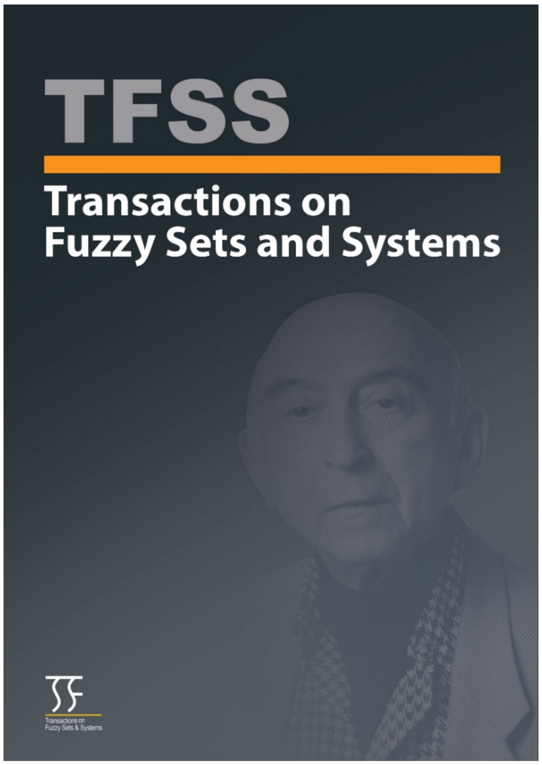
Transactions on Fuzzy Sets and Systems



Transactions on Fuzzy Sets and Systems

ISSN: **2821-0131**

https://sanad.iau.ir/journal/tfss/

Super Neutrosophic 10^p- Based Graceful Labeling Graphs and its Application

Vol.5, No.1, (2026), 18-34. DOI: https://doi.org/10.71602/tfss.2026.1196061

Author(s):

Govindan Vetrivel, Department of Mathematics, Alagappa University, Karaikudi, Tamilnadu, India. E-mail: menakagovindan@gmail.com
Murugappan Mullai, Department of Mathematics, Alagappa University, Karaikudi, Tamilnadu, India & Research Fellow, INTI International University, Nilai Campus, Malaysia.
E-mail: mullaim@alagappauniversity.ac.in

Transactions on Fuzzy Sets and Systems (TFSS)
URL: https://sanad.iau.ir/journal/tfss/
Online ISSN: 2821-0131
Vol.5, No.1, (2026), 18-34
DOI: https://doi.org/10.71602/tfss.2026.1196061

Article Type: Original Research Article
(Journal Pre-proof)

Super Neutrosophic 10^p- Based Graceful Labeling Graphs and its Application

Govindan Vetrivel , **Murugappan Mullai***

Abstract. Neutrosophic graph is used to deal with numerous real world problems and the attained solution is much more accurate than the previous fuzzy models. In this manuscript, a kind of graceful labeling based on 10^p is applied in intuitionistic and neutrosophic framework of graphs with super behaviour, that is quite useful to generalize the labeling structure. In addition, a methodology and an application for this labeling approach are discussed briefly.

AMS Subject Classification 2020: 05C78
Keywords and Phrases: Graceful labeling, Intuitionistic fuzzy 10^p- based graceful labeling, Neutrosophic 10^p- based graceful labeling, Super intuitionistic fuzzy 10^p- based graceful labeling.

1 Introduction

Graph theory approaches real-world issues after Euler's graphical solution for the Konigsberg bridge problem. It represents the data using the components- vertices and edges. The classical graph theory fails to reduce the uncertainty in the final results, which affects the accuracy. Zadeh [1] enacted a definition of fuzzy sets and relations, which paved the way for Kaufmann's [2] fuzzy graph and Rosenfeld's [3] fuzzy graphical model. Bhattacharya [4] and Bhuttani [5] came up with some exemplifying fuzzy graph output. An irreplaceable and notable author, Nagoorgani [6], enforced many astonishing results on fuzzy graphs and their labeling properties. Graceful labeling is nothing but the assignment of integer values to the edges of a graph using the absolute difference of adjacent vertices. Jahir Hussain et al. [7] initiated the work on fuzzy graceful graphs, where fuzzy values are used for the assignment of edges and vertex labels. Nagoor Gani et al. [8] executed the edge graceful labeling concept in a fuzzy graphical scenario. Jebesty Shajila and Vimala [9, 10, 11] established fuzzy graceful labeling approach for complete bipartite graphs and also shared their insight on fuzzy vertex & edge-vertex graceful labeling of some graphs. Solairaju et al. [12] introduced super behaviour in the fuzzy graceful labeling concept, which targets the vertex and edge count of the graph for vertex labeling.

To rectify the issues raised during the fuzzy approach to graph theory, Atanassov [13] enriched the theory and accomplished an "Intuitionistic fuzzy set", a generalized form of fuzzy set theory. Parvathi and Karunambigai [14] applied this theory to develop an intuitionistic fuzzy graph (In-FG), which segregates the false part of an event as a membership. Akram [15], Nagoorgani [16] and Sahoo et al. [17] encountered many features and a brief structural discussion on In-FG and its labeling. Rabeeh Ahamed et al. [18] learned about

*Corresponding Author: Murugappan Mullai, Email: mullaim@alagappauniversity.ac.in, ORCID: 0000-0001-5762-1308
Received: 10 January 2025; **Revised**: 10 February 2025; **Accepted**: 11 February 2025; **Available Online**: 15 February 2025; **Published Online**: 7 May 2026.

How to cite: Vetrivel G, Mullai M. Super Neutrosophic 10^p-Based Graceful Labeling Graphs and its Application. *Transactions on Fuzzy Sets and Systems.* 2026; 5(1): 18-34. DOI: https://doi.org/10.71602/tfss.2026.1196061

graceful labeling graphs using triangular intuitionistic fuzzy numbers. Devie Abirami et al. [19] considered the implementation of graceful labeling in picture fuzzy graphs (modified version of In-FG).

Smarandache [20, 21] established an extension of existing fuzzy set theories in the name "Neutrosophic set" by introducing a new membership that organizes the overall uncertain cases in a situation with an extended total sum of memberships. A neutrosophic set is extensible and flexible compared to classical and fuzzy set theory. The emergence of neutrosophic graph theory is based on the definition and theories of the neutrosophic set. Vasantha Kandasamy et al. [22] and Broumi et al. [23] excelled in their work on neutrosophic graph theory (NeGT) and its approach to decision-making. The general labeling concept in the neutrosophic graph was first accomplished by Gomathi and Keerthika [24]. Ajay and Chellamani [25] conducted a study on the labeling features and introduced the magic labeling idea in the neutrosophic graph background, which is used to estimate the student's academic performance [26]. Krishnaraj and Vikramaprasad [27] formulated the bi-magic labeling behaviour in single valued neutrosophic graphs. Logasoundarya and Vimala [28] used neutrosophic labeling to analyze the electric circuit energy. Ajay et al. [29, 30] furnished some set theory concepts like pythagorean neutrosophic set (PYNS) and complex neutrosophic set. By keeping PYNS as a base, Ajay and Chellamani [31, 32, 33] developed Pythagorean neutrosophic graph (PYNG) and learnt its regularity in detail. Vetrivel and Mullai [34] applied the anti-behavioural insight on PYNG and carried over some product operations on it. Though various enhancements regarding NeGT are established, the types of labeling concept is not yet done extensively with NeGT.

In this article, super 10^p graceful labeling of intuitionistic and neutrosophic kind is explored with some graphs and its application for a company relationship with its branches is portrayed. The super behaviour denotes the total count of vertices and edges of a graph and it is used to fix the starting membership value. The 10^p labeling is used to wider the range of each membership value, thereby helps for the generalization of the graph labeling. It is applied to neutrosophic graceful labeling to attain the distinct membership values, when the absolute difference of adjacent vertices (edge values) is processed.

