Finding the Most Efficient Unit in Data Envelopment Analysis

In data envelopment analysis, identifying the most efficient decision-making unit (DMU) is crucial for gaining insights into efficient DMUs. Various approaches have been suggested in the literature to determine the most efficient DMU in data envelopment analysis. These approaches aim to develop a model with enhanced discriminatory ability among DMUs. This study introduces a new model based on a common set of weights approach using mixed integer linear programming to select the most efficient DMU. The proposed model ensures that the efficiency score of only one DMU (the most efficient) is strictly greater than one, while the efficiency scores of other DMUs are less than or equal to one. This model demonstrates a strong discriminatory capability, enabling the full ranking of all DMUs with fewer constraints than models that allow complete ranking. To validate the proposed model and compare its performance with recent approaches, two numerical examples from the literature are utilized.

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Introduction

Charnes, Cooper [1] introduced data envelopment analysis (DEA), a mathematical approach to assess the relative efficiency of a homogeneous group of DMUs. DEA categorizes DMUs into efficient and inefficient groups. While it's not possible to rank efficient units solely based on their efficiency score of one, so several methods in the DEA literature have been explored for this purpose. These methods offer varied perspectives for ranking efficient units. Notable methods include cross-efficiency ranking methods [2-12], super efficiency ranking methods[11], the common set of weights (CWS) methods[13-21], benchmark ranking methods[22], the linear discriminant analysis[23], discriminant analysis of ratios[24-25].

In some instances, the decision-maker has to choose just one DMU from a group of efficient DMUs, known as the most efficient DMU. This has led to various studies in DEA aiming to identify the most efficient unit. Karsak and Ahiska [12] introduced an integrated multi-criteria decision-making (MCDM) DEA model to evaluate the most efficient DMU in Advanced Manufacturing Technology (AMT). Amin, Toloo [26] developed an enhanced MCDM model to address convergence issues raised in [12]. Amin and Toloo [27] proposed an integrated DEA model for finding the most CCR-efficient. Toloo and Nalchigar [28] extended the model in [27] for selecting the most BCC-efficient DMU by solving only one linear programming.

Later, a new mixed integer nonlinear programming (MINLP) model was introduced by Amin [29] to overcome the drawback of determining more than one most efficient DMU by the model of Amin and Toloo [27]. While the model in [29] can identify the most optimal unit, it is nonlinear and consequently challenging to solve. Toloo, Sohrabi [30] researched data mining, where they discovered that determining the most pertinent association rule by taking various factors into account is a pivotal undertaking. They developed an algorithm for giving priority to association rules, albeit with certain shortcomings. This was subsequently enhanced by Toloo and Nalchigar [28] to address some of its limitations.

Foroughi [31] proposed a new mixed integer linear programming (MILP) model to identify the most efficient DMU by maximizing the minimum possible distance between a chosen DMU and the next highest-ranked DMU. By removing additional constraints in Foroughi's model, Wang and Jiang [32] proposed a new model to identify the most efficient DMU, which is less complex than Foroughi's model.

Toloo [33] proposed a new MILP model to find the most efficient DMU without explicit input. Another model by Toloo [34] removes non-Archimedean epsilon, reducing computations needed, to identify the most efficient DMU while emphasizing epsilon selection.

Toloo [35] emphasized the crucial challenge of selecting and ranking suppliers accurately in the supply chain with imprecise data. Toloo [36] proposed a novel minimax MILP model that employs the CSW method to choose the most efficient DMU.

Lam [37] proposed a MILP model similar to the super-efficiency model, aiming to directly identify the most efficient DMU. Salahi and Toloo [38] illustrated that Lam’s model may be infeasible, and they proposed a modified model to cope with this issue. Toloo [39] proposed a method for finding the most cost-efficient DMU by utilizing the proposed approach in [40] when the prices are fixed and known. Toloo and Salahi [41] developed a new two-step MINP model utilizing epsilon to identify a single efficient DMU with an efficiency score exceeding one. Both non-linear models of Toloo and Salahi [41] can be turned into linear models. Based on the proposed model in [41], Özsoy, Örkcü [42] presented a MINP without epsilon, streamlining the process to select the most efficient DMU. This model singles out one DMU as the most efficient with fewer constraints compared to [41].

Given the limitations of the existing proposed models, like non-linear nature, incomplete ranking, and two-stage process, this study presents a new MILP model, which selects the most efficient unit in a single step. The proposed model assigns an efficiency score greater than one for the most efficient DMU, while other DMUs have efficiency scores that are strictly less than or equal to one. This model has several computational advantages such as high discriminative power, fewer constraints, and greater simplicity compared to similar models. Furthermore, the model is compared with several recent models on two examples from the literature, demonstrating its high discrimination power and potential application in various real-world scenarios such as facility layout design in manufacturing systems and the banking industry.

The following is an outline of the paper: Section 2 provides a brief overview of current models used to identify the most efficient DMU. In Section 3, an alternative MILP model is proposed to determine the most efficient DMU. Furthermore, Section 4 demonstrates the potential applications of the proposed MILP model through two numerical examples and its effectiveness in identifying the most efficient DMU. Finally, the paper concludes in Section 5.

