Collocation method for solving systems of fractional differential equations; a case study of HIV infection by using Müntz wavelets basis

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Abstract

This work was written with the aim of solving the fractional differential equation system using the Müntz wavelets. Müntz wavelets are modified using polynomials, respectively. The error of the proposed method is evaluated. This method applies to the fractional version of the HIV infection model. Numerical results confirm the accuracy of the proposed method.

Keywords: Müntz wavelels; Fractional differential equations systems;Collocation method; Jacobi polynomials; HIV infection.

1 Introduction

Wavelets as powerful and practical tools in solving integral and differential equations, are used in various applications. So far, many wavelets have been introduced among which one can mention to Haar [15], Legendre wavelets [16] and Chebyshev wavelets of the first, second, third and forth kinds [17, 28, 29] ets. In fact, wavelets are orthogonal bases that are used to solve integral and differential equations as well as many mathematical, engineering, and medical problems. Although the number of operations in the Haar wavelet is few, it is mostly not very accurate in solving problems. Legendre wavelets with weight function $w(x) = 1$, are known as simple, low-operating basis with relatively good computational accuracy in many problems. Due to the weight functions of Chebyshev wavelets, they have relatively high operations. However in the integral equations where the nucleus is inversely proportional to the weight function, they have shown a good accuracy. This article intends to introduce the Müntz wavelet and examine its application. An important difference of Müntz wavelet and above-mentioned ones is the degree of the extended sentences. For more explanation, all the former wavelets are in the form of

$$\Sigma_{i=1}^{\infty} c_i x^i,$$

while the form of the Müntz wavelet is as follows:

$$\Sigma_{i=1}^{\infty} c_i x^{\lambda_i},$$

in which λ_i is a complex number. So it can be argued that, Müntz wavelet not only gives a good approximation for fractional power and complex functions, but also covers a wide range of functions. Fractional calculus first emerged as a pure mathematical theory in the mid-nineteenth century. [1].One hundred years later, engineers and physicists faced practical problems with fractional arithmetic [2, 3]. A good way to describe the memory and hereditary properties of different materials and processes is to use fractional derivatives. [4]. In some cases, fractional order models of real systems perform better than correct order ones. To illustrate, researchers have used fractional derivatives in many fields related to science and engineering, including fluid flow, rheology, diffusion-like diffusion, electrical networks, electromagnetic theory, and probability. [7, 11, 13, 14]. Most of these equations do not have exact analytical answers.This forces us to use approximate and numerical techniques. So far, several analytical and numerical methods

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for solving fractional differential equations have been proposed. As the main examples one can mention to domain decomposition method [8], Linear B-spline method [12], Product integration method [10], multistep method [9], Predictor Corrector method [6], Extrapolation method [5]. In this paper, we present an approximate solution for a system of differential equations in $t \in [0, T]$. The general form of this type of equation is as follows:

$$\begin{cases} D_*^{\omega_1} y_1(t) = g_1(t, y_1(t), y_2(t), \ldots, y_n(t)) \\ D_*^{\omega_2} y_2(t) = g_2(t, y_1(t), y_2(t), \ldots, y_n(t)) \\ \vdots \\ D_*^{\omega_n} y_n(t) = g_n(t, y_1(t), y_2(t), \ldots, y_n(t)) \\ y_i(0) = \mu_i \end{cases} \quad 0 < \omega_i < 1, i = 1, 2, \ldots, n. \tag{1}$$

Where $D_*^{\omega_i} y_i(t)$ and g_i are respectively the fractional derivative of order ω_i and function by t and $y_i(t)$ and μ_i are known constant number. The aim of solving this equation is the calculation of $y_i(t)$.

In the present work, in the first stage the Müntz wavelets are introduced. These wawelets are definitely faster and more accurate than the Müntz Legendre polynomials.In the second stage the Müntz Legendre polynomials in the interval $[0, 1]$ are introduced.In the continuation of the stages the definitions and properties of wavelets will be discussed.The Müntz wavelets are then presented in the range $[0, T]$. Subsequently, with the help of Jacobi polynomials, a more stable formula for Müntz wavelets is obtained. Finally the fractional differential equations are solved and the error analysis is reviewed. Also, to investigate of the accuracy of the presented method, mathematical and practical examples are presented.

2 Müntz Legendre Polynomials

Assume that the set $\Lambda_n = \{\lambda_1, \lambda_2, ..., \lambda_n\}$ is a sequence of complex numbers, provided that $Re(\lambda_k > -1/2)$. So one can define Müntz Legendre polynomials in the interval $[0, 1]$ as follows [19]:

$$L_n(x) = L_n(x, \Lambda_n) = \Sigma_{k=1}^{n} c_{n,k} x^{\lambda_k}, \tag{2}$$

$$c_{n,k} = \frac{\prod_{v=0}^{n-1} \lambda_k + \bar{\lambda}_v + 1}{\prod_{v=0, v\neq k}^{n-1} \lambda_k - \lambda_v}. \tag{3}$$

These polynomials are orthogonal with respect to the weight function $w(x) = 1$.
As a result, one can say that:

$$(L_n(x), L_m(x)) = \int_0^T L_n(x) L_m(x) dx = \frac{\delta_{m,n}}{\lambda_n + \bar{\lambda}_{n+1}}. \tag{4}$$

Where $(.,.)$ represents the inner product and $\delta_{m,n}$ is Kronecker delta function.
In this work, it assumes that $\lambda_k = \gamma k$, then

$$L_n(x) = L_n(x, \gamma) = \Sigma_{k=1}^{n} c_{n,k} x^{\gamma_k}, \tag{5}$$

$$c_{n,k} = \frac{(-1)^{n-k}}{\gamma^n k(n-k)} \prod_{v=0}^{n-1} ((k+v)\gamma + 1). \tag{6}$$

