_1^{i-1}, x_{i+1}^m, x \in R$ and $1 \le j \le m$, there is $l \in R$ that

$$f(x_1^{j-1}, x, x_{j+1}^m) = f(x_1^{i-1}, l, x_{i+1}^m),$$
$$\bigvee_{i\in\delta} \vartheta_i(l) = \sup\{\vartheta_i(l) \mid i \in \delta\} \le \sup\{\vartheta_i(x) \mid i \in \delta\} = \bigvee_{i\in\delta} \vartheta_i(x),$$

(4) for all $v_1, v_2, ..., v_n \in R$,

$$\begin{aligned}\bigvee_{i\in\delta} \vartheta_i\big(g(v_1, v_2, ..., v_n)\big) &= \sup\{\vartheta_i\big(g(v_1, v_2, ..., v_n)\big) \mid i \in \delta\}\\ &\le \sup\{\max\{\vartheta_i(v_1), \vartheta_i(v_2), ..., \vartheta_i(v_n) \mid i \in \delta\}\}\\ &= \max\{\sup\{\vartheta_i(v_1) \mid i \in \delta\}, ..., \sup\{\vartheta_i(v_n) \mid i \in \delta\}\}\\ &= \max\{\bigvee_{i\in\delta} \vartheta_i(v_1), \bigvee_{i\in\delta} \vartheta_i(v_2), ..., \bigvee_{i\in\delta} \vartheta_i(v_n)\},\end{aligned}$$

(5) for all $v_1^n, v \in R$ and $1 \le i \le n$,

$$\begin{aligned}\bigvee_{i\in\delta} \vartheta_i\big(g(v_1^{i-1}, v, v_{i+1}^n)\big) &= \sup\{\vartheta_i\big(g(v_1^{i-1}, v, v_{i+1}^n)\big) \mid i \in \delta\}\\ &\le \sup\{\vartheta_i(v) \mid i \in \delta\}\\ &= \bigvee_{i\in\delta} \vartheta_i(v),\end{aligned}$$

(6) for all $k \neq i$, $1 \le k \le m$, $j_1^{i-1}, j_{i+1}^n, l_1^m \in R$ and $l_k \in R$ there is $x_k \in R$ so that $g\big(j_1^{i-1}, f(l_1, l_2, ..., l_m), j_{i+1}^n\big) = f\big(g(j_1^{i-1}, l_1, j_{i+1}^n), g(j_1^{i-1}, l_2, j_{i+1}^n), ..., g(j_1^{i-1}, l_{k-1}, j_{i+1}^n), x_k, g(j_1^{i-1}, l_{k+1}, j_{i+1}^n), ..., g(j_1^{i-1}, l_m, j_{i+1}^n)\big)$ and

$$\bigvee_{i\in\delta} \vartheta_i(x_k) = \sup\{\vartheta_i(x_k) \mid i \in \delta\} \le \sup\{\vartheta_i(l_k) \mid i \in \delta\} = \bigvee_{i\in\delta} \vartheta_i(l_k).$$

□

Example 2.18. In Example 2.13, we let

$$\vartheta_i(o) = \begin{cases} 0 & \text{if } o \in i\mathbb{Z} \\ 1 & \text{if } o \notin i\mathbb{Z} \end{cases}$$

in this case, $\{\vartheta_i \mid i \in \mathbb{Z}\}$ is a family of anti-fuzzy ideals of (m,n)-near ring $(\mathbb{Z}, f, g)$, so by previous theorem $\bigvee_{i\in\mathbb{Z}} \vartheta_i$ is an anti-fuzzy ideal of $\mathbb{Z}$.

3 Homomorphisms

See [11] for the definitions of (m,n)-near ring homomorphism and h-invariant.

Definition 3.1. *Suppose that h is a mapping from the set R to the set S and ϑ and υ are fuzzy sets in R and S, respectively. Then, $h(\vartheta)$, the image of ϑ under h, is a fuzzy set in S:*

$$h(\vartheta)(k) = \begin{cases} \sup\limits_{l\in h^{-1}(k)} \vartheta(l) & \text{if } h^{-1}(k) \neq \phi \\ 0 & \textit{otherwise,} \end{cases}$$

for all $k \in S$.

$h^{-1}(\upsilon)$, the preimage of υ under h, is a fuzzy set in R so that for all $l \in R$, $h^{-1}(\upsilon)(l) = \upsilon\big(h(l)\big)$.

Theorem 3.2. *An (m,n)-near ring monomorphism preimage of an anti-fuzzy ideal is an anti-fuzzy ideal.*

Proof. Suppose that $\delta : R \longrightarrow S$ is an (m,n)-near ring monomorphism, υ is an anti-fuzzy ideal of S and ϑ the preimage of υ under δ.

(1) For all $o_1, o_2, ..., o_m \in R$ have

$$\begin{aligned} \vartheta\big(f(o_1,o_2,...,o_m)\big) &= \upsilon\big(\delta(f(o_1,o_2,...,o_m))\big) \\ &= \upsilon f\big(\delta(o_1),\delta(o_2),...,\delta(o_m)\big) \\ &\leq \max\{\upsilon\big(\delta(o_1)\big),\upsilon\big(\delta(o_2)\big),...,\upsilon\big(\delta(o_m)\big)\} \\ &= \max\{\vartheta(o_1),\vartheta(o_2),...,\vartheta(o_m)\}. \end{aligned}$$

So

$$\vartheta\big(f(o_1,o_2,...,o_m)\big) \leq \max\{\vartheta(o_1),\vartheta(o_2),...,\vartheta(o_m)\}.$$

