

The fractional-order differential model of the pollution for a system of lakes

Bijan Hasani Lichae*, Mehran Nemati

Abstract

Pollution produced by human is a serious danger to the planet Earth in our time. In recent decades, a lot of efforts have been made to monitor and control the pollution to save the environment. In this paper, the fractional-order differential model of the pollution for a system of lakes has been introduced. There are three components; the amount of the pollution in lake 1, x , the amount of the pollution in lake 2, y , and the amount of the pollution in lake 3, z , at any time $t \geq 0$. The aim of this work is to get numerical solution of the proposed fractional-order model by Laplace Adomian decomposition method (LADM). The numerical solution has been obtained in a series form. The solution has been compared with the solutions of some other numerical approaches. The results illustrate the ability and accuracy of the present method. The Caputo form has been applied for fractional derivatives. All of computations have been done in Maple.

Keywords: Numerical solution, Laplace Adomian decomposition method, System of fractional-order differential equations of the pollution, Caputo fractional derivative.

1. Introduction

Many natural phenomena in biology, medicine, physics and other branches of science can be explained by system of differential equations [1-7]. In [8], J. Hoggard presented a model of the pollution for a system of lakes. Proposed model is simulated for three lakes with interconnecting channels. Each lake has been assumed to be a large compartment. First, a pollutant enters the first lake and then infects two other lakes (See Fig. 1). The function $p(t)$ denotes the rate of the pollutant that enters the lake 1 for $t \geq 0$. The rate of the pollutant may be vary or constant with any time.

*Corresponding author

Bijan Hasani Lichae¹
E-mail address: Bi.Hasani@iau.ac.ir, b_hasani2004@yahoo.com.

Mehran Nemati²
E-mail address: Mehran.Nemati@iau.ac.ir.

¹ *Department of Mathematics, fouman and shaft Branch, Islamic Azad University, fouman, Iran*

² *Department of Mathematics, roudbar Branch, Islamic Azad University, roudbar, Iran*

It is important to know the amount of the pollutant in each lake at per time. The amount of the pollution in lake 1, 2, and 3 are defined $x(t)$, $y(t)$, and $z(t)$, respectively. Constants F_{ji} denote the flow rate of water from lake i into lake j , V_i denote the volume of water in lake i , and $r_{ji}(t)$ denote the flow rate of contamination from lake i into lake j at any time t . If there is no flow of water between Lake i into Lake j , then $F_{ji} = 0$. The flux of pollution from lake i into lake j , called $r_{ji}(t)$, is defined as follows

$$\begin{cases} r_{j1}(t) = \frac{F_{j1}x(t)}{V_1}, \\ r_{j2}(t) = \frac{F_{j2}y(t)}{V_2}, \\ r_{j3}(t) = \frac{F_{j3}z(t)}{V_3}. \end{cases}$$

In other words, $r_{ji}(t)$ determines the rate of concentration of contamination in Lake i flows into lake j .

The referred model is modeled as the following simple principle:

Rate of change of contamination = Input rate of contamination - output rate of contamination.

So, the proposed model will be obtained as the following form

$$\frac{dx(t)}{dt} = \frac{F_{13}}{V_3} z(t) + p(t) - \frac{F_{31}}{V_1} x(t) - \frac{F_{21}}{V_1} x(t), \quad (1)$$

$$\frac{dy(t)}{dt} = \frac{F_{21}}{V_1} x(t) - \frac{F_{32}}{V_2} y(t), \quad t \geq 0$$

$$\frac{dz(t)}{dt} = \frac{F_{31}}{V_1} x(t) + \frac{F_{32}}{V_2} y(t) - \frac{F_{13}}{V_3} z(t),$$

with initial conditions $x(0) = 0$, $y(0) = 0$, and $z(0) = 0$, which means the lakes are not contaminant from the beginning. In order to keep constant the volume of water in each lake, the following conditions have been assumed:

$$\text{Lake 1: } F_{13} = F_{21} + F_{31},$$

$$\text{Lake 2: } F_{21} = F_{32},$$

$$\text{Lake 3: } F_{31} + F_{32} = F_{13}, \text{ (see [9]).}$$

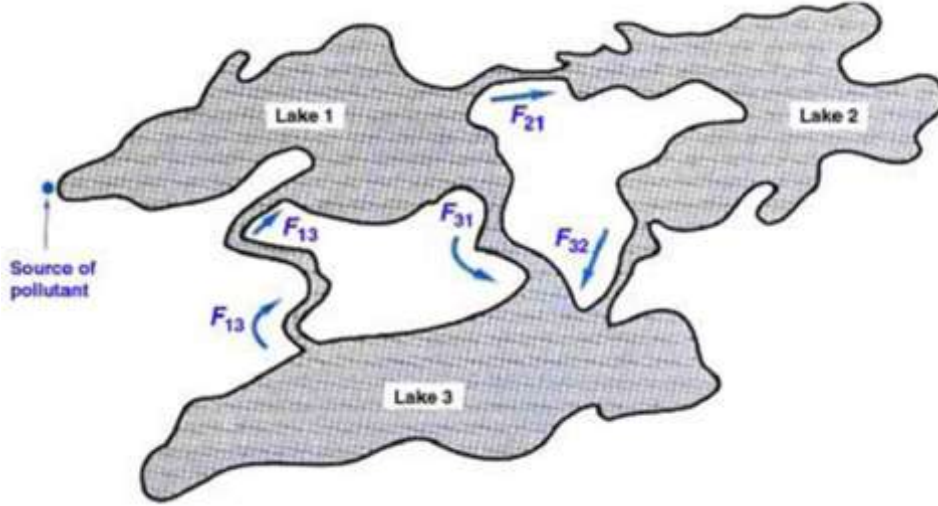


Fig. 1. System of three lakes with interconnecting channels [9].