2. Preliminaries

Throughout this paper, we assume that n homogeneous DMUs, $DMU_j \, (j = 1, 2, ..., n)$, which consume m various inputs, $x_{ij} \, (i = 1, 2, ..., m)$, to produce s different outputs, $y_{rj} \, (r = 1, 2, ..., s)$. Let $v_i \, (i = 1, 2, ..., m)$ and $u_r \, (r = 1, 2, ..., s)$ be the weights of ith input and rth output, respectively. The efficiency score of $DMU_j \, (j = 1, 2, ..., n)$ can be calculated mathematically as[43]:

$$e_j = \frac{\sum_{r=1}^{s} u_r y_{rj}}{\sum_{i=1}^{m} v_i x_{ij}}, \ j = 1,2,...,n$$

Sueyoshi [22] proposed the following DEA model under constant return to scale (CRS) for estimating the best relative efficiency score of the DMU under evaluation, DMU_p :

$$\begin{aligned}
e_p^* = Max \ & \sum_{r=1}^{s} y_{rp} \\
s.t. \ & \sum_{r=1}^{s} u_r y_{rj} - \sum_{i=1}^{m} v_i x_{ij} \le 0, \quad j = 1,2,...,n \\
& \sum_{i=1}^{m} v_i x_{ip} = 1 \\
& u_r \ge \frac{1}{(m+s)\max_j\{y_{rj}\}}, \quad r = 1,2,...,s \\
& v_i \ge \frac{1}{(m+s)\max_j\{x_{ij}\}}, \quad i = 1,2,...,m
\end{aligned} \tag{1}$$

Let $v_i^*(i = 1,2,...,m)$ and $u_r^*(r = 1,2,...,s)$ be the optimal weights of *ith* input and *rth* output in model (1), respectively. If the DMU_p is CCR-efficient ($e_p^* = 1$), then $\sum_{r=1}^{s} u_r^* y_{rp} - \sum_{i=1}^{m} v_i^* x_{ip} = 0$; otherwise there exists at least one other index $j \in \{1,2,...,n\}$ such that $\sum_{r=1}^{s} u_r^* y_{rj} - \sum_{i=1}^{m} v_i^* x_{ij} = 0$.

Definition 1. If there is a common set of optimal weights, $\left(\mathbf{u}_r^*(r = 1,2,...,s), \mathbf{v}_i^*(i = 1,2,...,m)\right) > 0$, such that $\sum_{r=1}^{s} u_r^* y_{rp} - \sum_{i=1}^{m} v_i^* x_{ip} = 0$ and moreover $\sum_{r=1}^{s} u_r^* y_{rj} - \sum_{i=1}^{m} v_i^* x_{ij} < 0, j \ne p$, then DMU_p is called the most (best) efficient unit[36].

In the following, we will review some well-known existing models in the literature for finding the most efficient DMU.Wang and Jiang [32] proposed the following MILP model for finding the most CCR-efficient DMU under CRS.

$$
\begin{aligned}
& Min \sum_{i=1}^{m} v_i \left(\sum_{j=1}^{n} x_{ij} \right) - \sum_{r=1}^{s} u_r \left(\sum_{j=1}^{n} y_{rj} \right) \\
& s.t. \quad \sum_{r=1}^{s} u_r y_{rj} - \sum_{i=1}^{m} v_i x_{ij} \le \delta_j, \quad j = 1, 2, \ldots, n \\
& \qquad \sum_{j=1}^{n} \delta_j = 1, \qquad\qquad\qquad\qquad\qquad\qquad (2) \\
& \qquad u_r \ge l_r^u, \qquad\qquad r = 1, 2, \ldots, s \\
& \qquad v_i \ge l_i^v, \qquad\qquad i = 1, 2, \ldots, m \\
& \qquad \delta_j \in \{0, 1\}, \qquad\quad j = 1, 2, \ldots., n
\end{aligned}
$$

Where $l_r^u = ((m+s)\max_j\{y_{rj}\})^{-1}$ and $l_i^v = ((m+s)\max_j\{x_{ij}\})^{-1}$ lower bounds borrowed from model (1). Model (2) is feasible and aims to maximize the overall efficiency of all of the DMUs. In this model, DMU_p is determined as the most efficient DMU if and only if $\delta_p^* = 1$.

Toloo [36] proposed the following minimax model for finding the most efficient DMU under CRS.

$$
\begin{aligned}
& Min \quad d_{\max} \\
& s.t. \quad d_{\max} - d_j + \beta_j \ge 0, \qquad\qquad j = 1, 2, \ldots, n \\
& \qquad \sum_{r=1}^{s} u_r y_{rj} - \sum_{i=1}^{m} v_i x_{ij} + d_j - \beta_j = 0, \quad j = 1, 2, \ldots, n \\
& \qquad \sum_{j=1}^{n} d_j = n - 1, \qquad\qquad\qquad\qquad\qquad (3) \\
& \qquad d_j \in \{0, 1\}, \qquad\qquad j = 1, 2, \ldots, n \\
& \qquad u_r \ge l_r^u, \qquad\qquad r = 1, 2, \ldots, s \\
& \qquad v_i \ge l_i^v, \qquad\qquad i = 1, 2, \ldots, m \\
& \qquad \beta_j \le 1, \qquad\qquad j = 1, 2, \ldots, n \\
& \qquad d_{\max} \ \textit{free in sign}
\end{aligned}
$$

The model (3) is always feasible and the optimal objective value of model (3) is bounded. This model determines DMU_p as the most efficient DMU if $d_p^* = 0$; in this case, DMU_p has the highest efficiency score that can be greater than 1, whereas those of the other DMUs are bounded by 1.