(2) For all $l_1^{i-1}, l_{i+1}^m, b \in R$ have
$\delta(l_1),\delta(l_2),...,\delta(l_{i-1}),\delta(l_{i+1}),...,\delta(l_m),\delta(b) \in \delta(R)$ so there is $\delta(m_i) \in \delta(R)$ that
$f\big(\delta(l_1),\delta(l_2),...,\delta(l_{i-1}),\delta(m_i),\delta(l_{i+1}),...,\delta(l_m)\big) = \delta(b)$ and
$\upsilon\big(\delta(m_i)\big) \leq \max\{\upsilon\big(\delta(l_1)\big),\upsilon\big(\delta(l_2)\big),...,\upsilon\big(\delta(l_{i-1})\big),\upsilon\big(\delta(l_{i+1})\big),...,\upsilon\big(\delta(l_m)\big)\}$. So
$\delta\big(f(l_1^{i-1},m_i,l_{i+1}^m)\big) = \delta(b)$ hence

$$\begin{gathered} f(l_1^{i-1},m_i,l_{i+1}^m) = b, \\ \vartheta(m_i) \leq \max\{\vartheta(l_1),\vartheta(l_2),...,\vartheta(l_{i-1}),\vartheta(l_{i+1}),...,\vartheta(l_m)\}. \end{gathered}$$

(3) For all $l_1^{i-1}, l_{i+1}^m, v \in R$ have $\delta(l_1),\delta(l_2),...,\delta(l_{i-1}),\delta(l_{i+1}),...,\delta(l_m),\delta(v) \in \delta(R)$ so there is $\delta(a) \in \delta(R)$ that
$f\big(\delta(l_1),\delta(l_2),...,\delta(l_{i-1}),\delta(a),\delta(l_{i+1}),...,\delta(v_m)\big) = f\big(\delta(l_1),\delta(l_2),...,\delta(l_{i-1}),\delta(v),$
$\delta(l_{i+1}),...,\delta(l_m)\big)$. So
$\delta\big(f(l_1^{i-1},a,l_{i+1}^m)\big) = \delta\big(f(l_1^{i-1},v,l_{i+1}^m)\big)$, hence

$$\begin{gathered} f(l_1^{i-1},a,l_{i+1}^m) = f(l_1^{i-1},v,l_{i+1}^m), \\ \vartheta(a) = \upsilon\big(\delta(a)\big) \leq \upsilon\big(\delta(v)\big) = \vartheta(v). \end{gathered}$$

(4) For all $o_1, o_2, ..., o_n \in R$ have

$$\begin{aligned} \vartheta\big(g(o_1,o_2,...,o_n)\big) &= \upsilon\big(\delta(g(o_1,o_2,...,o_n))\big) \\ &= \upsilon\big(g(\delta(o_1),\delta(o_2),...,\delta(o_n))\big) \\ &\leq \max\{\upsilon\big(\delta(o_1)\big),\upsilon(\delta(o_2)),...,\upsilon(\delta(o_n))\} \\ &= \{\vartheta(o_1),\vartheta(o_2),...,\vartheta(o_n)\}. \end{aligned}$$

So

$$\vartheta\big(g(o_1,o_2,...,o_n)\big) \leq \{\vartheta(o_1),\vartheta(o_2),...,\vartheta(o_n)\}.$$

(5) For all $l_1^{i-1}, l_{i+1}^n, v \in R$ have

$$\begin{aligned} \vartheta\big(g(l_1^{i-1},v,l_{i+1}^n)\big) &= \upsilon\big(\delta(g(l_1^{i-1},v,l_{i+1}^n))\big) \\ &= \upsilon\big(g(\delta(l_1),\delta(l_2),...,\delta(l_{i-1}),\delta(v),\delta(l_{i+1}),...,\delta(l_n))\big) \\ &\leq \upsilon\big(\delta(v)\big) \\ &= \vartheta(v). \end{aligned}$$

Therefore

$$\vartheta\big(g(l_1^{i-1},v,l_{i+1}^n)\big) \leq \vartheta(v).$$

(6) For $j \neq i$ and $1 \leq j \leq n$ and $o_1^{i-1}, o_{i+1}^n, l_1^m \in R$ have $\delta(o_1), \delta(o_2), ..., \delta(o_{i-1})$, $\delta(o_{i+1}), ..., \delta(o_n), \delta(l_1), \delta(l_2), ..., \delta(l_m) \in S$. So there is $\delta(s_j) \in \delta(R)$ that
$\delta\big(g(o_1^{i-1}, f(l_1, l_2, ..., l_m), o_{i+1}^n)\big)$
$= g\big(\delta(o_1), \delta(o_2), ..., \delta(o_{i-1}), f(\delta(l_1), \delta(l_2), ..., \delta(l_m)), \delta(o_{i+1}), ..., \delta(o_n)\big)$
$= f\big(g(\delta(o_1), \delta(o_2), ..., \delta(o_{i-1}), \delta(l_1), \delta(o_{i+1}), ..., \delta(o_n)), g(\delta(o_1), \delta(o_2), ..., \delta(o_{i-1}),$
$\delta(l_2), \delta(o_{i+1}), ..., \delta(o_n)), ..., g(\delta(o_1), \delta(o_2), ..., \delta(o_{i-1}), \delta(l_{j-1}), \delta(o_{i+1}), ..., \delta(o_n)),$
$\delta(s_k), g(\delta(o_1), \delta(o_2), ..., \delta(o_{i-1}), \delta(l_{j+1}), \delta(o_{i+1}), ..., \delta(o_n)), ..., g(\delta(o_1), ..., \delta(o_{i-1}),$
$\delta(l_m), \delta(o_{i+1}), ..., \delta(o_n))\big) = \delta\big(f(g(o_1^{i-1}, l_1, o_{i+1}^n), g(o_1^{i-1}, l_2, o_{i+1}^n), ..., g(o_1^{i-1}, l_{j-1},$
$o_{i+1}^n), s_j, g(o_1^{i-1}, l_{j+1}, o_{i+1}^n), ..., g(o_1^{i-1}, l_m, o_{i+1}^n))\big)$.
Hence we have
$g\big(o_1^{i-1}, f(l_1, l_2, ..., l_m), o_{i+1}^n\big) = f\big(g(o_1^{i-1}, l_1, o_{i+1}^n), g(o_1^{i-1}, l_2, o_{i+1}^n), ..., g(o_1^{i-1}, l_{j-1},$
$o_{i+1}^n), s_k, g(o_1^{i-1}, l_{j+1}, o_{i+1}^n), ..., g(o_1^{i-1}, l_m, o_{i+1}^n)\big)$
and $\upsilon(\delta(s_j)) \leq \upsilon\big(\delta(l_j)\big)$, therefore $\vartheta(s_j) \leq \vartheta(l_j)$.
The condition stated in the definition 2.7 is for ϑ, thus ϑ is an anti-fuzzy ideal of R. □