1.1 Novelty and Contribution

Most of the fuzzy labeling structure and its extension are devised with specific value assignment for each membership, that fails in the increased structure of graph vertices. But the super 10^p graceful labeling holds an initial general assignment of membership value based on total count of vertices and edges, with which the other memberships are formed. The super behaviour of In-FG & NeGT states the labeling of first (truth) membership of graph vertices using the total vertex and edge count and the following membership labels are defined using the previous membership stated. This labeling approach surpasses the previous fuzzy labelings, since super behaviour is inserted under the 10^p graceful labeling and it bridges the gap between fuzzy and the fuzzy extensions. This type of labeling has more flexibility and applicability. An application based on financial risk management of a core company is illustrated in a neutrosophic graph environment, with the help of [35, 36]. This is a novel and new approach in the intuitionistic and neutrosophic labeling framework, where better results can be attained that are quite useful to reach the fine accuracy.

1.2 Limitation of the work

There are no limitations since the proposed work is based on the 10^p model structure and the initial membership assignment is done on the basis of super behaviour(vertex and edge count of the graph). It surely advances the existing works in the conventional fuzzy approach as because of the total sum limit of memberships is greater in NeGT.

1.3 Article structure

This article comprises of following sections: Section 1 bears the introductory part of the fuzzy related works and its extension. Some prior definitions of the base work are furnished in Section 2. The section 3 contains the outline of super 10^p graceful labeling and its result on In-FG. Section 4 is the essential work done on neutrosophic graphs with the super 10^p graceful labeling. An application which relates to the study of super neutrosophic 10^p graceful labeling is demonstrated in Section 5. Section 6 list down the conclusion and results obtained on this labeling work.

2 Preliminaries

Some terminologies and definitions of fuzzy extension graphs are listed below, which acts as the base for our proposed labeling technique and the results we acquire.

Definition 2.1. [10] *An injection $f : V(G) \to 0, 1, 2, \cdots, b$ that causes all of the edge labels to be distinct when each edge $xy \in E(G)$ is given by the label $|f(x) - f(y)|$ is a graceful labeling of a graph G with b edges.*

Definition 2.2. [15] *An intuitionistic fuzzy graph is illustrated as $Gr = (P, Q)$, where $\sigma = (\alpha_P, \beta_P)$ and $\mu = (\alpha_Q, \beta_Q)$ with the following conditions:*

(i) *$V = \{v_1, v_2, ..., v_n\}$ such that $\alpha_P : V \to [0, 1]$ and $\beta_P : V \to [0, 1]$ denote the degree of membership and nonmembership of the element $v_i \in V$ respectively, and $0 \leq \alpha_P(v_i) + \beta_P(v_i) \leq 1$ for every $v_i \in V$, $(i = 1, 2, ..., n)$,*

(ii) *$E \subseteq V \times V$, where $\alpha_Q : V \times V \to [0, 1]$ and $\beta_Q : V \times V \to [0, 1]$ are such that*
$\alpha_Q(v_i, v_j) \leq min[\alpha_P(v_i), \alpha_P(v_j)]$,
$\beta_Q(v_i, v_j) \leq max[\beta_P(v_i), \beta_P(v_j)]$ *and*
$0 \leq \alpha_Q(v_i, v_j) + \beta_Q(v_i, v_j) \leq 1$, *for every* $(v_i, v_j) \in E$, $(i, j = 1, 2, \cdots, n)$

Definition 2.3. [17] *A graph $Gr = (P, Q)$, where $\sigma = (\alpha_P, \beta_P)$ and $\mu = (\alpha_Q, \beta_Q)$ is said to be intuitionistic fuzzy labeling graph if $\alpha_P : V \to [0, 1]$, $\beta_P : V \to [0, 1]$, $\alpha_Q : V \times V \to [0, 1]$ and $\beta_Q : V \times V \to [0, 1]$ are bijective such that $\alpha_P(v_i)$, $\beta_P(v_i)$, $\alpha_Q(v_i, v_j)$, $\beta_Q(v_i, v_j) \in [0, 1]$ all are distinct for each node and edge, where α_P is the degree of membership and β_P is the degree of non-membership of nodes. Similarly, α_Q and β_Q are the degrees of membership and non-membership of edges.*

Definition 2.4. [31] *A neutrosophic graph on Z is a pair $Gr = (P, Q)$ with neutrosophic set P on Z and a neutrosophic relation Q on Z , where $P = \{a_1, a_2, \cdots, a_n\}$ such that*

(i) *α_P, β_P and γ_P defined from Z to $[0, 1]$ represent the membership degree of existence, uncertain and non-existence function of the vertex $a \in Z$ respectively with $0 \leq \alpha_P(a) + \beta_P(a) + \gamma_P(a) \leq 3$, for all $a \in Z(i = 1, 2, \cdots, n)$.*

(ii) *$Q \subseteq Z \times Z$ with α_Q, β_Q and γ_Q defined from $Z \times Z$ to $[0, 1]$ represent the membership degree of existence, uncertain and non-existence function of the edges $ab \in Z \times Z$ respectively such that*
$\alpha_Q(ab) \leq min[\alpha_P(a), \alpha_P(b)]$,
$\beta_Q(ab) \leq min[\beta_P(a), \beta_P(b)]$,
$\gamma_Q(ab) \leq max[\gamma_P(a), \gamma_P(b)]$ *and*
$0 \leq \alpha_Q(ab) + \beta_Q(ab) + \gamma_Q(ab) \leq 3$, *for every* ab.

Definition 2.5. [31] *A neutrosophic graph $Gr = (P, Q)$, where $\sigma = (\alpha_P, \beta_P, \gamma_P)$ and $\mu = (\alpha_Q, \beta_Q, \gamma_Q)$ is said to be an neutrosophic labeling graph, if $\alpha_P : V \to [0, 1]$, $\beta_P : V \to [0, 1]$ $\gamma_P : V \to [0, 1]$ and $\alpha_Q : V \times V \to [0, 1]$, $\beta_Q : V \times V \to [0, 1]$, $\gamma_Q : V \times V \to [0, 1]$ are bijective such that the truth, indeterminacy and false membership functions of the vertices and edges are distinct and*
$\alpha_Q(v_i, v_j) \leq min[\alpha_P(v_i), \alpha_P(v_j)]$,
$\beta_Q(v_i, v_j) \leq min[\beta_P(v_i), \beta_P(v_j)]$,
$\gamma_Q(v_i, v_j) \leq max[\gamma_P(v_i), \gamma_P(v_j)]$ *and* $0 \leq \alpha_Q(v_i, v_j) + \beta_Q(v_i, v_j) + \gamma_Q(v_i, v_j) \leq 3$,
for every edge (v_i, v_j).

3 An Outline of Super Intuitionistic Fuzzy 10^p- Based Graceful Labeling Graphs

Graceful labeling and its componential discussion is not yet widely flourished with intuitionistic fuzzy graphical approach. To create a strong foundation for various labeling concepts in intuitionistic fuzzy graph(In-FG), graceful labeling was taken initially and implemented with In-FG. The super behaviourial property is introduced to deal with intuitionistic fuzzy graceful labeling graphs that shows comparatively best outcome than the fuzzy approach. Here, an extended value of membership is attained to deal with intuitionistic case, which can be used to solve many real world problems.