Fig.1 shows system of three lakes with interconnecting channels. The source of pollutant and the constants F_{ji} have been marked.

System of differential equations (1) has been solved by Adomian and RK4 methods [9]. In [10], numerical solution of (1) has been obtained by variational iteration method. In [11], M. Merdan by modified differential transformation method has been achieved the numerical solution of (1). A collocation approach has been introduced to solve (1) in [12]. Polluted lakes system (1) has been solved by PIM in [13].

In this work, we introduce fractional-order of (1) and solve it by LADM. Generalizing system of differential equations (1) to system of fractional-order differential equations (2) indicates the novelty of the paper. Fractional order differential equations are related to fractals [14-16], save memory on themselves [17], has freedom on degree of derivative operator, and can explain many phenomena in sciences [18-20]. We introduce fractional-order of (1) as follows:

$$\begin{aligned}
 D^{\alpha_1} x(t) &= \frac{F_{13}}{V_3} z(t) + p(t) - \frac{F_{31}}{V_1} x(t) - \frac{F_{21}}{V_1} x(t), \\
 D^{\alpha_2} y(t) &= \frac{F_{21}}{V_1} x(t) - \frac{F_{32}}{V_2} y(t), \\
 D^{\alpha_3} z(t) &= \frac{F_{31}}{V_1} x(t) + \frac{F_{32}}{V_2} y(t) - \frac{F_{13}}{V_3} z(t),
 \end{aligned}
 \tag{2}$$

$t \geq 0$

with the same initial conditions, where $0 < \alpha_i \leq 1$, $i = 1,2,3$. When $\alpha \rightarrow 1$, $D^\alpha x(t) \rightarrow Dx(t)$, therefore system of fractional-order differential equations of the pollution (2) reduces to traditional model (1). The development of numerical methods, especially for the solution of fractional differential equations, has led to an increasing interest in fractional calculus [21]. Some numerical methods that can be implemented to solve system of fractional-order differential equations are: optimal homotopy asymptotic [22-25], predictor-corrector [26], homotopy analysis [27, 28], variational iteration [10], generalized Euler [17], Laplace Adomian [18, 29], homotopy perturbation [30, 31], differential transformation [11], and Rung-Kutta method [32]. The rest of this paper is organized as follows: in section 2 a brief review of fractional calculus has been presented. Section 3 will be devoted to solving (2) by LADM in three phases. In Section 4, the convergence of the method will be discussed. In the last section, we present the conclusion.

2. Fractional calculus

The purpose of this section is to remind the reader of some fundamental preliminaries of fractional calculus.

Definition 2.1. The fractional integral of Riemann-Liouville type of order α for a function $f: (0, \infty) \rightarrow R$ is defined by

$$J^\alpha f(s) = \frac{1}{\Gamma(\alpha)} \int_0^s (s-t)^{\alpha-1} f(t) dt, \quad (3)$$

where $\alpha \in (0, \infty)$, (See[20]).

Definition 2.2. The Caputo fractional derivative of a function $f: (0, \infty) \rightarrow R$ on the closed interval $[0, T]$ is defined as

$$D^\alpha f(s) = \frac{1}{\Gamma(m-\alpha)} \int_0^s (s-t)^{m-\alpha-1} f^{(m)}(t) dt, \quad m = [\alpha] + 1. \quad (4)$$

where $[\alpha]$ is the integer part of α .

Definition 2.3. The Caputo fractional derivative has another presentation that can be shown as follows

$$D^\alpha f(s) = J^{m-\alpha}(D^m f(s)), \quad (5)$$

(See[20]).

Lemma 2.1. If $\alpha \in (0, \infty)$, then the following result holds for fractional calculus

$$J^\alpha [D^\alpha f](s) = f(s) + \sum_{j=0}^{m-1} \frac{f^{(j)}(0)}{j!} s^j, \quad (6)$$

where $m = [\alpha] + 1$.

Proof. (See[20, 33]).

Definition 2.4. We remind that the Laplace transform of Caputo fractional derivative is defined as follows

$$\mathcal{L}\{D^\alpha f(t)\} = s^\alpha F(s) - \sum_{j=0}^{m-1} s^{\alpha-k-1} f^{(j)}(0), \quad m-1 < \alpha < m, m \in \mathbb{N}, \quad (7)$$

3. Solution of system of fractional-order differential equations (2)

In this section, system of fractional-order differential equations (2) will be solved by LADM and the results will be compared with the results of some other numerical methods. Because of the proposed model (2) to better describe system of polluted lakes, three types of input models such as, impulse, step, and sinusoidal has been considered [34].

3.1. Impulse input

The impulse input model describes pollutants that released very quickly into the lake. The impulse input functions are zero everywhere except when contamination enters the lake. Impulse input functions have a spike. The spike indicates the time at which the pollution has been evacuated.

For an example, suppose a barrel of oil drains into the lake suddenly, therefore, we assume that input function is equal to 100 at interval of 0 to 10. The values of parameters in (2) are reported in [34],

$$V_1 = 2900 \text{ mi}^3, V_2 = 850 \text{ mi}^3, V_3 = 1180 \text{ mi}^3,$$

$F_{21} = 18 \text{ mi}^3/\text{year}$, $F_{32} = 18 \text{ mi}^3/\text{year}$, $F_{31} = 20 \text{ mi}^3/\text{year}$, $F_{13} = 38 \text{ mi}^3/\text{year}$.