Lam [37] introduced a MILP model for selecting the most efficient unit that has an objective similar to that of the super-efficiency model in DEA.

$$
\begin{aligned}
& Max \quad h \\
& s.t. \\
& \quad \sum_{r=1}^{s} u_r y_{rj} - \sum_{i=1}^{m} v_i x_{ij} - MI_j \le 0, \qquad j = 1,2,\dots,n \\
& \quad -\sum_{r=1}^{s} u_r y_{rj} + \sum_{i=1}^{m} v_i x_{ij} + MI_j + h \le M, \quad j = 1,2,\dots,n \\
& \quad \sum_{i=1}^{m} v_i x_{ij} + MI_j \le 1 + M, \qquad j = 1,2,\dots,n \\
& \quad \sum_{i=1}^{n} I_j = 1, \qquad\qquad (4) \\
& \quad I_j \in \{0,1\}, \qquad j = 1,2,\dots,n \\
& \quad u_r \ge \varepsilon^*, \qquad r = 1,2,\dots,s \\
& \quad v_i \ge \varepsilon^*, \qquad i = 1,2,\dots,m
\end{aligned}
$$

where M is a large number. In model (4), it is assumed that all inputs and outputs are strictly positive. This model aims to maximize h, which is the difference between the weighted sums of the outputs and inputs of the chosen DMU (most efficient DMU). The most efficient DMU ($I_p^* = 1$) is the DMU with the highest efficiency score, and its efficiency score can be greater than 1, while the scores of other DMUs are bounded by 1. In this model, ε^* is the maximum non-Archimedean[44].

Toloo and Salahi [41] suggested a MINP model for selecting the best DMU with two steps as follows:

$$
\begin{aligned}
& h^* = Max \quad h \\
& s.t. \\
& \quad \sum_{r=1}^{s} u_r y_{rj} - \sum_{i=1}^{m} v_i x_{ij} \le MI_j - h(1 - I_j), \quad j = 1,2,\dots,n, \\
& \quad \sum_{r=1}^{s} u_r y_{rj} - \sum_{i=1}^{m} v_i x_{ij} \ge hI_j - M(1 - I_j), \quad j = 1,2,\dots,n, \\
& \quad \sum_{i=1}^{n} I_j = 1, \qquad\qquad (5) \\
& \quad I_j \in \{0,1\}, \; j = 1,2,\dots,n, \\
& \quad u_r \ge \varepsilon^*, \quad r = 1,2,\dots,s, \\
& \quad v_i \ge \varepsilon^*, \quad i = 1,2,\dots,m,
\end{aligned}
$$

where M is a large positive number. Toloo and Salahi [41] revealed that the minimum possible interval between the first two top-ranking DMUs is [−h*, h*]. They also proved that h* is strictly positive. As per model (5), only one DMU can be identified as the most efficient DMU ($I_p^* = 1$).

ε^* in model (5) can be obtained through the following model in the second step of Toloo and Salahi's procedure.

$$
\begin{aligned}
&\varepsilon^* = Max \quad \varepsilon \\
&s.t. \\
&\quad \sum_{r=1}^{s} u_r y_{rj} - \sum_{i=1}^{m} v_i x_{ij} \le MI_j - h(1-I_j), \quad j = 1,2,\dots,n, \\
&\quad \sum_{r=1}^{s} u_r y_{rj} - \sum_{i=1}^{m} v_i x_{ij} \ge hI_j - M(1-I_j), \quad j = 1,2,\dots,n, \\
&\quad \sum_{i=1}^{n} I_j = 1, \\
&\quad I_j \in \{0,1\}, \ j = 1,2,\dots,n, \\
&\quad u_r \ge \varepsilon, \quad r = 1,2,\dots,s, \\
&\quad v_i \ge \varepsilon, \quad i = 1,2,\dots,m,
\end{aligned}
\tag{6}
$$

Toloo and Salahi [41] introduced a continuous variable z_j to replace hI_j in models (5-6) and by adding the following constraints to model (5-6), transformed these models to MILP models.

$$
\begin{aligned}
&z_j \le MI_j, \quad j = 1,2,\dots,n \\
&z_j \le h \le z_j + M(1-I_j), j = 1,2,\dots,n \\
&z_j \ge 0, j = 1,2,\dots,n
\end{aligned}
$$