Example 3.3. We know $(\mathbb{Z}, f, g)$ and $(2\mathbb{Z}, f, g)$ with $f(l_1, l_2, ..., l_m) = l_1 + l_2 + ... + l_m$ and $g(w_1, w_2, ..., w_n) = w_1$ are (m,n)-near rings and $f : \mathbb{Z} \to 2\mathbb{Z}$, that $f(x) = 2x$ is an (m,n)-near ring monomorphism. We define

$$\eta(x) = \begin{cases} 0.2 & \text{if } x \in 6\mathbb{Z} \\ 0.6 & \text{if } x \notin 6\mathbb{Z}. \end{cases}$$

Because level subsets $\eta_{0.2}^{\leq} = 6\mathbb{Z}$ and $\eta_{0.6}^{\leq} = 2\mathbb{Z}$ are ideals of $(2\mathbb{Z}, f, g)$ thus by Theorem 2.8, η is an anti-fuzzy ideal of $(2\mathbb{Z}, f, g)$. Hence by using the previous theorem $f^{-1}(\eta)$ is an anti-fuzzy ideal of $\mathbb{Z}$.

An anti-fuzzy set ϑ in R has the inf property, if for any subset T of R there exists $l_0 \in T$ such that $\vartheta(l_0) = \inf \vartheta(t)_{t \in T}$.

Theorem 3.4. *An (m,n)-near ring homomorphism image of an anti-fuzzy ideal, which has the inf property is an anti-fuzzy ideal.*

Proof. Let $\delta : R \longrightarrow S$ be an (m,n)-near ring homomorphism and ϑ be an anti-fuzzy ideal of R with the inf property and υ be the image of ϑ under δ.

(1) Given $\delta(l_1), \delta(l_2), ..., \delta(l_m) \in \delta(R)$, let for all $1 \leq i \leq m$, $x_{i_0} \in \delta^{-1}(\delta(l_i))$ be such that $\vartheta(l_{i_0}) = \inf \vartheta(t)_{t \in \delta^{-1}(\delta(l_i))}$,

$$\begin{aligned} \upsilon\big(f(\delta(l_1), \delta(l_2), ..., \delta(l_m))\big) &= \inf \vartheta(t)_{t \in \delta^{-1}\left(f(\delta(l_1), \delta(l_2), ..., \delta(l_m))\right)} \\ &\leq \vartheta\big(f(l_{1_0}, l_{2_0}, ..., l_{m_0})\big) \\ &\leq \max\{\vartheta(l_{1_0}), \vartheta(l_{2_0}), ..., \vartheta(l_{m_0})\} \\ &= \max\{\inf \vartheta(t)_{t \in \delta^{-1}\left(\delta(l_1)\right)}, ..., \inf \vartheta(t)_{t \in \delta^{-1}\left(\delta(l_m)\right)}\} \\ &= \max\{\upsilon\big(\delta(l_1)\big), \upsilon\big(\delta(l_2)\big), ..., \upsilon\big(\delta(l_m)\big)\}. \end{aligned}$$

So

$$\upsilon\big(f(\delta(l_1), \delta(l_2), ..., \delta(l_m))\big) \leq \max\{\upsilon\big(\delta(l_1)\big), \upsilon\big(\delta(l_2)\big), ..., \upsilon\big(\delta(l_m)\big)\}.$$

(2) Given $\delta(l_1), \delta(l_2), ..., \delta(l_m), \delta(b) \in \delta(R)$, let for all $1 \leq i \leq m$, $l_{i_0} \in \delta^{-1}\big(\delta(l_i)\big)$,
$b_0 \in \delta^{-1}(\delta(b))$ such that $\vartheta(l_{i_0}) = \inf \vartheta(t)_{t \in \delta^{-1}\left(\delta(l_i)\right)}$, $\vartheta(b_0) = \inf \vartheta(t)_{t \in \delta^{-1}\left(\delta(b)\right)}$.
There is $a_i \in R$ so that

$$f(l_1^{i-1}, a_i, l_{i+1}^m) = b,$$
$$\vartheta(a_i) \leq \max\{\vartheta(l_1), \vartheta(l_2), ..., \vartheta(l_{i-1}), \vartheta(b), \vartheta(l_{i+1}), ..., \vartheta(l_m)\},$$

$$\delta(b) = \delta\big(f(l_1^{i-1}, a_i, l_{i+1}^m)\big) = f\big(\delta(l_1), \delta(l_2), ..., \delta(l_{i-1}), \delta(a_i), \delta(l_{i+1}), ..., \delta(l_m)\big),$$

$$\delta^{-1}\big(\delta(b)\big) = \delta^{-1}\big(f(\delta(l_1), \delta(l_2), ..., \delta(l_{i-1}), \delta(a_i), \delta(l_{i+1}), ..., \delta(l_m))\big)$$
$$= f\big(\delta^{-1}(\delta(l_1)), \delta^{-1}(\delta(l_2)), ..., \delta^{-1}(\delta(l_{i-1})), \delta^{-1}(\delta(a_i)), \delta^{-1}(\delta(l_{i+1})), ..., \delta^{-1}(\delta(l_m))\big)$$

and because $l_{1_0} \in \delta^{-1}\big(\delta(l_1)\big), l_{2_0} \in \delta^{-1}\big(\delta(l_2)\big), ..., l_{m_0} \in \delta^{-1}\big(\delta(l_m)\big), b_0 \in \delta^{-1}\big(\delta(b)\big)$ there is $a_{i_0} \in \delta^{-1}(\delta(a_i))$ that $f(l_{1_0}^{(i-1)_0}, a_{i_0}, l_{(i+1)_0}^{m_0}) = b_0$ and