So, model (2) will be obtained as the following form

$$\begin{aligned} D^{\alpha_1}x(t) &= \frac{38}{1180}z(t) + 100 - \frac{20}{2900}x(t) - \frac{18}{2900}x(t), \\ D^{\alpha_2}y(t) &= \frac{18}{2900}x(t) - \frac{18}{850}y(t), \quad t \geq 0 \\ D^{\alpha_3}z(t) &= \frac{20}{2900}x(t) + \frac{18}{850}y(t) - \frac{38}{1180}z(t), \end{aligned} \quad (3)$$

with the same initial conditions, where $0 < \alpha_i \leq 1$, $i = 1, 2, 3$.

Using Laplace transform on both sides of each equation of (3) gives

$$\begin{cases} \mathcal{L}\{D^{\alpha_1}x(t)\} = \mathcal{L}\left\{\frac{38}{1180}z(t) + 100 - \frac{38}{2900}x(t)\right\}, \\ \mathcal{L}\{D^{\alpha_2}y(t)\} = \mathcal{L}\left\{\frac{18}{2900}x(t) - \frac{18}{850}y(t)\right\}, \\ \mathcal{L}\{D^{\alpha_3}z(t)\} = \mathcal{L}\left\{\frac{20}{2900}x(t) + \frac{18}{850}y(t) - \frac{38}{1180}z(t)\right\}, \end{cases} \quad (9)$$

which implies that

$$\begin{cases} s^{\alpha_1}\mathcal{L}\{x(t)\} - s^{\alpha_1-1}x(0) = \frac{38}{1180}\mathcal{L}\{z(t)\} + \frac{100}{s} - \frac{38}{2900}\mathcal{L}\{x(t)\}, \\ s^{\alpha_2}\mathcal{L}\{y(t)\} - s^{\alpha_2-1}y(0) = \frac{18}{2900}\mathcal{L}\{x(t)\} - \frac{18}{850}\mathcal{L}\{y(t)\}, \\ s^{\alpha_3}\mathcal{L}\{z(t)\} - s^{\alpha_3-1}z(0) = \frac{20}{2900}\mathcal{L}\{x(t)\} + \frac{18}{850}\mathcal{L}\{y(t)\} - \frac{38}{1180}\mathcal{L}\{z(t)\}, \end{cases} \quad (10)$$

Substitution of initial conditions in (10) results in

$$\begin{cases} \mathcal{L}\{x(t)\} = \frac{38}{1180} \frac{1}{s^{\alpha_1}} \mathcal{L}\{z(t)\} + \frac{100}{s^{\alpha_1+1}} - \frac{38}{2900} \frac{1}{s^{\alpha_1}} \mathcal{L}\{x(t)\}, \\ \mathcal{L}\{y(t)\} = \frac{18}{2900} \frac{1}{s^{\alpha_2}} \mathcal{L}\{x(t)\} - \frac{18}{850} \frac{1}{s^{\alpha_2}} \mathcal{L}\{y(t)\}, \\ \mathcal{L}\{z(t)\} = \frac{20}{2900} \frac{1}{s^{\alpha_3}} \mathcal{L}\{x(t)\} + \frac{18}{850} \frac{1}{s^{\alpha_3}} \mathcal{L}\{y(t)\} - \frac{38}{1180} \frac{1}{s^{\alpha_3}} \mathcal{L}\{z(t)\}, \end{cases} \quad (11)$$

Applying inverse Laplace transform reads to

$$\begin{cases} x(t) = \frac{38}{1180} \mathcal{L}^{-1} \left[\frac{1}{s^{\alpha_1}} \mathcal{L}\{z(t)\} \right] + \frac{100t^{\alpha_1}}{\Gamma(\alpha_1+1)} - \frac{38}{2900} \mathcal{L}^{-1} \left[\frac{1}{s^{\alpha_1}} \mathcal{L}\{x(t)\} \right], \\ y(t) = \frac{18}{2900} \mathcal{L}^{-1} \left[\frac{1}{s^{\alpha_2}} \mathcal{L}\{x(t)\} \right] - \frac{18}{850} \mathcal{L}^{-1} \left[\frac{1}{s^{\alpha_2}} \mathcal{L}\{y(t)\} \right], \\ z(t) = \frac{20}{2900} \mathcal{L}^{-1} \left[\frac{1}{s^{\alpha_3}} \mathcal{L}\{x(t)\} \right] + \frac{18}{850} \mathcal{L}^{-1} \left[\frac{1}{s^{\alpha_3}} \mathcal{L}\{y(t)\} \right] - \frac{38}{1180} \mathcal{L}^{-1} \left[\frac{1}{s^{\alpha_3}} \mathcal{L}\{z(t)\} \right]. \end{cases} \quad (11)$$

Let's consider x , y , and z , are as the following series

$$x = \sum_{i=0}^{\infty} x_i, \quad y = \sum_{i=0}^{\infty} y_i, \quad z = \sum_{i=0}^{\infty} z_i. \quad (12)$$