3 Wavelets

Wavelet families are created by the expansion and transmission of a function called the mother wavelet. As a result of the continuous change of the expansion and transfer parameters, the following continuous wavelet families are raised [21]:

$$\psi_{a,b}(t) = |a|^{-1/2} \psi(\frac{t-b}{a}), a, b \in R, a \neq 0.$$

Where a and b are expansion and transfer parameters, respectively.
If the parameters a and b are bound to discrete values, i.e. $b_0 > 0, a_0 > 1, a = a_0^{-k}, b = nb_0a_0^{-k}$ and if n is a positive integer number:

$$\psi_{a,b}(t) = |a_0|^{k/2}\psi(a_0^k t - nb_0).$$

Thus the following discrete wavelet family is formed:

$$\psi_{k,n}(t) = |a_0|^{k/2}\psi(a_0^k t - nb_0). \tag{7}$$

Where $\psi_{k,n}(t)$is a basic wavelet for $L^2(R)$.
In general, if one assumes that $a_0 = 2$ and $b_0 = 1$, then $\psi_{k,n}(t)$ produces an orthonormal basis.

3.1 Müntz wavelets

Consider Müntz Legendre's polynomial of degree m is denoted by $L_m(x, \gamma)$.These polynomials are orthogonal to their weight function $w(t) = 1$. Müntz wavelets have four arguments $\psi_{n,m}(t) = \psi(k, n, m, t)$ in which $m = 0, 1, ..., M - 1$, $n = 1, 2, ..., 2^{k-1}$ and $k = 2, 3,$ (M is a positive integer number).
Eq. (8) shows a definition of the Müntz wavelets on $[0, T]$:

$$\psi_{n,m}(t) = \begin{cases} \sqrt{2^{k-1}(1-2m\gamma)}L_m[2^{k-1}t - T(n-1), \gamma], & \frac{(n-1)T}{2^{k-1}} \le t < \frac{nT}{2^{k-1}} \\ 0, \quad otherwise. \end{cases} \tag{8}$$

In Eq. (8), increasing M value causes the coefficients increase and become very large. However, the sum of the coefficients is always equal to $\sqrt{2^{k-1}(1-2m\gamma)}$.
In the next section it is shown that Müntz wavelets can be obtained using Jacobi polynomials in such a way that the wavelet coefficients to be stabilized.

3.2 Jacobi polynomials

Jacobi polynomials with the weight function $w(x) = (1-x)^\alpha(1+x)^\beta$ and $\alpha, \beta > -1$ are orthogonal. They are defined as follows [19, 20]:

$$J_k^{\alpha,\beta}(x) = \sum_{m=0}^{k} \frac{(-1)^{k-m}(1+\beta)^k(1+\alpha+\beta)^{k+m}}{m(k-m)(1+\beta)^m(1+\alpha+\beta)^k}(\frac{1+x}{2})^m, x \in [-1, 1]. \tag{9}$$

Also, one can say that:

$$\begin{cases} J_0^{\alpha,\beta}(x) = 1 \\ J_1^{\alpha,\beta}(x) = \frac{1}{2}((\alpha-\beta) + (\alpha+\beta+2)x) \\ J_{k+1}^{\alpha,\beta}(x) = \frac{b_k^{\alpha,\beta}(x)}{a_k^{\alpha,\beta}}J_k^{\alpha,\beta}(x) - c_k^{\alpha,\beta}J_{k-1}^{\alpha,\beta}(x). \end{cases} \tag{10}$$

Where

$$\begin{cases} a_k^{\alpha,\beta} = 2(k+1)(k+\alpha+\beta+1)(2k+\alpha+\beta) \\ b_k^{\alpha,\beta}(x) = (2k+\alpha+\beta+1)((2k+\alpha+\beta)(2k+\alpha+\beta+2)x + \alpha^2 - \beta^2) \\ c_k^{\alpha,\beta} = 2(k+\alpha)(k+\beta)(2k+\alpha+\beta+2). \end{cases} \tag{11}$$

Then, using Jacobi polynomials, we introduce the modified Müntz wavelets.

3.3 Modified Müntz wavelets

Theorem1: It assumes that $J_m^{\alpha,\beta}(x)$ is a Jacobi polynomial of degree m and $\alpha > 0$ is a real number and $t \in [0, 1]$, then we can rewrite Eq. (8) as follows:

$$\psi_{n,m}(t) = \begin{cases} \sqrt{2^{k-1}(1+2m\gamma)}J_m^{(0,\frac{1}{\gamma}-1)}[2\frac{(2^{k-1}t - T(n-1))}{T}^\gamma - 1], & \frac{(n-1)T}{2^{k-1}} \le t < \frac{nT}{2^{k-1}} \\ 0, \quad otherwise. \end{cases} \tag{12}$$

Proof: According to the Eq. (8) and Eq. (12), it suffices to show that:

$$L_m[2^{k-1}t - T(n-1), \gamma] = J_m^{(0,\frac{1}{\gamma}-1)}[2\frac{(2^{k-1}t - T(n-1))}{T}^{\gamma} - 1], \quad \frac{(n-1)T}{2^{k-1}} \le t < \frac{nT}{2^{k-1}}.$$

With the change of variables $x = 2^{k-1}t - T(n-1)$ we have:

$$L_m[x, \gamma] = J_m^{(0,\frac{1}{\gamma}-1)}[2(\frac{x}{T})^{\gamma} - 1], \quad 0 \le t < 1.$$

Putting $y = 2(\frac{x}{T})^{\gamma} - 1$ in Eq. (9):

$$J_m^{(0,\frac{1}{\gamma}-1)}[2(\frac{x}{T})^{\gamma} - 1] = \sum_{k=0}^{m} \frac{(-1)^{m-k}(\frac{1}{\gamma})^m(\frac{1}{\gamma})^{m+k}}{k!(m-k)!(\frac{1}{\gamma})^m(\frac{1}{\gamma})^k}(\frac{x}{T})^{k\gamma} = \sum_{k=0}^{m} c_{m,k}(\frac{x}{T})^{k\gamma} = L_m(x,\gamma).\blacksquare$$