Inspired by the work of Toloo and Salahi (2,018), Özsoy, Örkcü [42] proposed a MINP model without epsilon to choose the most efficient DMU as follows:

$$
\begin{aligned}
&h^* = Max \quad h \\
&s.t. \\
&\quad \sum_{r=1}^{s} u_r y_{rj} - \sum_{i=1}^{m} v_i x_{ij} \le MI_j - h(1-I_j), \quad j = 1,2,\dots,n, \\
&\quad \sum_{r=1}^{s} u_r y_{rj} - \sum_{i=1}^{m} v_i x_{ij} \ge hI_j - M(1-I_j), \quad j = 1,2,\dots,n, \\
&\quad \sum_{i=1}^{n} I_j = 1, \\
&\quad I_j \in \{0,1\}, \ j = 1,2,\dots,n, \\
&\quad u_r \ge ((m+s)\max_j\{y_{rj}\})^{-1}, \quad r = 1,2,\dots,s, \\
&\quad v_i \ge ((m+s)\max_j\{x_{ij}\})^{-1}, \quad i = 1,2,\dots,m,
\end{aligned}
\tag{7}
$$

The structure of model (7) is similar to model (5), but it does not require calculating epsilon.

3. The proposed model

Inspired by the model (2), we propose the following model for determining the most efficient DMU:

$$
\begin{aligned}
&Max \quad \sum_{j=1}^{n} s_j \\
&s.t. \quad \sum_{r=1}^{s} u_r y_{rj} - \sum_{i=1}^{m} v_i x_{ij} + s_j = \delta_j, \quad j = 1,2,\dots,n, \\
&\qquad -M\,\delta_j \le s_j \le M\,(1-\delta_j), \; j = 1,2,\dots,n, \qquad\qquad (8) \\
&\qquad \sum_{j=1}^{n} \delta_j = 1, \\
&\qquad \delta_j \in \{0,1\}\;, j = 1,2,\dots,n, \\
&\qquad u_r \ge l_r^u, \; r = 1,2,\dots,s, \\
&\qquad v_i \ge l_i^v, \; i = 1,2,\dots,m,
\end{aligned}
$$

Where M is a large positive number and $\delta_j\,(j = 1,2,\dots,n)$ are binary variables, only one of which can take a nonzero value of one. Constraints $u_r \ge l_r^u\,(r = 1,2,\dots,s)$ and $v_i \ge l_i^v\,(i = 1,2,\dots,m)$ are borrowed from (2) and have been extensively applied in DEA practice.

If $\delta_p = 1$ then

$$-M \le s_p \le 0 \qquad (9)$$

$$0 \le s_j \le M \quad (j = 1,2,\dots,n;\, j \ne p)$$

So

$$\sum_{r=1}^{s} u_r y_{rp} - \sum_{i=1}^{m} v_i x_{ip} \ge 1, \qquad (10)$$

$$\sum_{r=1}^{s} u_r y_{rj} - \sum_{i=1}^{m} v_i x_{ij} \le 0, \quad j = 1,2,\dots,n;\, j \ne p \quad (11)$$

Which (10) allows the efficiency of DMU_p to be larger than one, while (11) guarantee that the efficiencies of the other DMUs to be less than or equal one. So, only the most efficient DMU can have an efficiency score of over one($\delta_p = 1$).The objective function in model (8) aims to maximize the distance between the efficiency score of the most efficient DMU and other DMUs. This minimizes the probability of other decision-making units having an efficiency score of 1, leading to a complete ranking of the units in a single step with fewer restrictions compared to other models. Table 1 compares models (2-5), (7) and the proposed model based on the number of constraints, complete ranking ability and the number of steps of each model.

Table1: Comparison of models based on structure and ability

	Model 2	**Model 3**	**Model 4**	**Model 5**	**Model 7**	**Proposed model**
Number of constraints	n+1	2n+2	3n+1	5n+1	5n+1	4n+1
Full rank ability	×	×	×	✓	✓	✓
Number of steps	1	1	2	2	1	1

As can be seen in table1, the proposed model is one-step model that has enabling the full ranking of all DMUs, and has fewer constraints than models with full ranking.

Theorem1. Model (8) always has a feasible solution.

Proof. Let $(\hat{u},\hat{v},\hat{\delta})$ be an optimal solution to model (2). Note that Wang and Jiang [32] proved that such a solution exists. Toloo [36] proved that in optimality there exists an index k such that $\hat{u}y_k - \hat{v}x_k > 0$ and $\hat{u}y_j - \hat{v}x_j \le 0, (j = 1,2,...,n; j \ne k)$. Let $w = \hat{u}y_k - \hat{v}x_k > 0$, so we have

$$\frac{\hat{u}}{w}y_k - \frac{\hat{v}}{w}x_k = 1,$$
$$\frac{\hat{u}}{w}y_j - \frac{\hat{v}}{w}x_j \le 0, \ (j = 1,2,...,n; j \ne k)$$

Let $s_k = 0,\ s_j = \frac{\hat{v}}{w}x_j - \frac{\hat{u}}{w}y_j \ (j = 1,2,...,n; j \ne k),\ \hat{\delta}_k = 1,\ \hat{\delta}_j = 0 \ (j = 1,2,...,n; j \ne k)$. The proof is completed.

Theorem 2. The optimal objective value of model (8) is bounded.