To compute the Adomian polynomials, using an alternate algorithm [35], the following recursive sequence would be derived:

$$\begin{cases} x_0(t) = \frac{100t^{\alpha_1}}{\Gamma(\alpha_1+1)}, \\ y_0(t) = 0, \\ z_0(t) = 0, \end{cases} \quad (13)$$

$$\begin{cases} x_{n+1}(t) = \frac{38}{1180} \mathcal{L}^{-1} \left[\frac{1}{s^{\alpha_1}} \mathcal{L}\{z_n(t)\} \right] - \frac{38}{2900} \mathcal{L}^{-1} \left[\frac{1}{s^{\alpha_1}} \mathcal{L}\{x_n(t)\} \right], \\ y_{n+1}(t) = \frac{18}{2900} \mathcal{L}^{-1} \left[\frac{1}{s^{\alpha_2}} \mathcal{L}\{x_n(t)\} \right] - \frac{18}{850} \mathcal{L}^{-1} \left[\frac{1}{s^{\alpha_2}} \mathcal{L}\{y_n(t)\} \right], \\ z_{n+1}(t) = \frac{20}{2900} \mathcal{L}^{-1} \left[\frac{1}{s^{\alpha_3}} \mathcal{L}\{x_n(t)\} \right] + \frac{18}{850} \mathcal{L}^{-1} \left[\frac{1}{s^{\alpha_3}} \mathcal{L}\{y_n(t)\} \right] - \frac{38}{1180} \mathcal{L}^{-1} \left[\frac{1}{s^{\alpha_3}} \mathcal{L}\{z_n(t)\} \right]. \end{cases} \quad (14)$$

We will calculate four terms of infinite series of x , y , and z , as an approximate solution.

$$\begin{cases} x(t) = \sum_{i=0}^{\infty} x_i(t) \approx x_0(t) + x_1(t) + x_2(t) + x_3(t), \\ y(t) = \sum_{i=0}^{\infty} y_i(t) \approx y_0(t) + y_1(t) + y_2(t) + y_3(t), \\ z(t) = \sum_{i=0}^{\infty} z_i(t) \approx z_0(t) + z_1(t) + z_2(t) + z_3(t). \end{cases} \quad (15)$$

Let's take α_1 , α_2 , and α_3 equal to α , so the approximate solution of system (3) would be derived as follows

$$\begin{cases} x(t) = \frac{100t^{\alpha}}{\Gamma(\alpha+1)} - \frac{38}{29} \frac{t^{2\alpha}}{\Gamma(2\alpha+1)} + \frac{48849}{1240475} \frac{t^{3\alpha}}{\Gamma(3\alpha+1)} - \frac{1982603583}{1804084816250} \frac{t^{4\alpha}}{\Gamma(4\alpha+1)}, \\ y(t) = \frac{18}{29} \frac{t^{2\alpha}}{\Gamma(2\alpha+1)} - \frac{1521}{71485} \frac{t^{3\alpha}}{\Gamma(3\alpha+1)} + \frac{361275039}{519821048750} \frac{t^{4\alpha}}{\Gamma(4\alpha+1)}, \\ z(t) = \frac{20}{29} \frac{t^{2\alpha}}{\Gamma(2\alpha+1)} - \frac{381738}{21088075} \frac{t^{3\alpha}}{\Gamma(3\alpha+1)} + \frac{1238903361}{3066944187625} \frac{t^{4\alpha}}{\Gamma(4\alpha+1)}. \end{cases} \quad (16)$$

When $\alpha = 1$, the solution of (3) will be obtained as the following form

$$\begin{cases} x(t) = \frac{100t}{\Gamma(2)} - \frac{38}{29} \frac{t^2}{\Gamma(3)} + \frac{48849}{1240475} \frac{t^3}{\Gamma(4)} - \frac{1982603583}{1804084816250} \frac{t^4}{\Gamma(5)}, \\ y(t) = \frac{18}{29} \frac{t^2}{\Gamma(3)} - \frac{1521}{71485} \frac{t^3}{\Gamma(4)} + \frac{361275039}{519821048750} \frac{t^4}{\Gamma(5)}, \\ z(t) = \frac{20}{29} \frac{t^2}{\Gamma(3)} - \frac{381738}{21088075} \frac{t^3}{\Gamma(4)} + \frac{1238903361}{3066944187625} \frac{t^4}{\Gamma(5)}. \end{cases} \quad (17)$$

3.2. Step input

The step input model describes pollutants that are added to the lake at steady concentration. Before time zero, the pollutant concentration is zero. After time zero, the pollutant enters into the lake suddenly and input contaminant increases with constant rate. For an example, suppose a manufacturing plant begins to produce at time zero and dumps raw sewage on a constant rate, therefore, we assume input function is equal to $100t$. So, model (3) with parameters that given in subsection 3.1 will be obtained as the following form

$$D^{\alpha_1} x(t) = \frac{38}{1180} z(t) + 100t - \frac{20}{2900} x(t) - \frac{18}{2900} x(t), \quad (17)$$

$$D^{\alpha_2} y(t) = \frac{18}{2900} x(t) - \frac{18}{850} y(t), \quad t \geq 0$$

$$D^{\alpha_3} z(t) = \frac{20}{2900} x(t) + \frac{18}{850} y(t) - \frac{38}{1180} z(t),$$

$$x(0) = 0, y(0) = 0, z(0) = 0.$$

According to subsection 3.1, we derive

$$\begin{cases} x(t) = \frac{38}{1180} \mathcal{L}^{-1} \left[\frac{1}{s^{\alpha_1}} \mathcal{L}\{z(t)\} \right] + \frac{100t^{\alpha_1+1}}{\Gamma(\alpha_1+2)} - \frac{38}{2900} \mathcal{L}^{-1} \left[\frac{1}{s^{\alpha_1}} \mathcal{L}\{x(t)\} \right], \\ y(t) = \frac{18}{2900} \mathcal{L}^{-1} \left[\frac{1}{s^{\alpha_2}} \mathcal{L}\{x(t)\} \right] - \frac{18}{850} \mathcal{L}^{-1} \left[\frac{1}{s^{\alpha_2}} \mathcal{L}\{y(t)\} \right], \\ z(t) = \frac{20}{2900} \mathcal{L}^{-1} \left[\frac{1}{s^{\alpha_3}} \mathcal{L}\{x(t)\} \right] + \frac{18}{850} \mathcal{L}^{-1} \left[\frac{1}{s^{\alpha_3}} \mathcal{L}\{y(t)\} \right] - \frac{38}{1180} \mathcal{L}^{-1} \left[\frac{1}{s^{\alpha_3}} \mathcal{L}\{z(t)\} \right]. \end{cases} \quad (18)$$