Now, with respect to Theorem 1 and Eq. (10) and Eq. (11), the modified wavelet Müntz will be as follows:

$$\psi_{n,m}(t) = \begin{cases} \sqrt{\frac{2^{k-1}(1+2m\gamma)}{T}}\widetilde{L}_m[2^{k-1}t - T(n-1)), \gamma], & \frac{(n-1)T}{2^{k-1}} \le t < \frac{nT}{2^{k-1}} \\ 0, & otherwise. \end{cases} \tag{13}$$

Where $\widetilde{L}_m$ is the Müntz modified formula:

$$\begin{cases} \widetilde{L}_0(x,\gamma) = 1 \\ \widetilde{L}_1(x,\gamma) = (\frac{1}{\gamma}+1)(\frac{x}{T})^{\gamma} - \frac{1}{\gamma} \\ \widetilde{L}_{m+1}(x,\gamma) = \frac{1}{a_m}(b_m(x)\widetilde{L}_m(x,\gamma) - c_m\widetilde{L}_{m-1}(x,\gamma), \end{cases} \tag{14}$$

in which

$$\begin{cases} a_m = a_m^{(0,\frac{1}{\gamma}-1)} \\ b_m(x) = b_m^{(0,\frac{1}{\gamma}-1)}(2(\frac{x}{T})^{\gamma} - 1) \\ c_m = c_m^{(0,\frac{1}{\gamma}-1)}. \end{cases} \tag{15}$$

We have already mentioned that the coefficients in (8) become very large with increasing m, while the sum of the coefficients is always equal to $\sqrt{2^{k-1}(1+2m\gamma)}$. In fact, using (5) and calculating $L_m(1) = L_m(1,\gamma) = \sum_{k=1}^{m} c_{m,k}$, it can be concluded that the sum of the coefficients in $\psi_{n,m}(t)$ is always equal to $\sqrt{2^{k-1}(1+2m\gamma)}$. For example, when $k=1$, $m=5$ and $\gamma=0.5$, by using (8) we have ,

$$\psi_{1,5}(t) = \sqrt{6}\left(-6 + 105t^{1/2} - 560t + 1260t^{3/2} - 1260t^2 + 462t^{5/2}\right), \quad 0 \le t < 1.$$

On the other hand, using (13), we can write $\psi_{1,5}(t)$ differently,

$$\begin{aligned} \psi_{1,5}(t) \quad = \quad & \frac{1}{90\sqrt{6}}(\frac{1}{28}(\frac{1}{15}(-1+63(-1+2\sqrt{t})(\frac{1}{6}(-1+35(-1+2\sqrt{t}))(-20+4(-1+15(-1+2\sqrt{t})) \\ & (-2+3\sqrt{t})) - 84(-2+3\sqrt{t})) - 6(-20+4(-1+15(-1+2\sqrt{t})) \\ & (-2+3\sqrt{t})))(-1+99(-1+2\sqrt{t})) - \frac{11}{3}(\frac{1}{6}(-1+35(-1+2\sqrt{t})) \\ & (-20+4(-1+15(-1+2\sqrt{t}))(-2+3\sqrt{t})) - 84(-2+3\sqrt{t}))), \quad 0 \le t < 1. \end{aligned}$$

It is observed that in the modified formula (13), large coefficients do not appear. In order to show the difference between (8) and (13) in the numerical evaluation of Müntz wavelets, the values $\psi_{1,m}(t)$ by some selected values t and $\gamma = 1/2$, $k=2$, $m = 20, 40, 50$ are reported in Table 1. As can be seen, the values obtained using (8) for $m = 40$ and 50 have a high error rate, except where t is very close to zero. Also, the sum of the coefficients is equal to $\psi_{n,m}(\frac{n}{2^{k-1}}) = \sqrt{2^{k-1}(1+2m\gamma)}$, which at the end point of each section, i.e. $t = \frac{n}{2^{k-1}}$, the exact values of $\psi_{n,m}(t)$ are known. In Fig. 1, the absolute errors $\psi_{1,m}(0.5)$, for $\gamma = 1/2$, $k=2$ and different values of m using the formulas (8) and (13) are obtained.

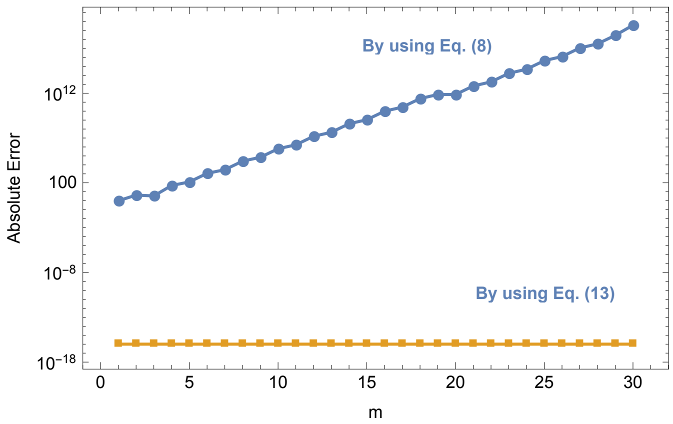

Figure 1: Absolute errors in the values $\psi_{1,m}(1/2)$ for $\gamma = 1/2$, $k = 2$ and various values of m.

Table 1: Calculated values of $\psi_{1,m}(t)$ for $\gamma = 1/2$, $k = 2$ and $m = 20, 40, 50$.