Definition 3.1. *Let a and b denotes the vertex and edge count of an intuitionistic fuzzy graph $Gr = (P, Q)$. The graph Gr with a least positive integer p and $0 < a + b < 10^p$ is said to hold an intuitionistic fuzzy 10^p-based graceful labeling if $\alpha_P, \beta_P : V \to [0, 1]$, $\alpha_Q, \beta_Q : V \times V \to [0, 1]$ is one-to-one and onto such that all vertex and edge memberships are distinct and the following conditions are satisfied: (i) Each vertex and edge labeling has decimal number in [0,1], having decimal places p, (ii) $\mu(u, v) \leq \sigma(u) \cap \sigma(v)$ for all $u, v \in V$, (iii) $\mu(u, v) = |\sigma(u) - \sigma(v)|$ for all $u, v \in V$, (iv) The set of $\sigma(v) \times 10^p$ for all $v \in V$ The set of $b+1, b+2, \cdots, b+a$. (v) The set of $\mu(e) \times 10^p$ for all $e \in Gr$= The set of $1, 2, \cdots, b$. Therefore, the graph Gr is called a super intuitionistic fuzzy 10^p- based graceful.*

Theorem 3.2. *A path graph $P_n, n \geq 3$ with intuitionistic fuzzy labeling is said to satisfy super intuitionistic 10^p- based graceful condition.*

Proof. Let P_n be an intuitionistic fuzzy path graph, composed of $a = n$ vertices and $b = (n - 1)-$edges. The vertex membership $(\alpha_P(v_i), \beta_P(v_i))$ of P_n is assigned as follows:

(i) For $i = 0$ and i is even; $j \in N$,

$$\begin{aligned}\alpha_P(v_i) &= \frac{(a+b)-i}{10^j}\\ \beta_P(v_i) &= \frac{(a+b+2)-i}{10^j}\end{aligned}$$

(ii) For i is odd; $j \in N$,

$$\begin{aligned}\alpha_P(v_i) &= \frac{a+i}{10^j}\\ \beta_P(v_i) &= \frac{(a+1)+i}{10^j}\end{aligned}$$

The edge membership $(\alpha_Q(v_i, v_j), \beta_Q(v_i, v_j))$ is obtained through the expression, $(\alpha_Q(v_i, v_j), \beta_Q(v_i, v_j)) = (|\alpha_P(v_i) - \alpha_P(v_j)|, |\beta_P(v_i) - \beta_P(v_j)|)$. When the total sum of the obtained edge membership of intuitionistic fuzzy path graph exceeds the value 1, then each membership value of vertices should be minimized by 10^j, where $j \in \mathbb{N}$. Since the vertex and edge memberships of the path graph follows the conditions of intuitionistic fuzzy labeling and every edges are distinct by each membership, we say that every intuitionistic fuzzy path graph P_n holds the super intuitionistic 10^p- based graceful labeling. $\square$

Example 3.3. Figure 1 is an example for super intuitionistic 10^p- based graceful P_6 labeling graph, where the value of j is taken as 2.

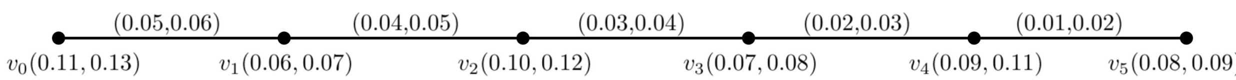


Figure 1: Super Intuitionistic Graceful 10^p- based P_6 Labeling Graph

Theorem 3.4. *A star graph S_n with intuitionistic fuzzy labeling admits super intuitionistic 10^p- based graceful condition.*

Proof. Consider an intuitionistic fuzzy star graph S_n with n vertices(one central vertex and remaining pendant vertices) and (n-1)-edges. The vertex membership $(\alpha_P(v_i), \beta_P(v_i))$ of P_n is assigned as follows:

(*i*) For central vertex v_1,

$$\begin{aligned} \alpha_P(v_1) &= \frac{a+b}{10^j} \\ \beta_P(v_1) &= \frac{(a+b)-n}{10^{j+1}} \end{aligned}$$

(*ii*) For pendant vertices v_i, where $i \in N-1$

$$\begin{aligned} \alpha_P(v_i) &= \alpha_P(v_{i-1}) - \frac{1}{10^j} \\ \beta_P(v_i) &= \beta_P(v_{i-1}) - \frac{1}{10^k} \end{aligned}$$

The edge membership $(\alpha_Q(v_i, v_j), \beta_Q(v_i, v_j))$ is obtained through the expression, $(\alpha_Q(v_i, v_j), \beta_Q(v_i, v_j)) = (|\alpha_P(v_i) - \alpha_P(v_j)|, |\beta_P(v_i) - \beta_P(v_j)|)$. Since the vertex and edge memberships of the star graph follows the conditions of intuitionistic fuzzy labeling and every edges are distinct by each membership, we say that every intuitionistic fuzzy star graph S_n holds the super intuitionistic 10^p- based graceful labeling. $\square$

Example 3.5. Figure 2 is an example for super intuitionistic 10^p- based graceful S_4 labeling graph, where the value of j is taken as 1.

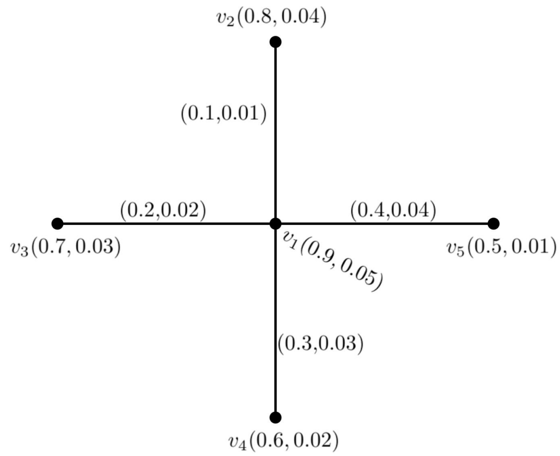


Figure 2: Super Intuitionistic Graceful 10^p- based S_4 Labeling Graph

4 Super Neutrosophic 10^p- Based Graceful Labeling Graphs

There are various types of labeling, that has a deep foundation and illustration with fuzzy graphical concept. But the establishment of these labelings to the improvised neutrosophic graphical concept is not yet done. Here, super behaviour of the neutrosophic graphs is pointed out to deal with the graceful labeling for the first time. It shows best result than the previous fuzzy works, since the memberships are segregated and extended specifically.

Definition 4.1. *A neutrosophic graph (NEG) $Gr = (P, Q)$, where $\sigma = (\alpha_P, \beta_P, \gamma_P)$ and $\mu = (\alpha_Q, \beta_Q, \gamma_Q)$ is said to satisfy graceful labeling if $\alpha_P, \beta_P, \gamma_P : V \to [0, 1]$, $\alpha_Q, \beta_Q, \gamma_Q : V \times V \to [0, 1]$ and all edges are distinct by each membership function when each edge label $(\alpha_Q(v_i, v_j), \beta_Q(v_i, v_j), \gamma_Q(v_i, v_j)) \in E(G)$ is obtained by the absolute difference between the adjacent vertices in an NEG (i.e.) $|(\alpha_P(v_i), \beta_P(v_i), \gamma_P(v_i)) - (\alpha_P(v_j), \beta_P(v_j), \gamma_P(v_j))|$, then the NEG is known as neutrosophic graceful labeling.*