To calculate approximate solution, using an alternate algorithm for Adomian polynomials [35], the following recursive sequence would be derived:

$$\begin{cases} x_0(t) = \frac{100t^{\alpha_1+1}}{\Gamma(\alpha_1+2)}, \\ y_0(t) = 0, \\ z_0(t) = 0, \end{cases} \quad (19)$$

$$\begin{cases} x_{n+1}(t) = \frac{38}{1180} \mathcal{L}^{-1} \left[\frac{1}{s^{\alpha_1}} \mathcal{L}\{z_n(t)\} \right] - \frac{38}{2900} \mathcal{L}^{-1} \left[\frac{1}{s^{\alpha_1}} \mathcal{L}\{x_n(t)\} \right], \\ y_{n+1}(t) = \frac{18}{2900} \mathcal{L}^{-1} \left[\frac{1}{s^{\alpha_2}} \mathcal{L}\{x_n(t)\} \right] - \frac{18}{850} \mathcal{L}^{-1} \left[\frac{1}{s^{\alpha_2}} \mathcal{L}\{y_n(t)\} \right], \\ z_{n+1}(t) = \frac{20}{2900} \mathcal{L}^{-1} \left[\frac{1}{s^{\alpha_3}} \mathcal{L}\{x_n(t)\} \right] + \frac{18}{850} \mathcal{L}^{-1} \left[\frac{1}{s^{\alpha_3}} \mathcal{L}\{y_n(t)\} \right] - \frac{38}{1180} \mathcal{L}^{-1} \left[\frac{1}{s^{\alpha_3}} \mathcal{L}\{z_n(t)\} \right]. \end{cases} \quad (20)$$

We take α_1 , α_2 , and α_3 equal to α . We will calculate four terms of infinite series of x , y , and z , as an approximate solution as the following form:

$$\begin{cases} x(t) = \frac{100 t^{\alpha+1}}{\Gamma(\alpha+2)} - \frac{38 t^{2\alpha+1}}{29 \Gamma(2\alpha+2)} + \frac{48849 t^{3\alpha+1}}{1240475 \Gamma(3\alpha+2)} - \frac{1982603583 t^{4\alpha+1}}{1804084816250 \Gamma(4\alpha+2)}, \\ y(t) = \frac{18 t^{2\alpha+1}}{29 \Gamma(2\alpha+2)} - \frac{1521 t^{3\alpha+1}}{71485 \Gamma(3\alpha+2)} + \frac{361275039 t^{4\alpha+1}}{519821048750 \Gamma(4\alpha+2)}, \\ z(t) = \frac{20 t^{2\alpha+1}}{29 \Gamma(2\alpha+2)} - \frac{381738 t^{3\alpha+1}}{21088075 \Gamma(3\alpha+2)} + \frac{1238903361 t^{4\alpha+1}}{3066944187625 \Gamma(4\alpha+2)}. \end{cases} \quad (21)$$

When $\alpha = 1$, we get the solution of (17) as follows

$$\begin{cases} x(t) = \frac{100 t^2}{\Gamma(3)} - \frac{38 t^{2\alpha+1}}{29 \Gamma(4)} + \frac{48849 t^4}{1240475 \Gamma(5)} - \frac{1982603583 t^5}{1804084816250 \Gamma(6)}, \\ y(t) = \frac{18 t^3}{29 \Gamma(4)} - \frac{1521 t^4}{71485 \Gamma(5)} + \frac{361275039 t^5}{519821048750 \Gamma(6)}, \\ z(t) = \frac{20 t^3}{29 \Gamma(4)} - \frac{381738 t^4}{21088075 \Gamma(5)} + \frac{1238903361 t^5}{3066944187625 \Gamma(6)}. \end{cases} \quad (22)$$

3.3. Sinusoidal input

The step input model describes pollutants that are entered to the lake periodically. For an example, we assume that $p(t) = \alpha + \beta \sin \frac{2\pi t}{T}$, where α is the average input concentration of pollutant and β is the

amplitude of fluctuations. Let's consider $\alpha = \beta$, and $T = 2\pi$, therefore, we have $p(t) = 1 + \sin t$. So, model (3) with parameters that given in subsection 3.1 will be obtained as follows

$$D^{\alpha_1} x(t) = \frac{38}{1180} z(t) + 1 + \sin t - \frac{20}{2900} x(t) - \frac{18}{2900} x(t), \quad (22)$$

$$D^{\alpha_2} y(t) = \frac{18}{2900} x(t) - \frac{18}{850} y(t), \quad t \geq 0$$

$$D^{\alpha_3} z(t) = \frac{20}{2900} x(t) + \frac{18}{850} y(t) - \frac{38}{1180} z(t),$$

with the same initial conditions as previous subsections.