Proof. Let $(\bar{u},\bar{v},\bar{\delta},\bar{s})$ be any arbitrarily feasible solution to model (8). From the constraints of this model, we have:

$$-M \le \bar{s}_k \le 0, \qquad (12)$$
$$0 \le \bar{s}_j \le M,\ j = 1,2,...,n; j \ne k \qquad (13)$$

(12) and (13) show that objective function of model (8) has lower and upper bounds for any feasible solution. Proof completed.

4. Numerical examples

The numerical examples below use models (2), (3), (7), and (8) to find the most efficient DMU. These datasets are sourced from prior research in the DEA literature, and the source is mentioned in each case.

Example 1. In this example, 19 facility layout plans (FLDs), including two inputs and four outputs in manufacturing systems, are considered to evaluate efficiency. Data retrieved from [36]is shown in Table 1.

Inputs: x_1= material handling cost, x_2 = adjacency score.

Outputs: y_1 = sharpe ratio, y_2 = flexibility,y_3 = quality, y_4 = hand-carry utility.

Table 1: Data set for 19 FLDs.

DMUs	x_1	x_2	y_1	y_2	y_3	y_4
FLD1	20,309.56	6,405	0.4697	0.0113	0.041	30.89
FLD2	20,411.22	5,393	0.438	0.0337	0.0484	31.34
FLD3	20,280.28	5,294	0.4392	0.0308	0.0653	30.26
FLD4	20,053.2	4,450	0.3776	0.0245	0.0638	28.03
FLD5	19,998.75	4,370	0.3526	0.0856	0.0484	25.43
FLD6	20,193.68	4,393	0.3674	0.0717	0.0361	29.11
FLD7	19,779.73	2,862	0.2854	0.0245	0.0846	25.29
FLD8	19,831	5,473	0.4398	0.0113	0.0125	24.8
FLD9	19,608.43	5,161	0.2868	0.0674	0.0724	24.45
FLD10	20,038.1	6,078	0.6624	0.0856	0.0653	26.45
FLD11	20,330.68	4,516	0.3437	0.0856	0.0638	29.46
FLD12	20,155.09	3,702	0.3526	0.0856	0.0846	28.07
FLD13	19,641.86	5,726	0.269	0.0337	0.0361	24.58
FLD14	20,575.67	4,639	0.3441	0.0856	0.0638	32.2
FLD15	20,687.5	5,646	0.4326	0.0337	0.0452	33.21
FLD16	20,779.75	5,507	0.3312	0.0856	0.0653	33.6
FLD17	19,853.38	3,912	0.2847	0.0245	0.0638	31.29
FLD18	19,853.38	5,974	0.4398	0.0337	0.0179	25.12
FLD19	20,355	17,402	0.4421	0.0856	0.0217	30.02

Table 2 displays the optimal solution obtained by solving model (8) with $M = 100$. Since $\delta_{10}^{*} = 1$, thus FLD10 is identified as the most efficient FLD by model(8).

Table 2: Results of the model (8) in Example 1.

v_1^* = 0.00873230639680301, v_2^* =9.57744320576179e-06,
u_1^* =265.112513351513, u_2^* =1.94704049844237, u_3^* = 1.97005516154452, u_4^* = 0.00496031746031746,
$\delta_{10}^{*} = 1, \delta_{j}^{*} = 0 (j \neq 10),\ s_{10} = 4.64050997732481\text{e-}14$

Table 3 shows the outcomes of models (1), (2), (3), (7), and (8) in Example 1. The highest efficiency scores achieved by the various models are highlighted in bold. The numbers in parentheses alongside the efficiency scores denote the FLDs rankings. The results of model 1 (CCR model) indicates that nine FLDs are efficient. The results from models (2), (3), (7), and (8) indicate that FLD10 is the most efficient FLD. Models (8), just like models (2), (3), and (7), allow the efficiency score of one DMU to exceed one (the most efficient DMU), while the efficiency scores of the other DMUs remain less than or equal to one. Models (2) cannot fully distinguish between the DMUs. In model (2), both FLD3 and FLD12 have the same rank value. However, models (3), (5), and (8) are capable of ranking all DMUs effectively. It's worth noting that FLD13 is identified as the worst unit in all models.

Table 3: Efficiency of FLDs by different models Example 1.