Definition 4.2. *Let a and b denotes the vertex and edge count of a neutrosophic graph $Gr = (P, Q)$. The graph Gr with a least positive integer p and $0 < a + b < 10^p$ is said to hold a neutrosophic 10^p- based graceful labeling if $\alpha_P, \beta_P, \gamma_P : V \to [0, 1]$, $\alpha_Q, \beta_Q, \gamma_Q : V \times V \to [0, 1]$ is one-to-one and onto such that all vertex and edge memberships are distinct and the following conditions are satisfied: (i) Each vertex and edge labeling has decimal number in [0,1], having decimal places p, (ii) $\mu(u, v) \leq \sigma(u) \cap \sigma(v)$ for all $u, v \in V$, (iii) $\mu(u, v) = |\sigma(u) - \sigma(v)|$ for all $u, v \in V$, (iv) The set of $\sigma(v) \times 10^p$ for all $v \in V =$ The set of $b+1, b+2, \cdots, b+a$, (v) The set of $\mu(e) \times 10^p$ for all $e \in Gr =$ The set of $1, 2, \cdots, b$. Therefore, the graph Gr is called a super neutrosophic 10^p- based graceful.*

Theorem 4.3. *A comb graph $P_n \odot K_1$ admits super neutrosophic 10^p- based graceful labeling.*

Proof. Consider a neutrosophic comb graph $P_n \odot K_1$, which comprises of $a = 2n$ vertices and $b = (2n - 1)$ edges. The vertex membership $(\alpha_P(v_i), \beta_P(v_i), \gamma_P(v_i))$ of $P_n \odot K_1$ are labeled as follows:

(i) For v_{n+1} vertex,

$$\begin{aligned}
\alpha_P(v_{n+1}) &= \frac{a+b}{10^j} \\
\beta_P(v_{n+1}) &= \alpha_P(v_{n+1}) - \frac{1}{10^{j+1}} \\
\gamma_P(v_{n+1}) &= 1 - \alpha_P(v_{n+1}) - \beta_P(v_{n+1})
\end{aligned}$$

(ii) For v_1 vertex,

$$\begin{aligned}
\alpha_P(v_1) &= \alpha_P(v_{n+1}) - \frac{b}{10^j} \\
\beta_P(v_1) &= \alpha_P(v_1) - \frac{2}{10^{j+1}} \\
\gamma_P(v_1) &= 1 - \alpha_P(v_1) - \beta_P(v_1)
\end{aligned}$$

(iii) For v_2 vertex,

$$\begin{aligned}
\alpha_P(v_2) &= \alpha_P(v_{n+1}) - \frac{1}{10^j} \\
\beta_P(v_2) &= \alpha_P(v_2) - \frac{1}{10^{j+1}} \\
\gamma_P(v_2) &= 1 - \alpha_P(v_2) - \beta_P(v_2)
\end{aligned}$$

(iv) For odd vertices v_i ; $i = 3, 5, \cdots, n$ $(if\ n\ is\ odd)$ (or) $n-1$ $(if\ n\ is\ even)$

$$\begin{aligned}
\alpha_P(v_i) &= \alpha_P(v_{i-2}) + \frac{2}{10^j} \\
\beta_P(v_i) &= \alpha_P(v_i) - \frac{2}{10^{j+1}} \\
\gamma_P(v_i) &= 1 - \alpha_P(v_i) - \beta_P(v_i)
\end{aligned}$$

(v) For even vertices v_i, where $i = 4, 6, \cdots, (n-1)$ $(if\ n\ is\ odd)$ (or) $n-1$ $(if\ n\ is\ even)$

$$\begin{aligned}
\alpha_P(v_i) &= \alpha_P(v_{i-2}) - \frac{2}{10^j} \\
\beta_P(v_i) &= \alpha_P(v_i) - \frac{1}{10^{j+1}} \\
\gamma_P(v_i) &= 1 - \alpha_P(v_i) - \beta_P(v_i)
\end{aligned}$$

(vi) For v_{n+2} vertex,

$$\begin{aligned}
\alpha_P(v_{n+2}) &= \alpha_P(v_1) + \frac{1}{10^j} \\
\beta_P(v_{n+2}) &= \alpha_P(v_{n+2}) - \frac{1}{10^j} \\
\gamma_P(v_{n+2}) &= 1 - \alpha_P(v_{n+2}) - \beta_P(v_{n+2})
\end{aligned}$$

(vii) For tooth vertices,

(a) v_{n+i}, where $i = 3, 5, \cdots, n$ $(if\ n\ is\ odd)$ (or) $n-1$ $(if\ n\ is\ even)$

$$\begin{aligned}
\alpha_P(v_{n+i}) &= \alpha_P(v_{n+i-2}) - \frac{2}{10^j} \\
\beta_P(v_{n+i}) &= \alpha_P(v_{n+i}) - \frac{1}{10^{j+1}} \\
\gamma_P(v_{n+i}) &= 1 - \alpha_P(v_{n+i}) - \beta_P(v_{n+i})
\end{aligned}$$

(b) v_{n+i}, where $i = 4, 6, \cdots, (n-1)$ $(if\ n\ is\ odd)$ (or) $n-1$ $(if\ n\ is\ even)$

$$\begin{aligned} \alpha_P(v_{n+i}) &= \alpha_P(v_{n+i-2}) + \frac{2}{10^j} \\ \beta_P(v_{n+i}) &= \alpha_P(v_{n+i}) - \frac{2}{10^{j+1}} \\ \gamma_P(v_{n+i}) &= 1 - \alpha_P(v_{n+i}) - \beta_P(v_{n+i}) \end{aligned}$$

The edge membership $(\alpha_Q(v_i, v_j), \beta_Q(v_i, v_j), \gamma_Q(v_i, v_j))$ is obtained through the expression, $(\alpha_Q(v_i, v_j), \beta_Q(v_i, v_j), \gamma_Q(v_i, v_j))$=$(|\alpha_P(v_i) - \alpha_P(v_j)|, |\beta_P(v_i) - \beta_P(v_j)|, |\gamma_P(v_i) - \gamma_P(v_j)|)$. Since the vertex and edge memberships of the comb graph follow the conditions of neutrosophic labeling and every edges are distinct by each membership, we say that every neutrosophic comb graph $P_n \odot K_1$ holds the super neutrosophic 10^p- based graceful labeling. □

Example 4.4. Figure 3 is an example for super neutrosophic 10^p- based graceful $P_3 \odot K_1$ labeling graph, where the value of j is taken as 2.

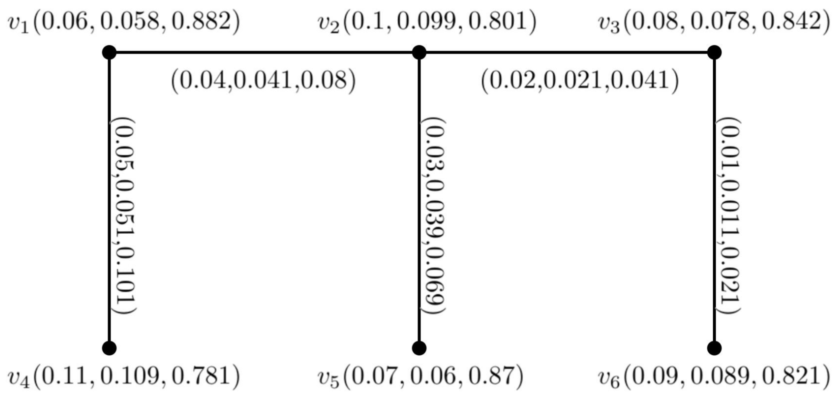


Figure 3: Super Neutrosophic 10^p- based Graceful $P_3 \odot K_1$ Labeling Graph

Theorem 4.5. *A caterpillar graph $P_n * nS_m$ admits super neutrosophic 10^p- based graceful labeling.*

Proof. Consider a neutrosophic caterpillar graph $P_n * nS_m$, which comprises of $a = n(m+1)$ vertices and $b = n(m+1) - 1$ edges. The vertex membership $(\alpha_P(v_i), \beta_P(v_i), \gamma_P(v_i))$ of $P_n * nS_m$ are labeled as follows:

(i) For starting vertex u_1,

$$\begin{aligned} \alpha_P(u_1) &= \frac{a+b}{10^j} \\ \beta_P(u_1) &= \alpha_P(u_1) - \frac{1}{10^{j+1}} \\ \gamma_P(u_1) &= 1 - \alpha_P(u_1) - \beta_P(u_1) \end{aligned}$$

(ii) For v_{i+1} vertices,

$$\begin{aligned} \alpha_P(v_{i+1}) &= \alpha_P(u_1) - \frac{b-i}{10^j} \\ \beta_P(v_{i+1}) &= \alpha_P(v_{i+1}) - \frac{2}{10^{j+1}} \\ \gamma_P(v_{i+1}) &= 1 - \alpha_P(v_{i+1}) - \beta_P(v_{i+1}) \end{aligned}$$

(iii) For u_2 vertex,

$$\begin{aligned}
\alpha_P(u_2) &= \alpha_P(v_m) + \frac{1}{10^j} \\
\beta_P(u_2) &= \alpha_P(u_2) - \frac{1}{10^{j+1}} \\
\gamma_P(u_2) &= 1 - \alpha_P(u_2) - \beta_P(u_2)
\end{aligned}$$

(iv) For odd vertices u_i ; $i = 3, 5, \cdots, n$ $(if\ n\ is\ odd)$ (or) $n-1$ $(if\ n\ is\ even)$

$$\begin{aligned}
\alpha_P(u_i) &= \alpha_P(u_{i-2}) - \frac{m+1}{10^j} \\
\beta_P(u_i) &= \alpha_P(u_i) - \frac{1}{10^{j+1}} \\
\gamma_P(u_i) &= 1 - \alpha_P(u_i) - \beta_P(u_i)
\end{aligned}$$

(v) For even vertices u_i, where $i = 4, 6, \cdots, (n-1)$ $(if\ n\ is\ odd)$ (or) $n-1$ $(if\ n\ is\ even)$

$$\begin{aligned}
\alpha_P(u_i) &= \alpha_P(u_{i-2}) - \frac{m+1}{10^j} \\
\beta_P(u_i) &= \alpha_P(u_i) - \frac{1}{10^{j+1}} \\
\gamma_P(u_i) &= 1 - \alpha_P(u_i) - \beta_P(u_i)
\end{aligned}$$

(vi) For vertices in the form $v_{(k-1)m+i}$,

(a) For $k = 3, 5, \cdots, n$ $(if\ n\ is\ odd)$ (or) $n-1$ $(if\ n\ is\ even)$ and $i = 1, 2, \cdots, m$

$$\begin{aligned}
\alpha_P(v_{(k-1)m+i}) &= \alpha_P(u_{k-1}) + \frac{i}{10^j} \\
\beta_P(v_{(k-1)m+i}) &= \alpha_P(v_{(k-1)m+i}) - \frac{2}{10^{j+1}} \\
\gamma_P(v_{(k-1)m+i}) &= 1 - \alpha_P(v_{(k-1)m+i}) - \beta_P(v_{(k-1)m+i})
\end{aligned}$$

(b) For $k = 2, 4, \cdots, (n-1)$ $(if\ n\ is\ odd)$ (or) $n-1$ $(if\ n\ is\ even)$ and $i = 1, 2, \cdots, m$

$$\begin{aligned}
\alpha_P(v_{(k-1)m+i}) &= \alpha_P(u_{k-1}) - \frac{i}{10^j} \\
\beta_P(v_{(k-1)m+i}) &= \alpha_P(v_{(k-1)m+i}) - \frac{2}{10^{j+1}} \\
\gamma_P(v_{(k-1)m+i}) &= 1 - \alpha_P(v_{(k-1)m+i}) - \beta_P(v_{(k-1)m+i})
\end{aligned}$$

The edge membership $(\alpha_Q(v_i, v_j), \beta_Q(v_i, v_j), \gamma_Q(v_i, v_j))$ is obtained through the expression, $(\alpha_Q(v_i, v_j), \beta_Q(v_i, v_j), \gamma_Q(v_i, v_j))$=$(|\alpha_P(v_i) - \alpha_P(v_j)|, |\beta_P(v_i) - \beta_P(v_j)|, |\gamma_P(v_i) - \gamma_P(v_j)|)$. Since the vertex and edge memberships of the caterpillar graph follows the conditions of neutrosophic labeling and every edges are distinct by each membership, we say that every neutrosophic caterpillar graph $P_n * nS_m$ holds the super neutrosophic 10^p- based graceful labeling. □

Example 4.6. Figure 4 is an example for super neutrosophic 10^p- based graceful $P_3 * 3S_3$ labeling graph, where the value of j is taken as 2.

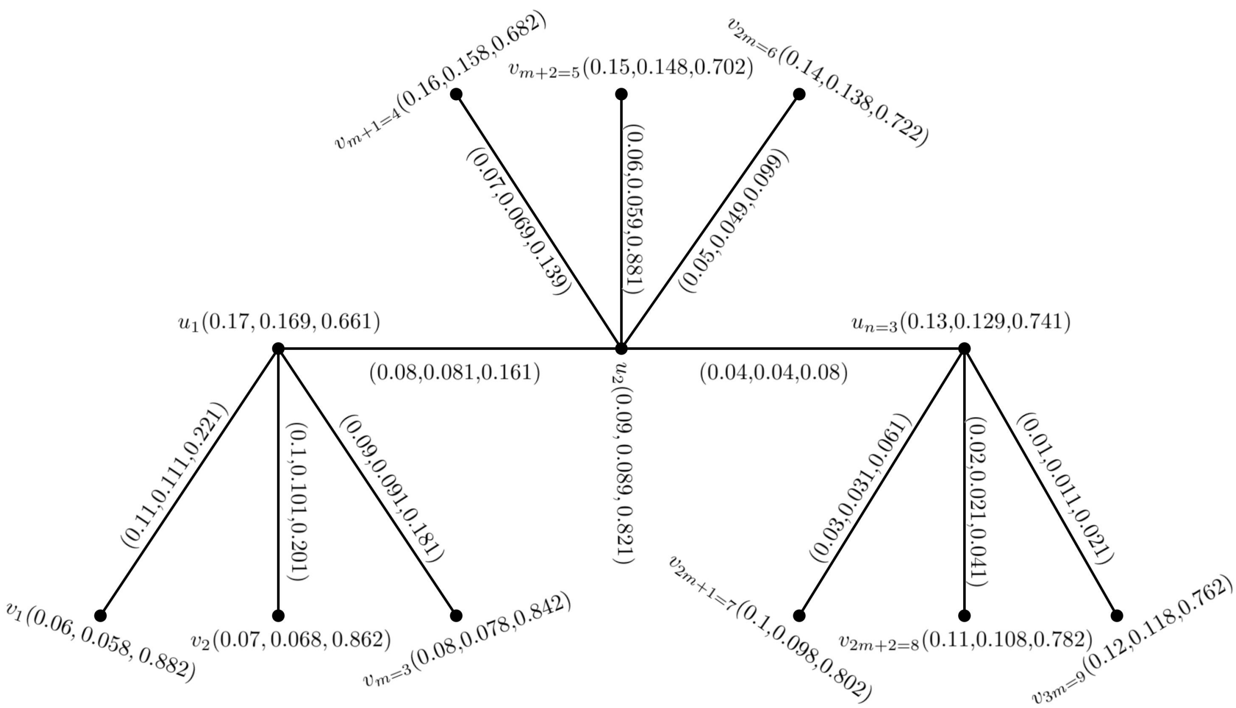


Figure 4: Super Neutrosophic 10^p- based Graceful $P_3 * 3S_3$ Labeling Graph

Theorem 4.7. *A broom graph $P_n * S_m$; $n, m \geq 2$ admits super neutrosophic 10^p- based graceful labeling.*