According to 3.1,

$$\begin{cases} x(t) = \frac{38}{1180} \mathcal{L}^{-1} \left[\frac{1}{s^{\alpha_1}} \mathcal{L}\{z(t)\} \right] + \frac{t^{\alpha_1+1}}{\Gamma(\alpha_1+1)} + \sum_{k=0}^{\infty} (-1)^k \frac{t^{\alpha_1+2k+1}}{\Gamma(\alpha_1+2k+1)} - \frac{38}{2900} \mathcal{L}^{-1} \left[\frac{1}{s^{\alpha_1}} \mathcal{L}\{x(t)\} \right], \\ y(t) = \frac{18}{2900} \mathcal{L}^{-1} \left[\frac{1}{s^{\alpha_2}} \mathcal{L}\{x(t)\} \right] - \frac{18}{850} \mathcal{L}^{-1} \left[\frac{1}{s^{\alpha_2}} \mathcal{L}\{y(t)\} \right], \\ z(t) = \frac{20}{2900} \mathcal{L}^{-1} \left[\frac{1}{s^{\alpha_3}} \mathcal{L}\{x(t)\} \right] + \frac{18}{850} \mathcal{L}^{-1} \left[\frac{1}{s^{\alpha_3}} \mathcal{L}\{y(t)\} \right] - \frac{38}{1180} \mathcal{L}^{-1} \left[\frac{1}{s^{\alpha_3}} \mathcal{L}\{z(t)\} \right]. \end{cases} \quad (23)$$

The recursive sequence would be derived:

$$\begin{cases} x_0(t) = \frac{t^{\alpha_1+1}}{\Gamma(\alpha_1+1)} + \sum_{k=0}^{\infty} (-1)^k \frac{t^{\alpha_1+2k+1}}{\Gamma(\alpha_1+2k+1)}, \\ y_0(t) = 0, \\ z_0(t) = 0, \end{cases} \quad (24)$$

$$\begin{cases} x_{n+1}(t) = \frac{38}{1180} \mathcal{L}^{-1} \left[\frac{1}{s^{\alpha_1}} \mathcal{L}\{z_n(t)\} \right] - \frac{38}{2900} \mathcal{L}^{-1} \left[\frac{1}{s^{\alpha_1}} \mathcal{L}\{x_n(t)\} \right], \\ y_{n+1}(t) = \frac{18}{2900} \mathcal{L}^{-1} \left[\frac{1}{s^{\alpha_2}} \mathcal{L}\{x_n(t)\} \right] - \frac{18}{850} \mathcal{L}^{-1} \left[\frac{1}{s^{\alpha_2}} \mathcal{L}\{y_n(t)\} \right], \\ z_{n+1}(t) = \frac{20}{2900} \mathcal{L}^{-1} \left[\frac{1}{s^{\alpha_3}} \mathcal{L}\{x_n(t)\} \right] + \frac{18}{850} \mathcal{L}^{-1} \left[\frac{1}{s^{\alpha_3}} \mathcal{L}\{y_n(t)\} \right] - \frac{38}{1180} \mathcal{L}^{-1} \left[\frac{1}{s^{\alpha_3}} \mathcal{L}\{z_n(t)\} \right]. \end{cases} \quad (25)$$

Let's take α_1, α_2 , and α_3 equal to α .

for $n = 0$,

$$\begin{cases} x_1(t) = -\frac{38}{2900} \left\{ \frac{t^{2\alpha}}{\Gamma(2\alpha+1)} + \sum_{k=0}^{\infty} (-1)^k \frac{t^{2\alpha+2k+1}}{\Gamma(2\alpha+2k+2)} \right\}, \\ y_1(t) = \frac{18}{2900} \left\{ \frac{t^{2\alpha}}{\Gamma(2\alpha+1)} + \sum_{k=0}^{\infty} (-1)^k \frac{t^{2\alpha+2k+1}}{\Gamma(2\alpha+2k+2)} \right\}, \\ z_1(t) = \frac{20}{2900} \left\{ \frac{t^{2\alpha}}{\Gamma(2\alpha+1)} + \sum_{k=0}^{\infty} (-1)^k \frac{t^{2\alpha+2k+1}}{\Gamma(2\alpha+2k+2)} \right\}, \end{cases} \quad (26)$$

for $n = 1$,

$$\begin{cases} x_2(t) = \frac{48849}{124047500} \left\{ \frac{t^{3\alpha}}{\Gamma(3\alpha+1)} + \sum_{k=0}^{\infty} (-1)^k \frac{t^{3\alpha+2k+1}}{\Gamma(3\alpha+2k+2)} \right\}, \\ y_2(t) = -\frac{1521}{7148500} \left\{ \frac{t^{3\alpha}}{\Gamma(3\alpha+1)} + \sum_{k=0}^{\infty} (-1)^k \frac{t^{3\alpha+2k+1}}{\Gamma(3\alpha+2k+2)} \right\}, \\ z_2(t) = -\frac{190869}{1054403750} \left\{ \frac{t^{3\alpha}}{\Gamma(3\alpha+1)} + \sum_{k=0}^{\infty} (-1)^k \frac{t^{3\alpha+2k+1}}{\Gamma(3\alpha+2k+2)} \right\}, \end{cases} \quad (27)$$

for $n = 2$,

$$\begin{cases} x_3(t) = -\frac{1982603583}{180408481625000} \left\{ \frac{t^{4\alpha}}{\Gamma(4\alpha+1)} + \sum_{k=0}^{\infty} (-1)^k \frac{t^{4\alpha+2k+1}}{\Gamma(4\alpha+2k+2)} \right\}, \\ y_3(t) = \frac{361275039}{51982104875000} \left\{ \frac{t^{4\alpha}}{\Gamma(4\alpha+1)} + \sum_{k=0}^{\infty} (-1)^k \frac{t^{4\alpha+2k+1}}{\Gamma(4\alpha+2k+2)} \right\}, \\ z_3(t) = \frac{1238903361}{306694418762500} \left\{ \frac{t^{4\alpha}}{\Gamma(4\alpha+1)} + \sum_{k=0}^{\infty} (-1)^k \frac{t^{4\alpha+2k+1}}{\Gamma(4\alpha+2k+2)} \right\}, \end{cases} \quad (28)$$