$$\vartheta(a_{i_0}) \leq \max\{\vartheta(l_{1_0}), \vartheta(l_{2_0}), ..., \vartheta(l_{(i-1)_0}), \vartheta(b_0), \vartheta(l_{(i+1)_0}), ..., \vartheta(l_{m_0})\}.$$

$$\begin{aligned}&\upsilon\big(\delta(a_i)\big) = \inf \vartheta(t)_{t \in \delta^{-1}\big(\delta(a_i)\big)} \leq \vartheta(a_{i_0}) \leq \max\{\vartheta(l_{1_0}), \vartheta(l_{2_0}), ..., \vartheta(l_{(i-1)_0}), \vartheta(b_0),\\ &\vartheta(l_{(i+1)_0}), ..., \vartheta(l_{m_0})\} = \max\{\inf \vartheta(t)_{t \in \delta^{-1}\big(\delta(l_1)\big)}, \inf \vartheta(t)_{t \in \delta^{-1}\big(\delta(l_2)\big)}, ...,\\ &\inf \vartheta(t)_{t \in \delta^{-1}\big(\delta(l_{i-1})\big)}, \inf \vartheta(t)_{t \in \delta^{-1}\big(\delta(l_{i+1})\big)}, ..., \inf \vartheta(t)_{t \in \delta^{-1}\big(\delta(l_m)\big)}, \inf \vartheta(t)_{t \in \delta^{-1}\big(\delta(b)\big)}\}\\ &= \max\{\upsilon\big(\delta(l_1)\big), ..., \upsilon\big(\delta(l_{i-1})\big), \upsilon\big(\delta(l_{i+1})\big), ..., \upsilon\big(\delta(l_m)\big), \upsilon\big(\delta(b)\big)\}.\end{aligned}$$

Thus

$$\upsilon\big(\delta(a_i)\big) \leq \max\{\upsilon\big(\delta(l_1)\big), ..., \upsilon\big(\delta(l_{i-1})\big), \upsilon\big(\delta(l_{i+1})\big), ..., \upsilon\big(\delta(l_m)\big), \upsilon\big(\delta(b)\big)\}.$$

(3) Given $\delta(d_1), \delta(d_2), ..., \delta(d_{i-1}), \delta(d_{i+1}), ..., \delta(d_m), \delta(d) \in \delta(R)$. Because ϑ is an anti-fuzzy ideal of R so for all $d_1^{i-1}, d_{i+1}^m \in R$, $d \in R$ and $1 \leq j \leq m$, there is $a \in R$ so that $f(d_1^{j-1}, a, d_{j+1}^m) = f(d_1^{i-1}, d, d_{i+1}^m)$ and $\vartheta(a) \leq \vartheta(d)$. So

$$\begin{aligned} f\big(\delta(d_1), \delta(d_2), ..., \delta(d_{j-1}), \delta(a), \delta(d_{j+1}), ..., \delta(d_m)\big) &= \delta\big(f(d_1^{j-1}, a, d_{j+1}^m)\big)\\ &= \delta\big(f(d_1^{i-1}, d, d_{i+1}^m)\big)\\ &= f\big(\delta(d_1), ..., \delta(d_{i-1}), \delta(d), \delta(d_{i+1}), ..., \delta(d_m)\big),\end{aligned}$$

$$\upsilon\big(\delta(a)\big) = \vartheta(a) \leq \mu(d) = \upsilon\big(\delta(d)\big).$$

(4) Given $\delta(l_1), \delta(l_2), ..., \delta(l_n) \in \delta(R)$, let for all $1 \leq i \leq n$, $l_{i_0} \in \delta^{-1}\big(\delta(l_i)\big)$ be such that $\vartheta(l_{i_0}) = \inf \vartheta(t)_{t \in \delta^{-1}(\delta(l_i))}$.

$$\begin{aligned}\upsilon\big(g(\delta(l_1), \delta(l_2), ..., \delta(l_n)\big) &= \inf \vartheta(t)_{t \in \delta^{-1}\big(g(\delta(l_1), \delta(l_2), ..., \delta(l_n))\big)}\\ &\leq \vartheta\big(g(l_{1_0}, l_{2_0}, ..., l_{n_0})\big)\\ &\leq \max\{\vartheta(l_{1_0}), \vartheta(l_{2_0}), ..., \vartheta(l_{n_0})\}\\ &= \max\{\inf \vartheta(t)_{t \in \delta^{-1}\big(\delta(l_1)\big)}, \inf \vartheta(t)_{t \in \delta^{-1}\big(\delta(l_2)\big)}, ..., \inf \vartheta(t)_{t \in \delta^{-1}\big(\delta(l_n)\big)}\}\\ &= \max\{\upsilon\big(\delta(l_1)\big), \upsilon\big(\delta(l_2)\big), ..., \upsilon\big(\delta(l_n)\big)\}.\end{aligned}$$

So

$$\upsilon\big(g(\delta(l_1), \delta(l_2), ..., \delta(l_n))\big) \geq \max\{\upsilon\big(\delta(l_1)\big), \upsilon\big(\delta(l_2)\big), ..., \upsilon\big(\delta(l_n)\big)\}.$$

(5) Given $k = i$ and $\delta(l_1), \delta(l_2), ..., \delta(l_n), \delta(l) \in \delta(R)$, let for all $1 \leq i \leq n$, $x_{i_0} \in \delta^{-1}\big(\delta(l_i)\big), l_0 \in \delta^{-1}\big(\delta(l)\big)$ such that $\vartheta(l_{i_0}) = \inf \vartheta(t)_{t \in \delta^{-1}\big(\delta(l_i)\big)}$, $\vartheta(l_0) = \inf \vartheta(t)_{t \in \delta^{-1}\big(\delta(l)\big)}$. Have

$$\vartheta(l_0) \leq \vartheta\big(g(l_{1_0}^{(i-1)_0}, l_0, l_{(i+1)_0}^{n_0})\big).$$

$$\begin{aligned}
\upsilon\big(\delta(l)\big) &= \inf \vartheta(t)_{t\in\delta^{-1}\left(\delta(l)\right)}\\
&\leq \vartheta(l_0)\\
&\leq \vartheta\big(g(l_{1_0}^{(i-1)_0}, l_0, l_{(i+1)_0}^{n_0})\big)\\
&= \inf \vartheta(t)_{t\in\delta^{-1}\left(\delta(g(l_1^{i-1}, l, l_{i+1}^{n}))\right)}\\
&= \upsilon\big(\delta(g(l_1^{i-1}, l, l_{i+1}^{n})\big)\\
&= \upsilon\big(g(\delta(l_1), \delta(l_2), ..., \delta(l_{i-1}), \delta(l), \delta(l_{i+1}), ..., \delta(l_n))\big).
\end{aligned}$$