	$m = 20$		$m = 40$		$m = 50$	
t	Eq. (8)	Eq. (13)	Eq. (8)	Eq. (13)	Eq. (8)	Eq. (13)
0.0005	5.061460	5.061460	10.4346	10.4346	-10.5350	-10.5350
0.005	0.868465	0.868465	2.12626	2.12626	2.70601	2.70945
0.05	-1.521720	-1.521720	-144.886	-1.82540	-2.11802×10^{7}	-0.89844
0.25	-0.470313	-0.468590	-2.67387×10^{10}	-0.73353	-1.33610×10^{17}	0.77026
0.45	-1.499940	-1.411100	6.96955×10^{13}	1.07803	9.48981×10^{20}	-1.50813
0.49	-2.506220	-2.445380	-5.97390×10^{13}	1.03524	-5.23977×10^{21}	-2.53715

4 Müntz wavelets collocation method

Consider the system of fractional differential equations (1).For all $0 < \omega < 1$ Caputo fractional derivative formula for $y(t)$ function is:

$$D_*^{\omega} y(t) = \frac{1}{\Gamma(1-\omega)} \int_0^t (t-\tau)^{-\omega} y'(\tau) d\tau, 0 \le t \le T. \tag{16}$$

The aim of solving Eq. (1) is the calculation of $y_i(t)$. For solving this problem with help of Müntz wavelet, we let $y_n(t) = (y_{1,n}(t), y_{2,n}(t), ..., y_{N,n}(t))$ is the exact solution of Eq. (1) in the interval $[\frac{(n-1)T}{2^{k-1}}, \frac{nT}{2^{k-1}}]$, now we put:

$$y(t) \simeq (\widetilde{y}_{1,n}(t), \widetilde{y}_{2,n}(t), ..., \widetilde{y}_{N,n}(t)) = (\sum_{j=0}^{M-1} a_{n,j}^1 \psi_{n,j}(t), \sum_{j=0}^{M-1} a_{n,j}^2 \psi_{n,j}(t), ..., \sum_{j=0}^{M-1} a_{n,j}^N \psi_{n,j}(t)). \tag{17}$$

Where $\psi_{n,j}(t)$ is Müntz wavelet and $a_{n,j}^i$ are unknown constants. By limiting the problem Eq. (1) in the interval $[\frac{(n-1)T}{2^{k-1}}, \frac{nT}{2^{k-1}}]$ and $\omega = \omega_1 = \omega_2 = \ldots = \omega_n$, $0 < \omega < 1$ we have:

$$\begin{cases} D_*^{\omega}(\sum_{j=0}^{M-1} a_{n,j}^1 \psi_{n,j}(t)) = g_1(t, \sum_{j=0}^{M-1} a_{n,j}^1 \psi_{n,j}(t), \sum_{j=0}^{M-1} a_{n,j}^2 \psi_{n,j}(t), \ldots, \sum_{j=0}^{M-1} a_{n,j}^N \psi_{n,j}(t)) \\ D_*^{\omega}(\sum_{j=0}^{M-1} a_{n,j}^2 \psi_{n,j}(t)) = g_2(t, \sum_{j=0}^{M-1} a_{n,j}^1 \psi_{n,j}(t), \sum_{j=0}^{M-1} a_{n,j}^2 \psi_{n,j}(t), \ldots, \sum_{j=0}^{M-1} a_{n,j}^N \psi_{n,j}(t)) \\ \vdots \\ D_*^{\omega}(\sum_{j=0}^{M-1} a_{n,j}^N \psi_{n,j}(t)) = g_N(t, \sum_{j=0}^{M-1} a_{n,j}^1 \psi_{n,j}(t), \sum_{j=0}^{M-1} a_{n,j}^2 \psi_{n,j}(t), \ldots, \sum_{j=0}^{M-1} a_{n,j}^N \psi_{n,j}(t)) \\ y_{i,n}(\frac{(n-1)T}{2^{k-1}}) = \begin{cases} \widetilde{y}_{i,n-1}(\frac{(n-1)T}{2^{k-1}}), n \neq 1 \\ \mu_i, \quad n = 1. \end{cases} \end{cases} \tag{18}$$

For approximating $y_n(t)$, we consider $M-1$ collocation points regarding the roots of Chebyshev polynomials of degree M in the interval $[\frac{(n-1)T}{2^{k-1}}, \frac{nT}{2^{k-1}}]$:

$$\theta_i = 2^{-k}T[\cos(\frac{2i-1}{2M}\pi) + 2n - 1], \quad i = 1, 2, ..., M-1. \tag{19}$$

Therefore, we have the following system of equations for $v = 1, 2, ..., M-1$:

$$\begin{cases} D_*^{\omega}(\sum_{j=0}^{M-1} a_{n,j}^1 \psi_{n,j}(\theta_v)) = g_1(\theta_v, \sum_{j=0}^{M-1} a_{n,j}^1 \psi_{n,j}(\theta_v), \ldots, \sum_{j=0}^{M-1} a_{n,j}^N \psi_{n,j}(\theta_v)) \\ D_*^{\omega}(\sum_{j=0}^{M-1} a_{n,j}^2 \psi_{n,j}(\theta_v)) = g_2(\theta_v, \sum_{j=0}^{M-1} a_{n,j}^1 \psi_{n,j}(\theta_v), \ldots, \sum_{j=0}^{M-1} a_{n,j}^N \psi_{n,j}(\theta_v)) \\ \vdots \\ D_*^{\omega}(\sum_{j=0}^{M-1} a_{n,j}^N \psi_{n,j}(\theta_v)) = g_N(\theta_v, \sum_{j=0}^{M-1} a_{n,j}^1 \psi_{n,j}(\theta_v), \ldots, \sum_{j=0}^{M-1} a_{n,j}^N \psi_{n,j}(\theta_v)) \\ y_{i,n}(\frac{(n-1)T}{2^{k-1}}) = \begin{cases} \widetilde{y}_{i,n-1}(\frac{(n-1)T}{2^{k-1}}), n \neq 1 \\ \mu_i, \quad n = 1. \end{cases} \end{cases} \tag{20}$$