so the approximate solution of (22), by calculating four terms of infinite series of x , y , and z , is obtained as the following form:

$$\begin{cases} x(t) = \frac{t^{\alpha+1}}{\Gamma(\alpha+1)} + \sum_{k=0}^{\infty} (-1)^k \frac{t^{\alpha+2k+1}}{\Gamma(\alpha+2k+1)} - \frac{38}{2900} \left\{ \frac{t^{2\alpha}}{\Gamma(2\alpha+1)} + \sum_{k=0}^{\infty} (-1)^k \frac{t^{2\alpha+2k+1}}{\Gamma(2\alpha+2k+2)} \right\}, \\ y(t) = \frac{18}{2900} \left\{ \frac{t^{2\alpha}}{\Gamma(2\alpha+1)} + \sum_{k=0}^{\infty} (-1)^k \frac{t^{2\alpha+2k+1}}{\Gamma(2\alpha+2k+2)} \right\} - \frac{1521}{7148500} \left\{ \frac{t^{3\alpha}}{\Gamma(3\alpha+1)} + \sum_{k=0}^{\infty} (-1)^k \frac{t^{3\alpha+2k+1}}{\Gamma(3\alpha+2k+2)} \right\} \\ z(t) = \frac{20}{2900} \left\{ \frac{t^{2\alpha}}{\Gamma(2\alpha+1)} + \sum_{k=0}^{\infty} (-1)^k \frac{t^{2\alpha+2k+1}}{\Gamma(2\alpha+2k+2)} \right\} - \frac{190869}{1054403750} \left\{ \frac{t^{3\alpha}}{\Gamma(3\alpha+1)} + \sum_{k=0}^{\infty} (-1)^k \frac{t^{3\alpha+2k+1}}{\Gamma(3\alpha+2k+2)} \right\} \end{cases} \quad (29)$$

When $\alpha = 1$, the solution of (22) is as follows

$$\begin{cases} x(t) = \frac{t^2}{\Gamma(2)} + \sum_{k=0}^{\infty} (-1)^k \frac{t^{2k+2}}{\Gamma(2k+2)} - \frac{38}{2900} \left\{ \frac{t^2}{\Gamma(3)} + \sum_{k=0}^{\infty} (-1)^k \frac{t^{2k+3}}{\Gamma(2k+4)} \right\}, \\ y(t) = \frac{18}{2900} \left\{ \frac{t^2}{\Gamma(3)} + \sum_{k=0}^{\infty} (-1)^k \frac{t^{2k+3}}{\Gamma(2k+4)} \right\} - \frac{1521}{7148500} \left\{ \frac{t^3}{\Gamma(4)} + \sum_{k=0}^{\infty} (-1)^k \frac{t^{2k+4}}{\Gamma(2k+5)} \right\} \\ z(t) = \frac{20}{2900} \left\{ \frac{t^2}{\Gamma(3)} + \sum_{k=0}^{\infty} (-1)^k \frac{t^{2k+3}}{\Gamma(2k+4)} \right\} - \frac{190869}{1054403750} \left\{ \frac{t^3}{\Gamma(4)} + \sum_{k=0}^{\infty} (-1)^k \frac{t^{2k+4}}{\Gamma(2k+5)} \right\} \end{cases}$$

4. Convergence analysis of the method

In this section, convergence of the proposed method, using the idea presented in [36], is studied.

5. Conclusion

In this paper, a fractional-order model of HIV-1 with three components has been introduced. By applying Laplace transform and Adomian decomposition method (LADM) which is a strong approach to compute

numerical solution of fractional differential equations, we gain an approximate solution of the proposed model. The accuracy of the proposed approach has been made it a reliable method. The result of LADM has been compared with the results of some other methods such as GEM, HAM, RK4 [17], and HPM [30]. The results are presented in Tables1-3. When $\alpha \rightarrow 1$, then $D^\alpha x(t) \rightarrow Dx(t)$, therefore the fractional-order of presented model reduces to traditional model. Because of the fact that obtaining the exact solution for system of fractional equation is difficult or impossible, our suggestion for future research is solving them by such numerical methods.