So

$$\upsilon\big(\delta(l)\big) \geq \upsilon\big(g(\delta(l_1), \delta(l_2), ..., \delta(l_{i-1}), \delta(l), \delta(l_{i+1}), ..., \delta(l_n))\big).$$

(6) Given $\delta(l_1), \delta(l_2), ..., \delta(l_m), \delta(z_1), ..., \delta(z_n) \in \delta(R)$, let for all $1 \leq i \leq m$, $1 \leq j \leq n$, $l_{i_0} \in \delta^{-1}\big(\delta(l_i)\big)$, $z_{j_0} \in \delta^{-1}\big(\delta(z_j)\big)$ such that $\vartheta(l_{i_0}) = \inf \vartheta(t)_{t\in\delta^{-1}\left(\delta(l_i)\right)}$, $\vartheta(z_{j_0}) = \inf \vartheta(t)_{t\in\delta^{-1}\left(\delta(z_j)\right)}$, for all $k \neq i$, $1 \leq k \leq n$ and $z_1^{i-1}, z_{i+1}^{n}, l_1^{m} \in R$ there is $x_k \in I$, so that
$g\big(z_1^{i-1}, f(l_1, l_2, ..., l_m), z_{i+1}^{n}\big) = f\big(g(z_1^{i-1}, l_1, z_{i+1}^{n}), g(z_1^{i-1}, l_2, z_{i+1}^{n}), ..., g(z_1^{i-1}, l_{k-1}, z_{i+1}^{n}),$
$x_k, g(z_1^{i-1}, l_{k+1}, z_{i+1}^{n}), ..., g(z_1^{i-1}, l_m, z_{i+1}^{n})\big)$ and $\vartheta(l_k) \geq \vartheta(x_k)$. So
$g\big(\delta(z_1), \delta(z_2), ..., \delta(z_{i-1}), f(\delta(l_1), \delta(l_2), ..., \delta(l_m)), \delta(z_{i+1}), ..., \delta(z_n)\big)$
$= f\big(g(\delta(z_1), \delta(z_2), ..., \delta(z_{i-1}), \delta(l_1), \delta(z_{i+1}), ..., \delta(z_n)), ..., g(\delta(z_1), \delta(z_2), ..., \delta(z_{i-1}), \delta(l_{k-1}),$
$\delta(z_{i+1}), ..., \delta(z_n)), \delta(x_k), g(\delta(z_1), \delta(z_2), ..., \delta(z_{i-1}), \delta(l_{k+1}), \delta(z_{i+1}), ..., \delta(z_n)), ..., g(\delta(z_1),$
$\delta(z_2), ..., \delta(z_{i-1}), \delta(l_m), \delta(z_{i+1}), ..., \delta(z_n))\big)$.
Hence
$g\big(\delta^{-1}(\delta(z_1)), \delta^{-1}(\delta(z_2)), ..., \delta^{-1}(\delta(z_{i-1})), f(\delta^{-1}(\delta(l_1)), \delta^{-1}(\delta(l_2)), ..., \delta^{-1}(\delta(l_m))),$
$\delta^{-1}(\delta(z_{i+1})), ..., \delta^{-1}(\delta(z_n))\big) = f\big(g(\delta^{-1}(\delta(z_1)), \delta^{-1}(\delta(z_2)), ..., \delta^{-1}(\delta(z_{i-1})), \delta^{-1}(\delta(l_1)),$
$\delta^{-1}(\delta(z_{i+1})), ..., \delta^{-1}(\delta(z_n))), ..., g(\delta^{-1}(\delta(z_1)), \delta^{-1}(\delta(z_2)), ..., \delta^{-1}(\delta(z_{i-1})), \delta^{-1}(\delta(l_{k-1})),$
$\delta^{-1}(\delta(z_{i+1})), ..., \delta^{-1}(\delta(z_n))), \delta^{-1}(\delta(x_k)), g(\delta^{-1}(\delta(z_1)), \delta^{-1}(\delta(z_2)), ..., \delta^{-1}(\delta(z_{i-1})),$
$\delta^{-1}(\delta(l_{k+1})), \delta^{-1}(\delta(z_{i+1})), ..., \delta^{-1}(\delta(z_n))), ..., g(\delta^{-1}(\delta(z_1)), \delta^{-1}(\delta(z_2)), ..., \delta^{-1}(\delta(z_{i-1})),$
$\delta^{-1}(\delta(l_m)), \delta^{-1}(\delta(z_{i+1})), ..., \delta^{-1}(\delta(z_n)))\big)$.
Therefore, there is $x_{k_0} \in \delta^{-1}\big(\delta(x_k)\big)$ that
$g\big(z_{1_0}^{(i-1)_0}, f(l_{1_0}, l_{2_0}, ..., l_{m_0}), z_{(i+1)_0}^{n_0}\big) = f\big(g(z_{1_0}^{(i-1)_0}, l_{1_0}, z_{(i+1)_0}^{n_0}), ..., g(z_{1_0}^{(i-1)_0}, l_{(k-1)_0}, z_{(i+1)_0}^{n_0}),$
$x_{k_0}, g(z_{1_0}^{(i-1)_0}, l_{(k+1)_0}, z_{(i+1)_0}^{n_0}), ..., g(z_{1_0}^{(i-1)_0}, l_{m_0}, z_{(i+1)_0}^{n_0})\big)$
and $\vartheta(l_{k_0}) \geq \vartheta(x_{k_0})$
$\upsilon\big(\delta(x_k)\big) = \inf \vartheta(t)_{t\in\delta^{-1}\left(\delta(x_k)\right)} \leq \vartheta(x_{k_0}) \leq \vartheta(l_{k_0}) = \inf \vartheta(t)_{t\in\delta^{-1}\left(\delta(l_k)\right)} = \upsilon\big(\delta(l_k)\big)$.
So there is $\delta(x_k) \in \delta(R)$ that
$g\big(\delta(z_1), \delta(z_2), ..., \delta(z_{i-1}), f(\delta(l_1), \delta(l_2), ..., \delta(l_m)), \delta(z_{i+1}), ..., \delta(z_n)\big) = f\big(g(\delta(z_1), \delta(z_2), ...,$
$\delta(z_{i-1}), \delta(l_1), \delta(z_{i+1}), ..., \delta(z_n)), ..., g(\delta(z_1), \delta(z_2), ..., \delta(z_{i-1}), \delta(l_{k-1}), \delta(z_{i+1}), ..., \delta(z_n)),$
$\delta(x_k), g(\delta(z_1), \delta(z_2), ..., \delta(z_{i-1}), \delta(l_{k+1}), \delta(z_{i+1}), ..., \delta(z_n)), ..., g(\delta(z_1), \delta(z_2), ..., \delta(z_{i-1}),$
$\delta(l_m), \delta(z_{i+1}), ..., \delta(z_n))\big)$ and $\upsilon\big(\delta(x_k)\big) \leq \upsilon\big(\delta(l_k)\big)$.
The condition stated in the definition is for υ, so υ is an anti-fuzzy ideal of S. □