For $i = 1, 2, \ldots, M-1$. The system of Eq. (20) has $N \times M$ equations and $N \times M$ unknown $a_{n,j}^i, i = 1, 2, \ldots, N, j = 0, 1, \ldots, M-1$ that is computable. After calculating $a_{n,j}^i$ and placement Eq. (17), Approximately $y_n(t)$ can be calculated. So, $y(t)$ in $[\frac{(n-1)T}{2^{k-1}}, \frac{nT}{2^{k-1}}]$ is calculated. For all of the intervals are calculated $y_n(t)$ then we have:

$$y(t) = \begin{cases} (\widetilde{y}_{1,1}(t), \widetilde{y}_{2,1}(t), \ldots, \widetilde{y}_{N,1}(t)), & 0 \leq t < \frac{T}{2^{k-1}} \\ (\widetilde{y}_{1,2}(t), \widetilde{y}_{2,2}(t), \ldots, \widetilde{y}_{N,2}(t)), & \frac{T}{2^{k-1}} \leq t < \frac{T}{2^{k-2}} \\ \vdots \\ (\widetilde{y}_{1,2^{k-1}}(t), \widetilde{y}_{2,2^{k-1}}(t), \ldots, \widetilde{y}_{N,2^{k-1}}(t)), & T - \frac{T}{2^{k-1}} \leq t < T. \end{cases} \tag{21}$$

5 Error assessment:

To check syntactic error analysis, let $y_L(t)$ is the approximate solution obtained from Eq. (20).

$$y_L(t) = (\widetilde{y}_1(t), \widetilde{y}_2(t), \ldots, \widetilde{y}_N(t)) = (\sum_{j=0}^{L} a_{n,j}^1 \psi_{n,j}(t), \sum_{j=0}^{L} a_{n,j}^2 \psi_{n,j}(t), \ldots, \sum_{j=0}^{L} a_{n,j}^N \psi_{n,j}(t)),$$

and $y(t)$ is the exact solution:

$$y(t) = (y_1(t), y_2(t), \ldots, y_N(t)) = (\sum_{j=0}^{\infty} a_{n,j}^1 \psi_{n,j}(t), \sum_{j=0}^{\infty} a_{n,j}^2 \psi_{n,j}(t), \ldots, \sum_{j=0}^{\infty} a_{n,j}^N \psi_{n,j}(t)),$$

then we have:

$$\begin{cases} D_*^{\omega}\widetilde{y}_1(t) = g_1(t, \widetilde{y}_1(t), \widetilde{y}_2(t), \ldots, \widetilde{y}_N(t)) \\ D_*^{\omega}\widetilde{y}_2(t) = g_2(t, \widetilde{y}_1(t), \widetilde{y}_2(t), \ldots, \widetilde{y}_N(t)) \\ \vdots \\ D_*^{\omega}\widetilde{y}_N(t) = g_N(t, \widetilde{y}_1(t), \widetilde{y}_2(t), \ldots, \widetilde{y}_N(t)) \\ \widetilde{y}_i(0) = \mu_i, i = 1, 2, \ldots, N \end{cases} \quad 0 < \omega < 1, \tag{22}$$

we consider

$$e_i(t) = y_i(t) - \widetilde{y}_i(t). \tag{23}$$

Is the calculation error of $y(t)$, therefore

$$R_{i,j}(t) = \frac{1}{\Gamma(1-\omega)} \int_0^t (t-\tau)^{-\omega} y_j'(\tau) d\tau - g_i(t, y_1(t), y_2(t), ..., y_N(t)), \tag{24}$$

and

$$\widetilde{R}_{i,j}(t) = \frac{1}{\Gamma(1-\omega)} \int_0^t (t-\tau)^{-\omega} \widetilde{y}'_j(\tau) d\tau - g_i(t, \widetilde{y}_1(t), \widetilde{y}_2(t), ..., \widetilde{y}_N(t)). \tag{25}$$

Subtracting both sides of the Eq. (24) and Eq. (25),we have:

$$E_{i,j}(t) = \frac{1}{\Gamma(1-\omega)} \int_0^t (t-\tau)^{-\omega} e'_j(\tau) d\tau - [g_i(t, y_1(t), y_2(t), ..., y_N(t)) - g_i(t, \widetilde{y}_1(t), \widetilde{y}_2(t), ..., \widetilde{y}_N(t))].$$

We consider

$$\begin{aligned} G_i(t, e_1(t), e_2(t), ..., e_N(t)) &= g(t, y(t)) - g(t, y_N(t)) \\ &= g_i(t, y_1(t), y_2(t), ..., y_N(t)) - g_i(t, \widetilde{y}_1(t), \widetilde{y}_2(t), ..., \widetilde{y}_N(t)) \\ &= g_i(t, \widetilde{y}_1(t) + e_1(t), \widetilde{y}_2(t) + e_2(t), ..., \widetilde{y}_N(t) + e_N(t)) - g_i(t, \widetilde{y}_1(t), \widetilde{y}_2(t), ..., \widetilde{y}_N(t)). \end{aligned}$$

By using the Taylor expansion:

$$\begin{aligned} G_i(t, e_1(t), e_2(t), ..., e_N(t)) &\simeq e_1(t) \frac{\partial g_i(t, \widetilde{y}_1(t), \widetilde{y}_2(t), ..., \widetilde{y}_N(t))}{\partial \widetilde{y}_1(t)} \\ &+ e_2(t) \frac{\partial g_i(t, \widetilde{y}_1(t), \widetilde{y}_2(t), ..., \widetilde{y}_N(t))}{\partial \widetilde{y}_2(t)} + ... + e_N(t) \frac{\partial g_i(t, \widetilde{y}_1(t), \widetilde{y}_2(t), ..., \widetilde{y}_N(t))}{\partial \widetilde{y}_N(t)}, \end{aligned}$$

that is

$$\frac{\partial g_i(t, \widetilde{y}_1(t), \widetilde{y}_2(t), ..., \widetilde{y}_N(t))}{\partial \widetilde{y}_j(t)} = \frac{d g_i(t, \widetilde{y}_1(t), \widetilde{y}_2(t), ..., \widetilde{y}_N(t))}{dt} \times \frac{1}{\frac{d\widetilde{y}_j(t)}{dt}},$$