Proof. Consider a neutrosophic broom graph $P_n * S_m$, which comprises of $a = (n + m)$ vertices and $b = (n + m - 1)$ edges. The vertex membership $(\alpha_P(v_i), \beta_P(v_i), \gamma_P(v_i))$ of $P_n * S_m$ are labeled as follows:

(i) For starting vertex v_1,

$$\begin{aligned}
\alpha_P(v_1) &= \frac{a+b}{10^j} \\
\beta_P(v_1) &= \alpha_P(v_1) - \frac{1}{10^{j+1}} \\
\gamma_P(v_1) &= 1 - \alpha_P(v_1) - \beta_P(v_1)
\end{aligned}$$

(ii) For v_2 vertex,

$$\begin{aligned}
\alpha_P(v_2) &= \alpha_P(v_1) - \frac{b}{10^j} \\
\beta_P(v_2) &= \alpha_P(v_2) - \frac{2}{10^{j+1}} \\
\gamma_P(v_2) &= 1 - \alpha_P(v_2) - \beta_P(v_2)
\end{aligned}$$

(iii) For vertices v_i,

(a) When $i = 3, 5, \cdots, n$ $(if\ n\ is\ odd)$ (or) $n-1$ $(if\ n\ is\ even)$

$$\begin{aligned}\alpha_P(v_i) &= \alpha_P(v_{i-2}) - \frac{1}{10^j}\\ \beta_P(v_i) &= \alpha_P(v_i) - \frac{1}{10^{j+1}}\\ \gamma_P(v_i) &= 1 - \alpha_P(v_i) - \beta_P(v_i)\end{aligned}$$

(b) When $i = 4, 6, \cdots, (n-1)$ $(if\ n\ is\ odd)$ (or) $n-1$ $(if\ n\ is\ even)$

$$\begin{aligned}\alpha_P(v_i) &= \alpha_P(v_{i-2}) + \frac{1}{10^j}\\ \beta_P(v_i) &= \alpha_P(v_i) - \frac{2}{10^{j+1}}\\ \gamma_P(v_i) &= 1 - \alpha_P(v_i) - \beta_P(v_i)\end{aligned}$$

(iv) For vertices of S_m, in the form v_{n+i},

(a) When n is even,

$$\begin{aligned}\alpha_P(v_{n+i}) &= \alpha_P(v_{n-1}) - \frac{i}{10^j}\\ \beta_P(v_{n+i}) &= \alpha_P(v_{n+i}) - \frac{i}{10^{j+1}}\\ \gamma_P(v_{n+i}) &= 1 - \alpha_P(v_{n+i}) - \beta_P(v_{n+i})\end{aligned}$$

(b) When n is odd,

$$\begin{aligned}\alpha_P(v_{n+i}) &= \alpha_P(v_{n-1}) + \frac{i}{10^j}\\ \beta_P(v_{n+i}) &= \alpha_P(v_{n+i}) + \frac{i}{10^{j+1}}\\ \gamma_P(v_{n+i}) &= 1 - \alpha_P(v_{n+i}) - \beta_P(v_{n+i})\end{aligned}$$

The edge membership $(\alpha_Q(v_i, v_j), \beta_Q(v_i, v_j), \gamma_Q(v_i, v_j))$ is obtained through the expression, $(\alpha_Q(v_i, v_j), \beta_Q(v_i, v_j), \gamma_Q(v_i, v_j))$=$(|\alpha_P(v_i) - \alpha_P(v_j)|, |\beta_P(v_i) - \beta_P(v_j)|, |\gamma_P(v_i) - \gamma_P(v_j)|)$. Since the vertex and edge memberships of the broom graph follow the conditions of neutrosophic labeling and every edges are distinct by each membership, we say that every neutrosophic broom graph $P_n * S_m$ holds the super neutrosophic 10^p- based graceful labeling. □

Example 4.8. Figure 5 is an example for super neutrosophic 10^p- based graceful $P_4 * S_3$ labeling graph, where the value of j is taken as 2.

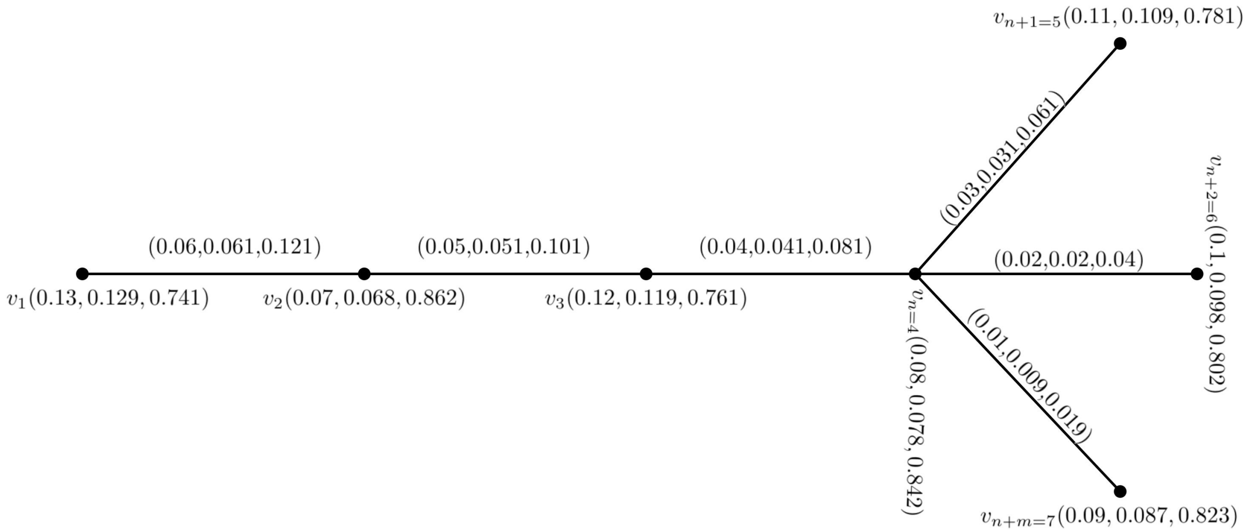


Figure 5: Super Neutrosophic 10^p- based Graceful $P_4 * S_3$ Labeling Graph

5 Application on Financial Risk Analysis of a Company

Financial risk analysis is an implementation process in business or by individuals to explore and adjust the potential risk. Some important risks like market risk, credit risk, liquidity risk and operational risk, etc happens almost in all situations. It is significant to safeguard the finance and economy of the customers and business people by identifying, assessing and monitoring for the possible risk causing factors & its impact. This scenario is structured with the above proposed concept in a neutrosophic graphical background. Since the neutrosophic graceful labeing is carried over with super and 10^p characteristics, it is widely applied for complex structures. To apply this in real time, consider a star graph structure S_3 with neutrosophic components. Assume the central vertex of S_3 as the core company and the other vertices as the branches of the core company. The edges between the central vertex and the other vertices represents the company relationship with the branches. The truth, indeterminacy and false membership of vertices are taken as surity in financial data & metrics, unpredicted shocks like market crash, and incomplete information/manipulated data respectively. Likewise, the three edge memberships are notified as financial tie to be strengthened, uncertainty due to market changes, and incorrect reporting respectively. The rise in surity of data between companies results in the low value of weak financial tie. This customised structure helps the company to have a good communication and overall financial balance. This model is used to expose the best branch of a company by analyzing the final results obtained by the core company from its branches. Also, the core company can make certain decisions to improvise the functioning of other branches. This application is extensible and can be applied for any n value of S_n structure.