Example 3.5. Similar to Example 2 in [11], it can be easily proven that $(\mathbb{Z}, f, g)$ and $(2\mathbb{Z}, f, g)$ with $f(l_1, l_2, ..., l_m) = l_1 + l_2 + ... + l_m$ and $g(w_1, w_2, ..., w_n) = w_1$ are (m,n)-near rings and $f : \mathbb{Z} \to 2\mathbb{Z}$, that $f(x) = 2x$ is an (m,n)-near ring homomorphism. Assume that

$$\eta(x) = \begin{cases} 0.1 & \text{if } x \in 6\mathbb{Z} \\ 0.7 & \text{if } x \notin 6\mathbb{Z} \end{cases}$$

is a fuzzy set of $\mathbb{Z}$. Because level subsets $\eta^{\leq}_{0.1} = 6\mathbb{Z}$ and $\eta^{\leq}_{0.7} = \mathbb{Z}$ are ideals of $(\mathbb{Z}, f, g)$ thus by Theorem 2.8, η is an anti-fuzzy ideal of $(\mathbb{Z}, f, g)$. Hence by using the previous theorem $f(\eta)$ is an anti-fuzzy ideal of $2\mathbb{Z}$.

Theorem 3.6. *If $f : R \longrightarrow S$ is an onto near ring homomorphism and ϑ is an anti-fuzzy ideal of R, then $f(\vartheta)$ is an anti-fuzzy ideal of S.*

Proof. Given Theorem 2.8 it is sufficient to show that $S^{\alpha}_{f(\vartheta)}$, $\alpha \in [0, \vartheta(0)]$, is an ideal of S. Note that $S^{0}_{f(\vartheta)} = S$ and if $\alpha \in (0, 1]$, then $S^{\alpha}_{f(\vartheta)} = \bigcap_{0<\beta<\alpha} f(\vartheta_{\alpha-\beta})$ by Theorem 1 in [6], because $\vartheta_{(\alpha-\beta)}$ is an ideal of R and f is onto, so $f(\vartheta_{(\alpha-\beta)})$ is a k-ideal of S. Therefore $f(\vartheta)_{\alpha} = S^{\alpha}_{f(\vartheta)}$ is an intersection of a family of ideals and is also an ideal of S. Hence by Theorem 2.8 $f(\vartheta)$ is an anti-fuzzy ideal of S. □

Example 3.7. According to Example 8 in [12], $f : \mathbb{Z} \to \mathbb{Z}_{mn}$, that $f(x) = \overline{x}$ is an (m, n)-near rings homomorphism. We define

$$\eta(x) = \begin{cases} 0.5 & \text{if } x \in m\mathbb{Z} \\ 0.9 & \text{if } x \notin m\mathbb{Z}. \end{cases}$$

We know level subsets $\eta^{\leq}_{0.5} = m\mathbb{Z}$ and $\eta^{\leq}_{0.9} = \mathbb{Z}$ are ideals of $(\mathbb{Z}, f, g)$ thus by Theorem 2.8, η is an anti-fuzzy ideal of $(\mathbb{Z}, f, g)$. Hence by using the previous theorem $f(\eta)$ is an anti-fuzzy ideal of $\mathbb{Z}_{mn}$.

Theorem 3.8. *Let $f : R \longrightarrow S$ be an onto homomorphism of near rings and let ϑ and υ be anti-fuzzy ideals of R and S, respectively such that*

$$\begin{aligned} \mathrm{Im}(\vartheta) &= \{\alpha_0, \alpha_1, ..., \alpha_n\} \textit{ with } \alpha_0 > \alpha_1 > ... > \alpha_n, \\ \mathrm{Im}(\upsilon) &= \{\beta_0, \beta_1, ..., \beta_m\} \textit{ with } \beta_0 > \beta_1 > ... > \beta_m. \end{aligned}$$

Then

(1) $\mathrm{Im}\big(f(\vartheta)\big) \subset \mathrm{Im}(\vartheta)$ *and the chain of level ideals of* $f(\vartheta)$ *is*

$$f(\vartheta^{\leq}_{\alpha_0}) \subset f(\vartheta^{\leq}_{\alpha_1}) \subset ... \subset f(\vartheta^{\leq}_{\alpha_n}) = S.$$