Where g_i and $\widetilde{y}_j(t)$ are known functions, then by solving the system of differential equations of the following fraction we can calculate the approximate error:

$$\begin{cases} D_*^{\omega} e_1(t) = G_1(t, y_1(t), y_2(t), \dots, y_N(t)) \\ D_*^{\omega} e_2(t) = G_2(t, y_1(t), y_2(t), \dots, y_N(t)) \\ \vdots & 0 < \omega < 1. \\ D_*^{\omega} e_N(t) = G_N(t, y_1(t), y_2(t) \dots, y_N(t)) \\ e_i(0) = 0, i = 1, 2, \dots, N. \end{cases} \tag{26}$$

6 Numerical solution of CD4+T cell HIV infection fractional order model

Human immunodeficiency virus (HIV) is a lentivirus (a subset of retroviruses) that is approximately spherical in shape and about 120 nanometers in diameter (about 60 times smaller than the size of a red blood cell). Attacks the immune system.Without a strong immune system, the body can not fight cancer or other infectious diseases effectively.HIV infects and kills certain white blood cells called CD4+T cells, which are an important part of the immune system. If CD4+T cells are destroyed too much, the body will no longer be able to defend itself against infections. Of course, early and timely treatment can slow or stop the progression of HIV infection.Medications can help the immune system return to a better state. The number of infected and non-infected CD4+T cells is important for measuring HIV progression and receiving the best treatment [22, 23].

Recently, various mathematical models have been proposed to study the dynamics of CD4+T cells. The model in [24] is one of them written with a system of differential equations:

$$\begin{cases} \frac{dT}{dt} = q - \eta T + rT(1 - \frac{T+1}{T_{max}}) - kVT \\ \frac{dI}{dt} = kVT - \beta I \\ \frac{dV}{dt} = \mu\beta I - \xi V & 0 < t < R < \infty. \\ T(0) = T_0, I(0) = I_0, V(0) = V_0 \end{cases} \tag{27}$$

Each of the parameters of this model is described in Table 2. Recently, many mathematicians have studied this model and proposed various numerical methods to solve it. For example,

Table 2: List of variable and parameters [18]

variable	Meaning
$T(t)$	The concentration of uninfected CD4+T in the blood
$I(t)$	The concentration of infected CD4+T in the blood
$V(t)$	The concentration of HIV virus particle in the blood
η	Turnover rate of uninfected CD+4T cells
β	Turnover rate of infected CD+4T cells
ξ	Turnover rate of HIV virus particle
$1-\frac{T+1}{T_{max}}$	Logistic growth indicator of unifected CD4+T cells
k	The infection rate of CD4+T cells by HIV virus
kVT	The incident of HIV infection of healthy CD4+T cell
μ	The number of virus particles produced by each infectedCD4+T cell during its life time
q	The generation rate of unifected CD4+T cells in the body
$\mu\beta$	The generation rate of virions through infected CD4+T cells
T_{max}	The maximal concentration of CD4+T cells in the blood
r	Tate of cells' duplication through the process of mitosis when they are stimulated by antigen and mitogen

In this paper, we consider the model presented in Eq. (27) as a form of fractional differential equations, so the model changes as follows:

$$\begin{cases} D_*^{\omega}T = q - \eta T + rT(1-\frac{T+1}{T_{max}}) - kVT \\ D_*^{\omega}I = kVT - \beta I \\ D_*^{\omega}V = \mu\beta I - \xi V \\ T(0)=T_0, I(0)=I_0, V(0)=V_0 \end{cases} \quad 0 < t < R < \infty, \quad 0 < \omega < 1 \tag{28}$$

The initial values and parameters described in the model are considered as follows:
$T_0 = 0.1, I_0 = 0, V_0 = 0.1, q = 0.1, \eta = 0.02, \beta = 0.3, r = 3, \xi = 2.4, k = 0.00027, T_{max} = 1500, \mu = 10.$
In Tables 3 ,4 and 5 we comparison $M = 15, k = 1, \gamma = 1, \omega = 1.$

Table 3: Numerical comparison for $T(t)$

t	Present method	Method in [27]	VIM [25]	LADM-Pade [26]
0	0.1	0.1	0.1	0.1
0.2	0.208808084	0.2038616561	0.2088073214	0.2088072731
0.4	0.406240543	0.3803309335	0.4061346587	0.4061052625
0.6	0.766442390	0.6954623767	0.7624530350	0.7611467713
0.8	1.414046852	1.2759624442	1.3978805880	1.3773198590
1	2.59155948	2.3832277428	2.5067466690	2.3291697610

Table 4: Numerical comparison for $I(t)$

t	Present method	Method in [27]	VIM [25]	LADM-Pade [26]
0	0	0	0	0
0.2	6.03270224e-6	0.6247872100e-5	0.6032634366e-5	0.603270728e-5
0.4	1.31583409e-5	0.1293552225e-4	0.1314878543e-4	0.131591617e-4
0.6	2.12237854e-5	0.2035267183e-4	0.2101417193e-4	0.212683688e-4
0.8	3.01774201e-5	0.2837302120e-4	0.2795130456e-4	0.300691867e-4
1	4.00378155e-5	0.3690842367e-4	0.2431562317e-4	0.398736542e-4

Table 5: Numerical comparison for $V(t)$

t	Present method	Method in [27]	VIM [25]	LADM-Pade [26]
0	0.1	0.1	0.1	0.1
0.2	0.061879843	0.06187991856	0.06187995314	0.06187996025
0.4	0.038294888	0.03829493490	0.03830820126	0.03831324883
0.6	0.023704550	0.02370431860	0.02392029257	0.02439174349
0.8	0.014680364	0.01467956982	0.01621704553	0.009967218934
1	0.009100845	0.02370431861	0.01608418711	0.003305076447