References

1. Culshaw, R.V. and S. Ruan, *A delay-differential equation model of HIV infection of CD4+ T-cells*. Mathematical biosciences, 2000. **165**(1): p. 27-39.
2. Arafa, A., S. Rida, and M. Khalil, *A fractional-order model of HIV infection with drug therapy effect*. Journal of the Egyptian Mathematical Society, 2014. **22**(3): p. 538-543.
3. Hindmarsh, J.L. and R. Rose, *A model of neuronal bursting using three coupled first order differential equations*. Proc. R. Soc. Lond. B, 1984. **221**(1222): p. 87-102.
4. Chen, T., H.L. He, and G.M. Church, *Modeling gene expression with differential equations*, in *Biocomputing'99*. 1999, World Scientific. p. 29-40.
5. Lichae, B.H., J. Biazar, and Z. Ayati, *The Fractional Differential Model of HIV-1 Infection of CD4+ T-Cells with Description of the Effect of Antiviral Drug Treatment*. Computational and mathematical methods in medicine, 2019. **2019**.
6. Lichae, B.H., J. Biazar, and Z. Ayati, *A class of Runge–Kutta methods for nonlinear Volterra integral equations of the second kind with singular kernels*. Advances in Difference Equations, 2018. **2018**(1): p. 349.
7. Hasani Lichae, B., J. Biazar, and Z. Ayati, *Asymptotic decomposition method for fractional order Riccati differential equation*. Computational Methods for Differential Equations, 2021. **9**(1): p. 63-78.
8. Hoggard, J., *Lake Pollution Modeling*, Virginia Tech, 2007.
9. Biazar, J., L. Farrokhi, and M.R. Islam, *Modeling the pollution of a system of lakes*. Applied Mathematics and Computation, 2006. **178**(2): p. 423-430.
10. Biazar, J., M. Shahbala, and H. Ebrahimi, *VIM for solving the pollution problem of a system of lakes*. Journal of Control Science and Engineering, 2010. **2010**: p. 8.
11. Merdan, M., *A new application of modified differential transformation method for modelling the pollution of a system of lakes*. Selçuk Journal of Applied Mathematics, 2010. **11**(2): p. 27-40.
12. Yüzbaşı, Ş., N. Şahin, and M. Sezer, *A collocation approach to solving the model of pollution for a system of lakes*. Mathematical and Computer Modelling, 2012. **55**(3-4): p. 330-341.
13. Khalid, M., et al., *Solving polluted lakes system by using perturbation-iteration method*. International Journal of Computer Applications, 2015. **114**(4).
14. Tatom, F.B., *The relationship between fractional calculus and fractals*. Fractals, 1995. **3**(01): p. 217-229.
15. Heymans, N. and J.-C. Bauwens, *Fractal rheological models and fractional differential equations for viscoelastic behavior*. Rheologica Acta, 1994. **33**(3): p. 210-219.
16. Giona, M. and H.E. Roman, *Fractional diffusion equation on fractals: One-dimensional case and asymptotic behaviour*. Journal of physics A: Mathematical and general, 1992. **25**(8): p. 2093.

17. Arafa, A., S. Rida, and M. Khalil, *The effect of anti-viral drug treatment of human immunodeficiency virus type 1 (HIV-1) described by a fractional order model*. Applied Mathematical Modelling, 2013. **37**(4): p. 2189-2196.
18. Haq, F., et al., *Numerical solution of fractional order smoking model via laplace Adomian decomposition method*. Alexandria Engineering Journal, 2017.
19. Ertürk, V.S., Z.M. Odibat, and S. Momani, *An approximate solution of a fractional order differential equation model of human T-cell lymphotropic virus I (HTLV-I) infection of T-cells*. Computers & Mathematics with applications, 2011. **62**(3): p. 996-1002.
20. Diethelm, K., *The analysis of fractional differential equations: An application-oriented exposition using differential operators of Caputo type*. 2010: Springer.
21. Garrappa, R., *Numerical solution of fractional differential equations: a survey and a software tutorial*. Mathematics, 2018. **6**(2): p. 16.
22. Marinca, V. and N. Herişanu, *Optimal Homotopy Asymptotic Method*, in *The Optimal Homotopy Asymptotic Method*. 2015, Springer. p. 9-22.
23. Marinca, V. and N. Herişanu, *Application of optimal homotopy asymptotic method for solving nonlinear equations arising in heat transfer*. International Communications in Heat and Mass Transfer, 2008. **35**(6): p. 710-715.
24. Marinca, V. and N. Herişanu, *Determination of periodic solutions for the motion of a particle on a rotating parabola by means of the optimal homotopy asymptotic method*. Journal of Sound and Vibration, 2010. **329**(9): p. 1450-1459.
25. Marinca, V., N. Herişanu, and I. Nemeş, *Optimal homotopy asymptotic method with application to thin film flow*. Central European Journal of Physics, 2008. **6**(3): p. 648.
26. Diethelm, K., N.J. Ford, and A.D. Freed, *A predictor-corrector approach for the numerical solution of fractional differential equations*. Nonlinear Dynamics, 2002. **29**(1-4): p. 3-22.
27. Ghoreishi, M., A.M. Ismail, and A. Alomari, *Application of the homotopy analysis method for solving a model for HIV infection of CD4+ T-cells*. Mathematical and Computer Modelling, 2011. **54**(11): p. 3007-3015.
28. Liao, S.J., *The proposed homotopy analysis technique for the solution of nonlinear problems*, 1992, Ph. D. Thesis, Shanghai Jiao Tong University.
29. Ongun, M.Y., *The Laplace adomian decomposition method for solving a model for HIV infection of CD4+ T cells*. Mathematical and Computer Modelling, 2011. **53**(5): p. 597-603.
30. Merdan, M. and T. Khan, *Homotopy perturbation method for solving viral dynamical model*. CÜ Fen-Edebiyat Fakültesi, Fen Bilimleri Dergisi, 2010. **31**: p. 65-77.
31. Odibat, Z. and S. Momani, *Modified homotopy perturbation method: application to quadratic Riccati differential equation of fractional order*. Chaos, Solitons & Fractals, 2008. **36**(1): p. 167-174.
32. Hong, J., C. Huang, and X. Wang, *Symplectic Runge-Kutta Methods for Hamiltonian Systems Driven by Gaussian Rough Paths*. arXiv preprint arXiv:1704.04144, 2017.
33. Kilbas, A.A., H.M. Srivastava, and J.J. Trujillo, *Preface*, 2006, Elsevier.
34. Aguirre, J. and D. Tully, *Lake pollution model*, 1999, May.
35. Biazar, J., et al., *An alternate algorithm for computing Adomian polynomials in special cases*. Applied Mathematics and Computation, 2003. **138**(2-3): p. 523-529.
36. Ayati, Z. and J. Biazar, *On the convergence of Homotopy perturbation method*. Journal of the Egyptian Mathematical Society, 2015. **23**(2): p. 424-428.