DMUs	CCR	Wang and Jiang (2,012)-Model(2)	Toloo (2,015) -Model(3)	Özsoy et al. (2,021) -Model(7)	Proposed model -Model(8)
FLD1	0.984592 (13)	0.964891 (5)	0.734523 (16)	0.761219 (7)	0.703336(2)
FLD2	0.988393 (12)	0.971531 (4)	0.804572 (10)	0.761527 (6)	0.653074(7)
FLD3	0.997428 (11)	1 (2)	0.844136 (7)	0.770702 (3)	0.659215(6)
FLD4	0.949290 (15)	0.894522 (14)	0.774063 (12)	0.673692 (15)	0.57332(9)
FLD5	**1 (1)**	0.925330 (9)	0.870627 (6)	0.751551 (8)	0.537374(11)
FLD6	0.973342 (14)	0.910794 (13)	0.825097 (9)	0.734339 (10)	0.554245(10)
FLD7	**1 (1)**	0.790849 (17)	0.76786 (13)	0.552031 (17)	0.439958(17)
FLD8	0.856831 (17)	0.868210 (15)	0.60761 (18)	0.723427 (13)	0.674081(3)
FLD9	0.889201 (16)	0.834482 (16)	0.833446 (8)	0.630595 (16)	0.446235(16)
FLD10	**1 (1)**	**1.440321 (1)**	**1.149501 (1)**	**1.230623 (1)**	**1.005713(1)**
FLD11	0.998328 (10)	0.940190 (8)	0.922867 (4)	0.732256 (11)	0.515594(13)
FLD12	**1 (1)**	1 (2)	1 (2)	0.766601 (5)	0.533705(12)
FLD13	0.775852 (19)	0.675683 (19)	0.605774 (19)	0.513299 (19)	0.417161(19)
FLD14	1 (1)	0.941034 (7)	0.931165 (3)	0.723855 (12)	0.510119(14)
FLD15	1 (1)	0.951281 (6)	0.791951 (11)	0.740819 (9)	0.63644(8)
FLD16	1 (1)	0.913958 (11)	0.915848 (5)	0.693781 (14)	0.4863(15)
FLD17	1 (1)	0.769322 (18)	0.735705 (15)	0.534852 (18)	0.437167(18)
FLD18	0.851718 (18)	0.913731 (12)	0.685373 (17)	0.767148 (4)	0.673624(4)
FLD19	1 (1)	0.923829 (10)	0.745818 (14)	0.790033 (2)	0.660799(5)
SUM		17.879452	16.491435	16.491435	11.11746

Fig. 1 provides an illustrative comparison between the results of models (2), (3), (7), and (8) in example 1 according to efficiency scores that are shown in Table 2.

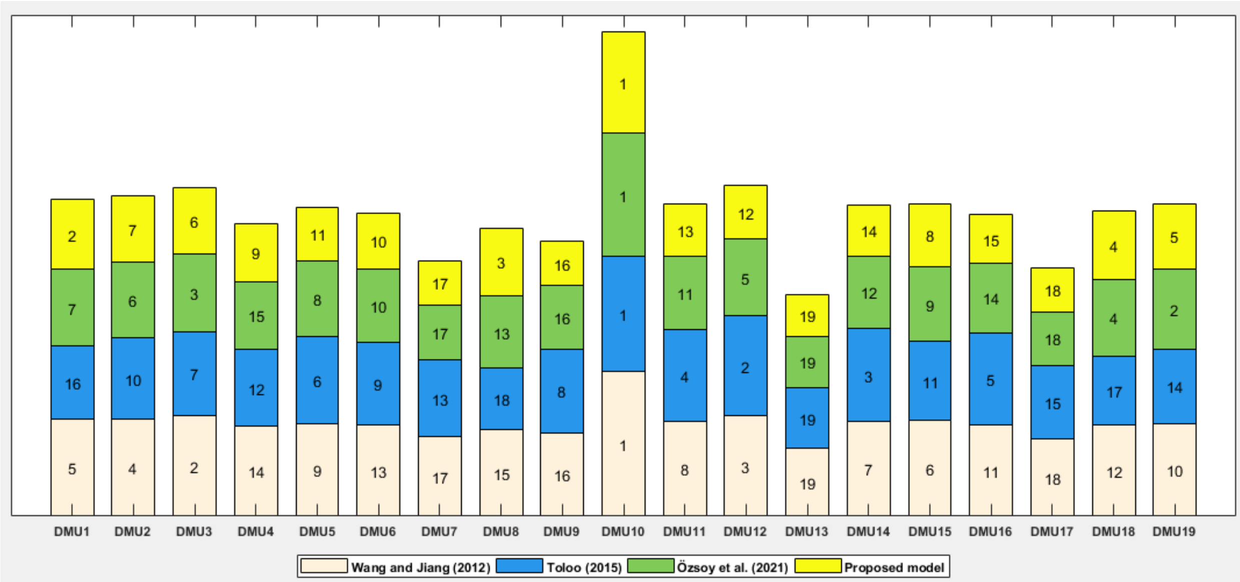

Fig.1. Illustrative comparison between the ranking results of different models in Example 1.

Table4: Ranking models correlation test in example 1.