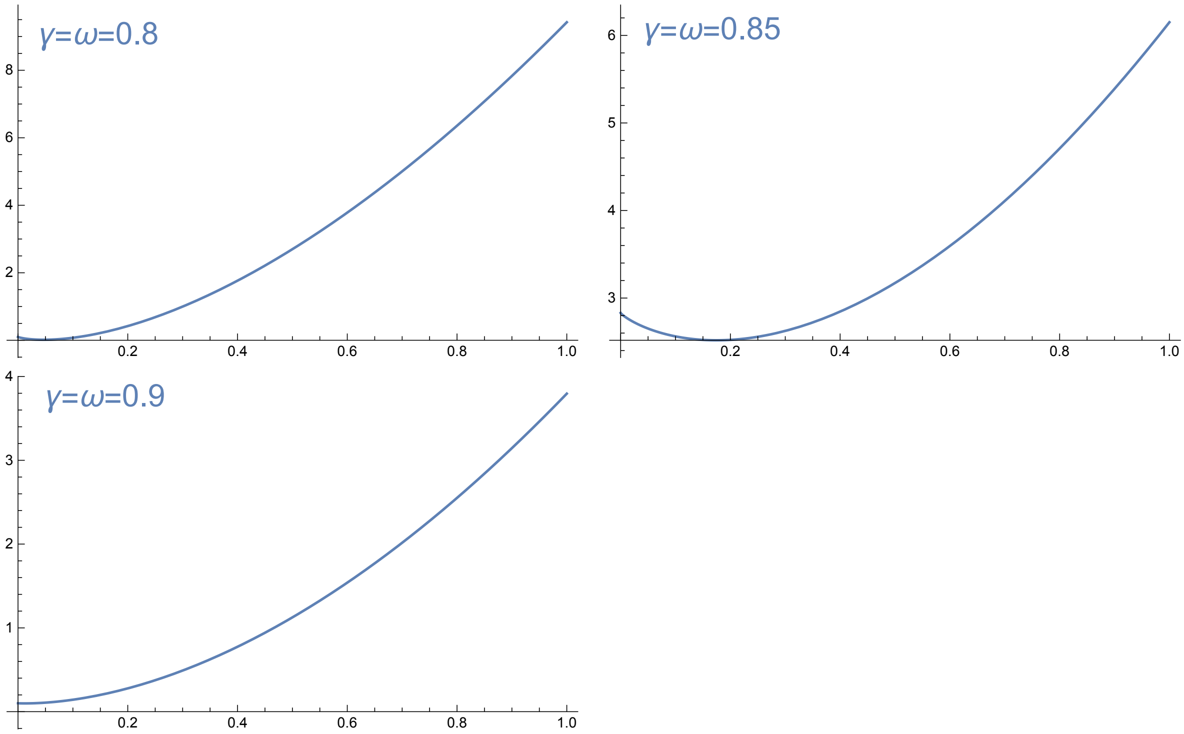

Figure 2: Numerical results for $T(t)$ by $\gamma = \omega = 0.85, 0.9, 0.95$ and $M = 3$ and $k = 1$.

In Figures 2 ,3 and 4 we cpmpare $T(t)$, $I(t)$ and $V(t)$ for $M = 3, k = 1, \gamma = \omega = 0.8, 0.85, 0.9$.
Tables 6, 7 and 8 show the values of $T(t)$, $I(t)$ and $V(t)$, for $M = 15, k = 1, \gamma = \omega = 0.75, 0.80, 0.85, 0.90, 0.95$.

7 Conclusion

The purpose of this work is to present the Müntz wavelet and use it as a basis for solving the system of fractional differential equations. The growth model of the HIV virus is definitely an important application of this method.
To continue our research, I propose solving other problems using this wavelet.

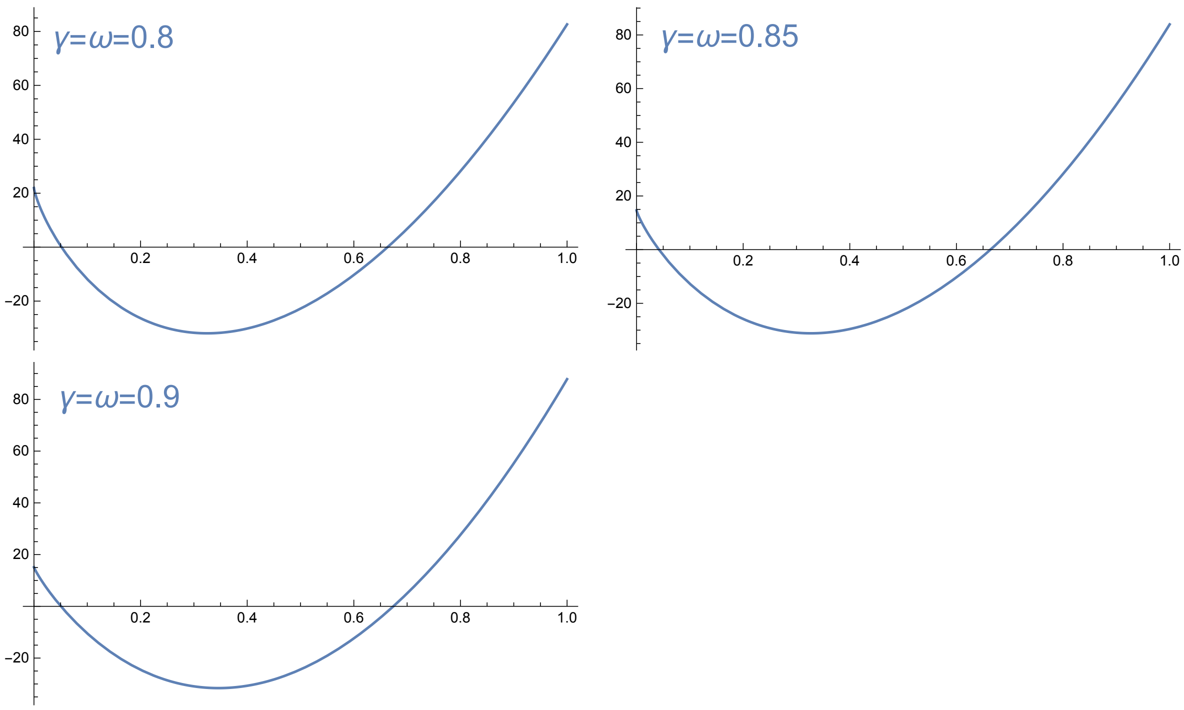

Figure 3: Numerical results for $I(t)$ by $\gamma = \omega = 0.85, 0.9, 0.95$ and $M = 3$ and $k = 1$.