Figure 6 portrays the relationship between core company and its branches. Here, the performance of the company and its branches can be clearly understood by analysing the membership value of each vertices. Through the edge membership values, one can identify the improvement needed branch. Among all branches, the branch 1 performs well after the core company with 89% surity in their data and metrics. It shows that the branch 1 tackles the unpredicted shocks and manipulated data to be lower than the other branches. Also, the low performing branch can be identified in default by checking the higher truth value of edges. Since the branch 3 has got higher score upto 33%, it should rectify the issues and to strengthen their relationship with the core company. Figure 7 demonstrates the best branch of a company by analyzing the performance of each branch with their corresponding membership values of the vertices (branches) in an individual manner.

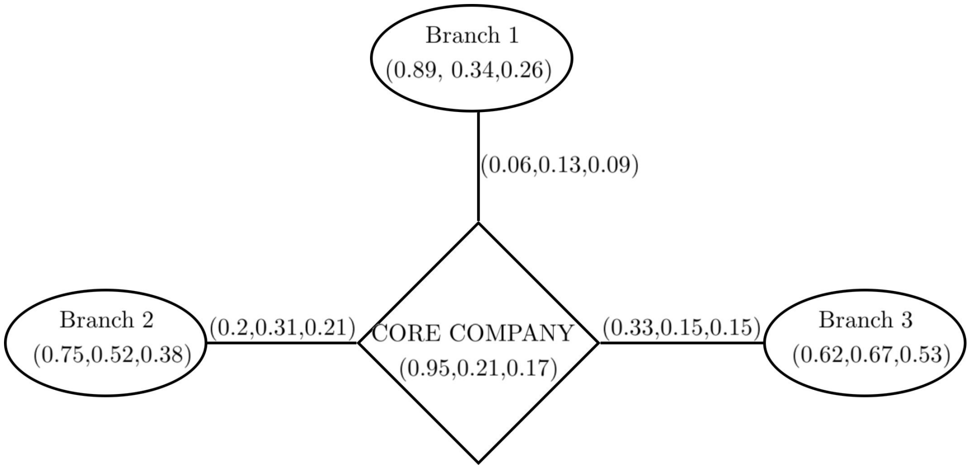


Figure 6: Company-Branch Relationship

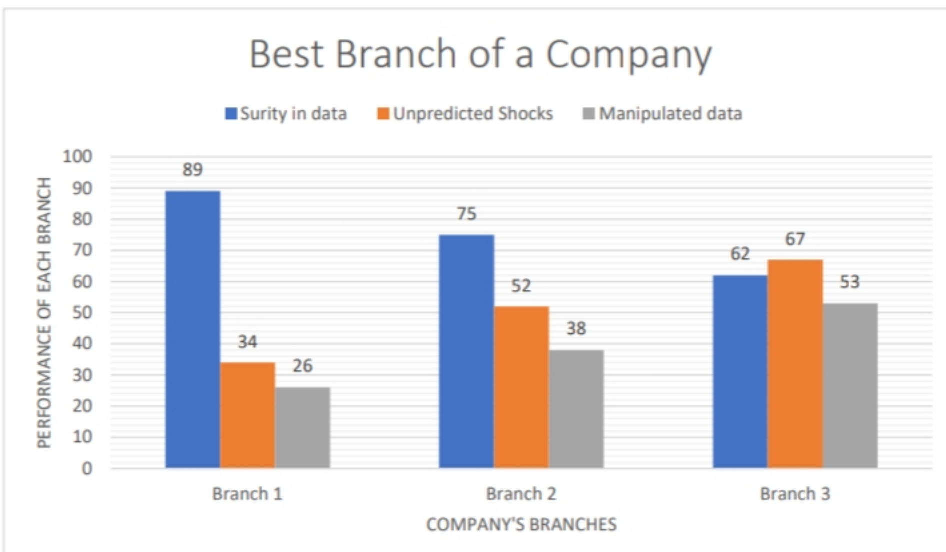


Figure 7: Branch performance

This simulation clearly says that branch 1 is the best performing branch among all branches.

Algorithm:

A clear step-by-step method is given below to understand the significance of super neutrosophic 10^p- based graceful labeling. It is the very flexible to apply for complex structures, when compared to the previous fuzzy models.

Step 1: The vertex membership is assigned separately for starting vertex, pendant vertex, central vertex, tooth vertex etc. using 10^p in the neutrosophic graph considered.

Step 2: Estimate the edge membership value of the graph using the expression

$|(\alpha_P(v_i), \beta_P(v_i), \gamma_P(a_i)) - (\alpha_P(v_j), \beta_P(v_j), \gamma_P(v_j))|$, where v_i and v_j denotes the vertices that are adjacent.

Step 3: Check the edge values attained between the adjacent vertices of a graph.

Step 4: The final result is obtained by recognising the edge membership values and its appropriate vertex memberships.

6 Conclusion

A graceful labeling based on 10^p structure is achieved with the intuitionistic and neutrosophic graphs with super behaviour. This labeling structure overcomes the uncertain issues and lack in flexibility that we faced in the classical structures. Also, a methodology and application for this labeling concept is portrayed. In future, we have planned to implement other labeling types such as harmonic labeling, skolem labeling etc. in the neutrosophic environment and to generalize the labeling structure using 10^p and n^p in each membership.

Acknowledgments: "We thank the reviewers and editor for their insightful comments to improve our manuscript."

Conflict of Interest: "The authors declare no conflict of interest."

References

[1] Zadeh LA. Fuzzy sets. *Information and control*. 1965; 8(3): 338-353. DOI: https://doi.org/10.1016/S0019-9958(65)90241-X

[2] Kaufmann A. *Introduction to the theory of fuzzy subsets.* New York: Academic Press Publishing; 1975.

[3] Rosenfeld A. Fuzzy graphs. In: Lotfi A. Zadeh, King-Sun Fu, Kokichi Tanaka, Masamichi Shimura. (eds.) *Fuzzy Sets and their Applications to Cognitive and Decision Processes: Proceedings of the USJapan Seminar on Fuzzy Sets and their Applications, Held at the University of California, Berkeley, California, July 14, 1974.* Amsterdam, The Netherlands: Elsevier; 1975. p.77-95. DOI: https://doi.org/10.1016/B978-0-12-775260-0.50008-6

[4] Bhattacharya P. Some remarks on fuzzy graphs. *Pattern Recognition Letters*. 1987; 6(5): 297-302. DOI: https://doi.org/10.1016/0167-8655(87)90012-2

[5] Bhuttani KR, Battou A. On M-strong fuzzy graphs. *Information Sciences*. 2003; (1-2): 103-109. DOI: https://doi.org/10.1016/S0020-0255(03)00157-9

[6] Nagoor Gani A, Rajalaxmi (a) Subahashini D. Properties of fuzzy labeling graph. *Applied Mathematical Sciences*. 2012; 6(70): 3461-3466. https://m-hikari.com/ams/ams-2012/ams-69-72-2012/nagoorganiAMS69-72-2012-1.pdf

[7] Jahir Hussain R, Karthikkeyan RM. Fuzzy Graceful Graphs. *Advances in Fuzzy Mathematics*. 2017; 12(2): 309-317.

[8] Nagoor Gani A, Fathima Kani B, Afya Farhana MS. Fuzzy Edge Graceful Labeling On Wheel Graph, Fan Graph And Friendship Graph. *American International Journal of Research in Science, Technology, Engineering & Mathematics, Special Issue of* 5^{th} *International Conference on Mathematical Methods and Computation (ICOMAC)*. 2019; 324-329.