			Spearmen			
			Wang and Jiang (2,012)-Model(2)	**Toloo (2,015) -Model(3)**	**Özsoy et al. (2,021) -model(7)**	**proposed model -model(8)**
Kendall	**Wang and Jiang (2,012)-Model(2)**	**Correlation**	1	0.6105	0.8175	0.5789
		p-value		(0.0065)	(0.0000)	(0.0107)
	Toloo (2,015) -Model(3)	**Correlation**	0.4854	1	0.3175	-0.0667
		p-value	(0.0032)		(0.1850)	(0.7868)
	Özsoy et al. (2,021)-Model(7)	**Correlation**	0.7076	0.2398	1	**0.7825**
		p-value	(0.0000)	(0.1637)		(0.0001)
	Proposed model-Model(8)	**Correlation**	0.4386	-0.0760	**0.6608**	1
		p-value	(0.0083)	(0.6787)	(0.0000)	

The rank correlation coefficient can be used to evaluate the significance of the relationship between the models mentioned previously. Kendall and Spearman are two commonly used nonparametric methods that use rank correlation. We use these methods to determine the strength of the relationship between the rankings of models (2), (3), (7), and (8). Correlation values for Spearman and Kendall methods are shown in Table 4. Values above and below the diagonal indicate rank coefficients for Spearman and Kendall, respectively. The p-value for the correlation test is shown in parentheses below the correlation value. Table 4 shows a positive correlation between the proposed model (model (8)), and model (7). The proposed model has fewer constraints than model (7) and can be solved through one-step linear programming. Also, unlike model (2), model (8) ranks all FLDs completely.

Example 2. As the second example, we use a real data set of 14 Czech Republic banks, adapted from [45]. Table 5 indicates the inputs and outputs of banks that are described below:
Inputs: X_1 =number of employees, X_2 =number of branches, X_3 =assets, X_4 =equity, X_5 =expenses
Outputs: Y_1 =deposits, Y_2 = loans, Y_3 = non-interest income, Y_4 = interest income.

Table 5: Inputs and Outputs of 14 branches.

DMUs	X_1	X2	X3	X4	X5	Y1	Y2	Y3	Y4
AIR	400	18	33,600	2,596	745	30,696	11,135	14	554
CMZRB	217	5	111,706	4,958	566	86,967	16,813	634	1,700
CS	10,760	658	920,403	93,190	18,259	629,622	479,516	8,747	32,697
CSOB	7,801	322	937,174	73,930	16,087	629,622	479,516	8,747	32,697
EQB	296	13	8,985	1,296	601	7,502	5,611	19	215
ERB	72	1	33,614	464	173	2,940	1,762	15	131
FIO	59	36	18,561	726	347	17,174	6,465	211	536
GEMB	3,346	260	135,474	34,486	5,276	97,063	101,898	3,943	11,026
ING	293	10	128,425	913	1,034	92,579	19,216	468	5,139
JTB	407	3	85,087	7,233	1,333	62,085	39,330	487	3,686
KB	8,758	399	786,836	100,577	13,511	579,067	451,547	8,834	35,972
LBBW	365	18	31,300	2,774	1,138	20,274	2,528	128	1,046
RB	2,927	125	197,628	18,151	57,112	144,143	150,138	2,829	8,563
UCB	2,004	98	318,909	38,937	13,804	195,120	192,046	2,740	8,891

Utilizing model (8) with the data set presented in Table 5, we arrive at the optimal solution($M = 500$) in Table 6. Since $\delta_{10}^* = 1$, thus JTB bank is recognized as the most efficient bank by model (8).

Table 6: Results of the model (8) in Example 2.

v_1^*= 1.03263114415531e-05, v_2^*=8.62395716185947, v_3^*=0.00839407430091381, $v_4^* = 0.0239030938097463$,
$v_5^* = 0.018156388032191$,
u_1^*= 1.6135236516751e-07, u_2^*=0.0119879515784093, u_3^*= 0.404132740901498, u_4^*= 0.0892887795654194,
$\delta_{10}^* = 1, \delta_j^* = 0\,(j \neq 10),\ s_{10}^* =$ -59.231022328601

Table 7: Efficiency of bank branches by different models in example 2.

DMUs	**CCR Model(1)**	**Wang and Jiang (2,012)-Model(2)**	**Toloo (2,015)-Model(3)**	**Özsoy et al. (2,021)-Model(7)**	**Proposed model-Model(8)**
AIR	1(1)	0.797188(11)	0.573436(11)	0.385261(12)	0.367774(12)
CMZRB	1(1)	1(2)	0.759709(10)	0.679386(9)	0.549378(9)
CS	1(1)	**1.13391(1)**	**1.098292(1)**	0.990725(2)	0.968671(2)
CSOB	1(1)	1(2)	1(2)	0.974095(4)	0.960639(3)
EQB	1(1)	0.618833(12)	0.522892(12)	0.352771(13)	0.410334(11)
ERB	0.473757(14)	0.131031(14)	0.13181(14)	0.136019(14)	0.127476(14)
FIO	1(1)	1(2)	0.859853(8)	0.540581(10)	0.429939(10)
GEMB	1(1)	1(2)	1(2)	0.92167(6)	0.883709(6)
ING	1(1)	0.943912(9)	0.773142(9)	0.874328(8)	0.729025(8)
JTB	1(1)	1(2)	0.977494(5)	**1.165441(1)**	**1.064267(1)**
KB	1(1)	1(2)	0.985409(4)	0.988263(3)	0.960615(4)
LBBW	0.824637(13)	0.604593(13)	0.415037(13)	0.410718(11)	0.347436(13)
RB	1(1)	1(2)	0.929992(7)	0.917785(7)	0.881171(7)
UCB	1(1)	0.906461(10)	0.940065(6)	0.965502(5)	0.893695(5)
SUM		12.135928	10.967131	10.302545	9.574129

Table 7 presents the results of models (1), (2), (3), (7) and (8) respectively. The results show that out of 14 branches, 12 are efficient. Models (2) and (3) identify CS bank as the most efficient bank, while models (7) and (8) select Bank JTB as the most efficient bank. Furthermore, all models agree that the ERB bank is the worst bank. It's important to note that models (2) and (3) do not fully rank all banks, while models (7) and (8) do. Among these models, (8) has a simpler structure than models (7) in terms of constraints.
Figure 2 shows the rank of each bank based on the efficiency score of models (2), (3), (7), and (8) in example 2.