Table 6: The values of $T(t)$ for $M = 15$, $k = 1$ and $\omega = \gamma$

t	$\omega = 0.75$	$\omega = 0.80$	$\omega = 0.85$	$\omega = 0.90$	$\omega = 0.95$	$\omega = 0.98$
0	0.1	0.1	0.1	0.1	0.1	0.1
0.2	0.2157344	0.2272514	0.2501158	0.2790315	0.3165184	0.3670560
0.4	0.4264249	0.4606339	0.5309877	0.6246844	0.7540629	0.9419131
0.6	0.8133480	0.8979687	1.0784967	1.3317891	1.7039759	2.2858520
0.8	1.5242915	1.8178109	2.1489168	2.7836259	3.7737946	5.4360892
1	2.8300579	3.2590105	4.2409152	5.7619568	8.2742007	12.784070

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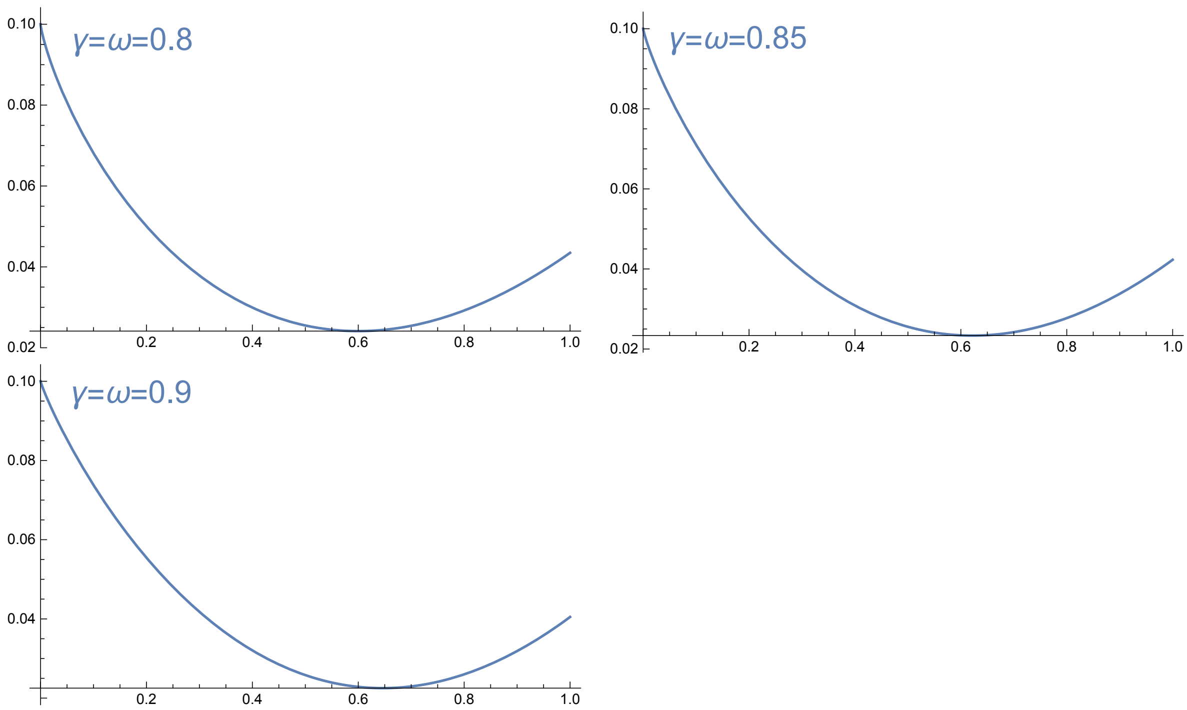

Figure 4: Numerical results for $V(t)$ by $\gamma=\omega=0.85, 0.9, 0.95$ and $M=3$ and $k=1$.

Table 7: The values of $I(t)$ for $M=15$, $k=1$ and $\omega=\gamma$

t	$\omega=0.75$	$\omega=0.80$	$\omega=0.85$	$\omega=0.90$	$\omega=0.95$	$\omega=0.98$
0	0	0	0	0	0	0
0.2	6.3363901e-6	6.8290042e-6	7.7692349e-6	8.9043994e-6	1.0315362e-5	1.2151979e-5
0.4	1.3753198e-5	1.4746964e-5	1.6753726e-5	1.9385818e-5	2.2989039e-5	2.8204527e-5
0.6	2.2294605e-5	2.4155672e-5	2.8164952e-5	3.3864401e-5	4.2343235e-5	5.5742298e-5
0.8	3.2111872e-5	3.5585428e-5	4.3464569e-5	5.5403434e-5	7.4433150e-5	1.0695931e-4
1	4.3511235e-5	4.9905893e-5	6.5030221e-5	8.9263409e-5	1.3048453e-4	2.0660978e-4

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Table 8: The values of $V(t)$ for $M = 15$, $k = 1$ and $\omega = \gamma$

t	$\omega = 0.75$	$\omega = 0.80$	$\omega = 0.85$	$\omega = 0.90$	$\omega = 0.95$	$\omega = 0.98$
0	0.1	0.1	0.1	0.1	0.1	0.1
0.2	0.0608295	0.0592643	0.0567019	0.0542264	0.0518676	0.0496485
0.4	0.0377472	0.0369869	0.0358789	0.0349582	0.0342057	0.0335988
0.6	0.0237253	0.0237983	0.02401462	0.0243227	0.0246948	0.0251079
0.8	0.0151098	0.0157587	0.0168397	0.0179038	0.0189386	0.0199388
1	0.0097708	0.0107509	0.0123158	0.0137958	0.0151977	0.0165367

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