(2) $\mathrm{Im}\big(f^{-1}(\upsilon)\big) = \mathrm{Im}(\upsilon)$ *and the chain of level k-ideals of* $f^{-1}(\upsilon)$ *is*

$$f^{-1}(S^{\beta_0}_{\upsilon}) \subset f^{-1}(S^{\beta_1}_{\upsilon}) \subset ... \subset f^{-1}(S^{\beta_m}_{\upsilon}) = R.$$

Proof. (1) Since $f(\vartheta)(y) = \sup \vartheta(l)_{l \in f^{-1}(y)}$ for all $y \in S$, obviously $\mathrm{Im}\big(f(\vartheta)\big) \subset \mathrm{Im}(\vartheta)$. Note that for any $y \in S$, $y \in f(l^{\alpha_i}_{\vartheta})$ there exists $l \in f^{-1}(y)$ such that

$$\begin{aligned} \vartheta(l) \geq \alpha_i \quad &\Longleftrightarrow \sup \vartheta(z)_{z \in f^{-1}(y)} \geq \alpha_i \\ &\Longleftrightarrow f(\vartheta)(y) \geq \alpha_i \\ &\Longleftrightarrow y \in S^{\alpha_i}_{f(\vartheta)}. \end{aligned}$$

Hence $f(l^{\alpha_i}_{\vartheta}) = S^{\alpha_i}_{f(\vartheta)}$ for $i \in \{0, 1, 2, ..., n\}$ and therefore the chain of level k-ideals of $f(\vartheta)$ is

$$f(l^{\alpha_0}_{\vartheta}) \subset f(l^{\alpha_1}_{\vartheta}) \subset ... \subset f(l^{\alpha_n}_{\vartheta}) = S.$$

(2) Since $f^{-1}(\upsilon)(l) = \upsilon(f(l))$ for all $l \in R$ and f is onto, we have $\text{Im}(f^{-1}(\upsilon)) = \text{Im}(\upsilon)$. Note that for all $l \in R$,

$$\begin{aligned} l \in f^{-1}(S_\upsilon^{\beta_i}) &\iff f(l) \in S_\upsilon^{\beta_i} \\ &\iff \upsilon(f(l)) \geq \beta_i \\ &\iff f^{-1}(\upsilon)(l) \geq \beta_i \\ &\iff l \in l_{f^{-1}(\upsilon)}^{\beta_i} \end{aligned}$$

so that $f^{-1}(S_\upsilon^{\beta_i}) = l_{f^{-1}(\upsilon)}^{\beta_i}$ for all $i = 0, 1, 2, ..., m$. Hence the chain of level k-ideals of $f^{-1}(\upsilon)$ is

$$f^{-1}(S_\upsilon^{\beta_0}) \subset f^{-1}(S_\upsilon^{\beta_1}) \subset ... \subset f^{-1}(S_\upsilon^{\beta_m}) = R.$$

□

Example 3.9. In Example 3.5, $f(\eta_{0.5}^{\leq}) \subset f(\eta_{0.9}^{\leq}) = 2\mathbb{Z}$.

Definition 3.10. *Let ϑ and υ be fuzzy sets of (m, n)-near ring R. We said $\vartheta \subseteq \upsilon$ if for all $x \in R$, $\vartheta(x) \leq \upsilon(x)$. If $\vartheta \subseteq \upsilon$ and there exists $x \in R$ such that $\vartheta(x) < \upsilon(x)$, then we said $\vartheta \subset \upsilon$ and we say that ϑ is a proper fuzzy set of υ. Note that $\vartheta = \upsilon$ if and only if for all $x \in R$, $\vartheta(x) = \upsilon(x)$.*

You can see more theorems in [8].

Definition 3.11. *An anti-fuzzy ideal ϑ is named the anti-fuzzy prime ideal of R if for any anti-fuzzy ideals $\sigma_1, \sigma_2, ..., \sigma_n$ of R such that $g(\sigma_1, \sigma_2, ..., \sigma_n) \geq \vartheta$ implies that $\sigma_1 \geq \vartheta$ or $\sigma_2 \geq \vartheta$ or ... or $\sigma_n \geq \vartheta$.*

Theorem 3.12. *Suppose that I is an ideal of X and ϑ is a fuzzy set in X defined by:*

$$\vartheta_I(l) = \begin{cases} s & \text{if } l \in I \\ 1 & \text{if } l \notin I \end{cases}$$

for all $l \in X$ and $s \in [0, 1)$. Then ϑ_I is an anti-fuzzy prime ideal of X if and only if I is a prime ideal of X.

Proof. Suppose I is a prime ideal of X. Then ϑ_I is an anti-fuzzy ideal of X. Assume that σ_1, σ_2, ..., σ_n are anti-fuzzy ideals of X such that $g(\sigma_1, \sigma_2, ..., \sigma_n) \geq \vartheta_I$. We prove $\sigma_1 \geq \vartheta_I$ or $\sigma_2 \geq \vartheta_I$ or ... or $\sigma_n \geq \vartheta_I$. Assume that $\sigma_1 < \vartheta_I$ and $\sigma_2 < \vartheta_I$ and ... and $\sigma_n < \vartheta_I$ then for all $l_1, l_2, ..., l_n \in X$, $\sigma_1(l_1) < \vartheta_I(l_1)$ and $\sigma_2(l_2) < \vartheta_I(l_2)$ and ... and $\sigma_n(l_n) < \vartheta_I(l_n)$. Now $\vartheta_I(l_1) \neq s$ and $\vartheta_I(l_2) \neq s$ and ... and $\vartheta_I(l_n) \neq s$. So $\vartheta_I(l_1) = \vartheta_I(l_2) = \vartheta_I(l_n) = 1$ thus $l_1, l_2, ..., l_n \notin I$. Because I is a prime ideal, $g(< l_1 >, < l_2 >, ..., < l_n >) \nsubseteq I$. Assume that $l \in g(< l_1 >, < l_2 >, ..., < l_n >)$ hence $1 = \vartheta_I(l) \leq g(\sigma_1, \sigma_2, ..., \sigma_n)(l)$