[9] Jebesty Shajila R, Vimala S. Fuzzy vertex graceful labeling on wheel and fan graph. *IOSR Journal of Mathematics*. 2016; 12(2):45-49. DOI: http://doi.org/10.9790/5728-12214549

[10] Jebesty Shajila R, Vimala S. Graceful labeling for complete bipartite fuzzy graphs. *British Journal of Mathematics & Computer Science*. 2017; 22(2): 1-9. DOI: http://doi.org/10.9734/BJMCS/2017/32242

[11] Vimala S, Jebesty Shajila R. A note on fuzzy edge-vertex graceful labeling of Star graph and Helm graph. *Advances in Theoretical and Applied Mathematics*. 2016; 11(4): 331-338. DOI: https://doi.org/10.26483/ijarcs.v8i6.4327

[12] Solairaju A, Narppasalai Arasu T. Super fuzzy 10^k-based graceful labeling of some classes of fuzzy trees related to path and star. *Journal of Computer and Mathematical Sciences*. 2018; 9(12): 2169-2177. DOI: https://doi.org/10.12988/pms.2020.91220

[13] Atanassov KT. Intuitionistic fuzzy sets. *Fuzzy Sets and Systems*. 1986; 20(1): 87-96. DOI: https://doi.org/10.1016/S0165-0114(86)80034-3

[14] Parvathi R, Karunambigai MG. Intuitionistic Fuzzy Graphs. In: Reusch B. (eds.) *Computational Intelligence, Theory and Applications.* Berlin, Heidelberg: Springer; 2006.

[15] Akram M, Akmal R. Intuitionistic Fuzzy Graph Structures. *Kragujevac Journal of Mathematics*. 2017; 41(2): 219-237. DOI: http://doi.org/10.1016/j.fiae.2017.01.001

[16] Nagoor Gani A, Shajitha Begum S. Degree, order and size in intuitionistic fuzzy graphs. *International Journal of Algorithms, Computing and Mathematics*. 2010; 3(3): 11-16.

[17] Sahoo S, Pal M. Intuitionistic fuzzy labeling graphs. *TWMS Journal of Applied and Engineering Mathematics*. 2018; 8(2): 466-476.

[18] Rabeeh Ahamed T, Ismail Mohideen S, Prasanna A. Triangular intuitionistic fuzzy graceful labeling in some path related graphs. *Advances and Applications in Mathematical Sciences*. 2021; 20(6): 1077-1084.

[19] Devie Abirami P, Pradeep A, Akhilesh M, Santhosh Kumar S. Picture fuzzy graceful labeling for some graphs. *Journal of Computational Analysis and Applications*. 2024; 33(05): 541-548.

[20] Smarandache F. Neutrosophic set- a generalization of the intuitionistic fuzzy set. *2006 IEEE International Conference on Granular Computing*, Atlanta, GA, USA: Springer; 2006, p. 38-42. DOI: http://doi.org/10.1109/GRC.2006.1635754

[21] Smarandache F. A geometric interpretation of the neutrosophic set- A generalization of the intuitionistic fuzzy set. *2011 IEEE International Conference on Granular Computing*, Kaohsiung, Taiwan: Springer; 2011, p. 602-606. DOI: http://doi.org/ 10.1109/GRC.2011.6122665

[22] Vasantha Kandasamy WB, Ilanthenral K, Smarandache F. (eds.) *Neutrosophic graphs: A New Dimension to Graph Theory.* Belgium: EuropaNova ASBL, Brussels; 2015.

[23] Broumi S, Smarandache F, Talea M, Bakali A. Single valued neutrosophic graphs: Degree, order and size. *IEEE International Conference on Fuzzy Systems(FUZZ-IEEE)*, Vancouver, BC, Canada, 2444-2451, 2016. DOI: http://doi.org/10.1109/FUZZ-IEEE.2016.7738000

[24] Gomathi M, Keerthika V. Neutrosophic Labeling Graph. *Neutrosophic Sets and Systems*. 2019; 30(1): 261-272. DOI: http://doi.org/ 10.5281/zenodo.3569706

[25] Ajay D, Chellamani P. Fuzzy magic labelling of neutrosophic path and star graph. *Advances in Mathematics: Scientific Journal*. 2020; 9(8): 6059-6070. DOI: http://doi.org/10.37418/amsj.9.8.74

[26] Sudha S, Lathamaheswari M, Broumi S, Smarandache F. Neutrosophic fuzzy magic labeling graph with its application in academic performance of the students. *Neutrosophic Sets and Systems*. 2023; 60: 88-112. DOI: http://doi.org/ 10.5281/zenodo.10224136

[27] Krishnaraj V, Vikramaprasad R. Some results on single valued neutrosophic bi-magic graphs. *International Journal of Mechanical Engineering and Technology*. 2018; 9(2):398-408.

[28] Logasoundarya ST, Vimala S. To analyse energy of the electric circuit by using neutrosophic labelling. *Indian Journal of Science and Technology*. 2024; 17(39):40304037. DOI: https://doi.org/10.17485/IJST/v17i39.2240

[29] Ajay D, Joseline Charisma J, Petaratip T, Hammachukiattikul P, Boonsatit N, Younis Jihad A, Rajchakit G. Pythagorean nanogeneralized closed sets with application in decision-making. *Journal of Function Spaces*. 2021; 2571301, 8 pages. DOI: https://doi.org/10.1155/2021/2571301

[30] Ajay D, Aldring J, Rajchakit G, Hammachukiattikul P, Boonsatit N. Sine trigonometry operational laws for complex neutrosophic sets and their aggregation operators in material selection. *Computer Modeling in Engineering and Sciences*. 2022; 130(2): 10331076. DOI: https://doi.org/10.32604/cmes.2022.018267

[31] Ajay D, Chellamani P. Pythagorean neutrosophic fuzzy graphs. *International Journal of Neutrosophic Science*. 2020; 11(2): 108-114. DOI: https://doi.org/10.54216/IJNS.0110205

[32] Ajay D, Karthiga S, Chellamani P. A study on labelling of pythagorean neutrosophic fuzzy graphs. *Journal of Computational Mathematica*. 2021; 5(1): 105-116. DOI: https://doi.org/10.26524/cm97

[33] Ajay D, Chellamani P, Rajchakit G, Boonsatit N, Hammachukiattikul P. Regularity of pythagorean neutrosophic graphs with an illustration in MCDM. *AIMS Mathematics*. 2022; 7(5): 94249442. DOI: https://doi.org/10.3934/math.2022523

[34] Vetrivel G, Mullai M, Rajchakit G. Product operations on pythagorean co-neutrosophic graphs and its application. *Neutrosophic Sets and Systems*. 2024; 72: 357-380. DOI: http://doi.org/10.5281/zenodo.13566972

[35] Li D-P, Cheng S-J, Cheng P-F, Wang J-Q, Zhang H-Y. A novel financial risk assessment model for companies based on heterogeneous information and aggregated historical data. *PLoS ONE*. 2018; 13(12): e0208166. DOI: https://doi.org/10.1371/journal.pone.0208166

[36] Shalihah M, Situmorang Tumpal P, Yusnita, Ridwan, Kalsum Ummy, Wahyudiyono. Implementation of financial risk management in improving the company's financial stability. *Maneggio*. 2024; 1(6): 1-13. DOI: http://doi.org/10.62872/wcsn8237

Govindan Vetrivel
Department of Mathematics
Alagappa University, Karaikudi
Tamilnadu, India
E-mail: menakagovindan@gmail.com

Murugappan Mullai
Department of Mathematics
Alagappa University, Karaikudi
Tamilnadu, India &
Research Fellow, INTI International University
Nilai Campus, Malaysia
E-mail: mullaim@alagappauniversity.ac.in