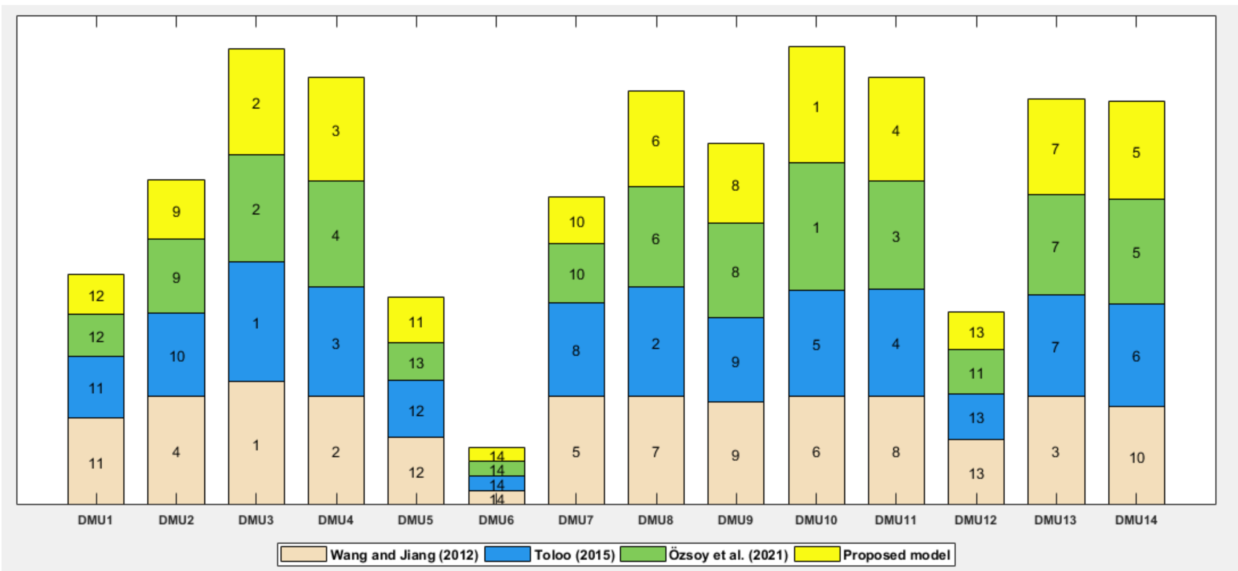

Fig.2. Illustrative comparison between the ranking results of different models in Example 2.

In table 8, we emphasized the coefficients of Spearman and Kendall. The findings indicated a positive relationship between between model (8) with models (2),(3) and (7). Data in Table 8 shows a strong correlation between model (8) and model (7). For the model (8) and model (7), the Spearman and Kendall's rank correlation coefficients are 0.978022 and 0.912087912 respectively. The proposed model with fewer constraints has completely ranked the banks in one step.

Table8: Ranking models correlation test in example 2.

			Spearmen			
			Wang and Jiang (2,012)-Model(2)	**Toloo (2,015)-Model(3)**	**Özsoy et al. (2,021)-model(7)**	**proposed model-model(8)**
Kendall	**Wang and Jiang (2,012)-Model(2)**	**Correlation**	1	0.736264	0.661538	0.696703
		p-value		(0.003819)	(0.012187)	(0.007343)
	Toloo (2,015) -Model(3)	**Correlation**	0.604395604	1	0.894505	0.907692
		p-value	(0.001957077)		(0)	(0)
	Özsoy et al. (2,021)-Model(7)	**Correlation**	0.516483516	0.736263736	1	**0.978022**
		p-value	(0.009753043)	(7.72E-05)		(0)
	Proposed model-Model(8)	**Correlation**	0.56043956	0.78021978	**0.912087912**	1
		p-value	(0.004565983)	(1.92E-05)	(5.18E-08)	

5. Conclusion

In this paper, we have proposed a new MILP model based on a common weight set for identifying the most efficient DMU. In the proposed model, the most efficient unit has an efficiency score greater than one, and other DMUs have efficiency scores less than or equal to one. In this model, the distance between the efficiency score of the most efficient unit and other units is enough to allow for a complete ranking. The proposed model has fewer constraints than some models with full ranking and is solved in one step. Two real examples with known ranks cited in the literature were selected to ensure the validity of the proposed model. The results illustrated that the proposed model has high discrimination power. Further important future research directions would be considering the effect of selecting a value for M on the finding of the most efficient DMU and incorporating negative data in the model.

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