$$\begin{aligned} g(\sigma_1, \sigma_2, ..., \sigma_n)(l) &= \inf_{l = g(a_1, a_2, ..., a_n)} \{\max\{\sigma_1(a_1), \sigma_2(a_2), ..., \sigma_n(a_n)\}\} \\ &\leq \max\{\sigma_1(a_1), \sigma_2(a_2), ..., \sigma_n(a_n)\} \\ &\leq \max\{\sigma_1(l_1), \sigma_2(l_2), ..., \sigma_n(l_n)\} \\ &< \max\{\vartheta_I(l_1), \vartheta_I(l_2), ..., \vartheta_I(l_n)\} = 1 = \vartheta_I(l) \end{aligned}$$

this gives that $g(\sigma_1, \sigma_2, ..., \sigma_n) > \vartheta_I$. Which is a contradiction. Therefore, ϑ_I is an anti-fuzzy prime ideal of X. □

Example 3.13. We know $(\mathbb{Z}, f, g)$ and $(2\mathbb{Z}, f, g)$ with $f(l_1, l_2, ..., l_m) = l_1 + l_2 + ... + l_m$ and $g(w_1, w_2, ..., w_n) = w_1$ are (m, n)-near rings and $2\mathbb{Z}$ is a prime ideal of $\mathbb{Z}$. We define

$$\eta_{2\mathbb{Z}}(x) = \begin{cases} 0/2 & \text{if } x \in 2\mathbb{Z} \\ 1 & \text{if } x \notin 2\mathbb{Z}. \end{cases}$$

Hence by using the previous theorem $\eta_{2\mathbb{Z}}$ is an anti-fuzzy prime ideal of $\mathbb{Z}$.

Theorem 3.14. *The arbitrary union of anti-fuzzy prime ideals of (m,n)-near ring (R,f,g) is also an anti-fuzzy prime ideal of R.*

Proof. Assume that $\{\vartheta_i \mid i \in \Gamma\}$ is the set of all anti-fuzzy prime ideals in R. $\vartheta = \bigcup\limits_{i\in\Gamma} \vartheta_i$ is also an anti-fuzzy prime ideal. Let $\sigma_1, \sigma_2, ..., \sigma_n$ be any anti-fuzzy ideals of R such that $g(\sigma_1, \sigma_2, ..., \sigma_n) \geq \bigcup\limits_{i\in\Gamma} \vartheta_i$ implies that $g(\sigma_1, \sigma_2, ..., \sigma_n) \geq \vartheta_i$ for some $i \in \Gamma$. Since each ϑ_i is an anti-fuzzy prime ideal. Therefore, $\sigma_1 \geq \vartheta_i$ or $\sigma_2 \geq \vartheta_i$ or ... or $\sigma_n \geq \vartheta_i$ for some $i \in \Gamma$. Thus $\sigma_1 \geq \bigcup\limits_{i\in\Gamma} \vartheta_i$ or $\sigma_2 \geq \bigcup\limits_{i\in\Gamma} \vartheta_i$ or ... or $\sigma_n \geq \bigcup\limits_{i\in\Gamma} \vartheta_i$ for some $i \in \Gamma$. □

Example 3.15. According Example 3.13, $\eta_{p\mathbb{Z}}$, p is the prime number in $\mathbb{Z}$, is anti-fuzzy prime ideal of $\mathbb{Z}$. Let $\eta = \bigcup\limits_{p\in\mathbb{Z}} \eta_{p\mathbb{Z}}$, hence by using the previous theorem η is an anti-fuzzy prime ideal of $\mathbb{Z}$.

Theorem 3.16. *The arbitrary intersection of anti-fuzzy prime ideals of (m,n)-near ring (R,f,g) is also an anti-fuzzy prime ideal of R.*

Proof. Suppose that $\{\vartheta_i \mid i \in \Gamma\}$ the set of all anti-fuzzy prime ideals in R. $\vartheta = \bigcap\limits_{i\in\Gamma} \vartheta_i$ is also an anti-fuzzy prime ideal. Let $\sigma_1, \sigma_2, ..., \sigma_n$ be any anti-fuzzy ideals of R such that $g(\sigma_1, \sigma_2, ..., \sigma_n) \geq \bigcap\limits_{i\in\Gamma} \vartheta_i$ implies that $g(\sigma_1, \sigma_2, ..., \sigma_n) \geq \vartheta_i$ for some $i \in \Gamma$. Since each ϑ_i is an anti-fuzzy prime ideal. Therefore, $\sigma_1 \geq \vartheta_i$ or $\sigma_2 \geq \vartheta_i$ or ... or $\sigma_n \geq \vartheta_i$ for all $i \in \Gamma$. Thus $\sigma_1 \geq \bigcap\limits_{i\in\Gamma} \vartheta_i$ or $\sigma_2 \geq \bigcap\limits_{i\in\Gamma} \vartheta_i$ or ... or $\sigma_n \geq \bigcap\limits_{i\in\Gamma} \vartheta_i$ for some $i \in \Gamma$. □

Example 3.17. According Example 3.13, $\eta_{p\mathbb{Z}}$, p is the prime number in $\mathbb{Z}$, is anti-fuzzy prime ideal of $\mathbb{Z}$. Let $\eta = \bigcap\limits_{p\in\mathbb{Z}} \eta_{p\mathbb{Z}}$, hence by using the previous theorem η is an anti-fuzzy prime ideal of $\mathbb{Z}$.

4 Conclusion

In this paper, we introduced and analyzed the concepts of fuzzy (m,n)-sub near rings, anti-fuzzy (m,n)-sub near rings, fuzzy ideals, and anti-fuzzy ideals in the context of (m,n)-near rings. Through rigorous theoretical exploration, we established key properties and proved several theorems related to these fuzzy structures. Our results provide a foundational understanding of how fuzziness interacts with the algebraic framework of (m,n)-near rings, extending classical notions to a broader and more flexible setting. Future research may explore further generalizations, applications in algebraic computing, and connections to other algebraic structures such as (m,n)-near fields and (m,n)-semirings.

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