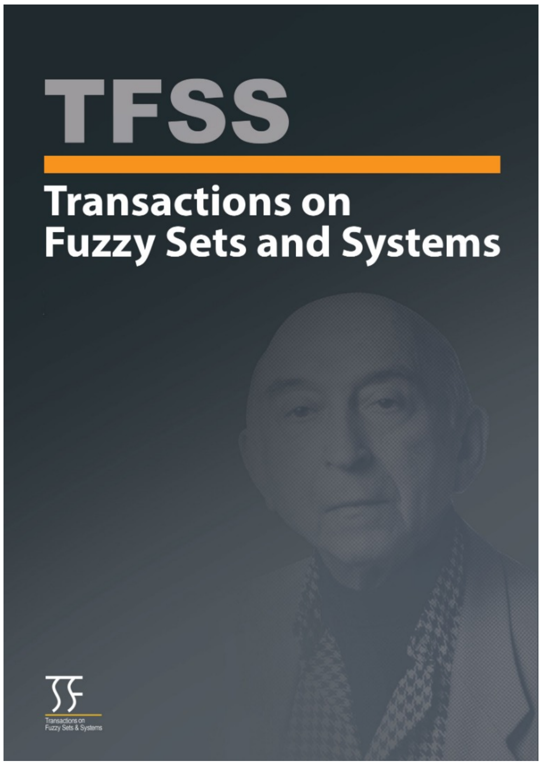
Transactions on Fuzzy Sets and Systems



Transactions on Fuzzy Sets and Systems

ISSN: **2821-0131**

https://sanad.iau.ir/journal/tfss/

Fuzzy Filters of Pre-ordered Residuated Systems

Vol.4, No.1, (2025), 30-41. DOI: https://doi.org/10.71602/tfss.2025.1129288

Author(s):

Young Bae Jun, Department of Mathematics Education, Gyeongsang National University, Jinju 52828, Korea. E-mail: skywine@gmail.com

Transactions on Fuzzy Sets and Systems (TFSS)
URL: https://sanad.iau.ir/journal/tfss/
Online ISSN: 2821-0131
Vol.4, No.1, (2025), 30-41
DOI: https://doi.org/10.71602/tfss.2025.1129288

Article Type: Original Research Article

Fuzzy Filters of Pre-ordered Residuated Systems

Young Bae Jun*

(This paper is dedicated to Professor "John N. Mordeson" on the occasion of his 91st birthday.)

Abstract. Using the concept of fuzzy points, the notion of fuzzy filters in pre-ordered residuated systems is introduced, and their relevant properties are investigated and analyzed. Characterization of fuzzy filters are displayed. Fuzzy filters are formed using filters. The concepts of positive set, $\in_t$-set, (extended) q_t-set are defined and the conditions under which they become filters are explored. The concept of fuzzy filter with thresholds is introduced and related properties are investigated.

AMS Subject Classification 2020: 08A02; 06A11; 06B75; 08A72
Keywords and Phrases: Residuated relational system, Pre-ordered residuated system, Filter, $\in_t$-set, (Extended) q_t-set, Positive set.

1 Introduction

The concept of a residuated relational system introduced by Bonzio et al. [1] is a mathematical structure used in the study of ordered algebraic systems, particularly in the fields of logic, lattice theory, and category theory. These systems generalize certain aspects of algebraic structures like lattices and posets, with a focus on the relationship between operations and their adjoints, often in the context of residuation. They developed the concept of a pre-ordered residuated system, which is nothing but a residuated relational system whose relation is pre-order, i.e., reflexive and transitive. Their work like this is based on generalizing the concept of residuated poset, by replacing the usual partial order to a pre-order. Romano [2, 3, 4, 5] called the pre-ordered residuated system a quasi-ordered residuated system. He introduced and analyzed the notion of filters in pre-ordered residuated systems. The purpose of this paper is to study the filter of a pre-ordered residuated system using the fuzzy set theory. For this, we will use the concept of fuzzy points. We introduce the concept of fuzzy filters in a pre-ordered residuated system, and investigate their relevant properties. We consider characterizations of fuzzy filter. We construct $\in_t$-set, (extended) q_t-set, positive set, etc., and explore the conditions under which these can be filters.

2 Preliminaries

Definition 2.1 ([1])**.** Let $(X, \odot, \to, 1)$ be an algebra of type (2,2,0) and let R be a binary operation on X. A structure $\mathbb{X} := (X, \odot, \to, 1, R)$ is called a *residuated relational system* if the following three conditions are valid.

*Corresponding Author: Young Bae Jun, Email: skywine@gmail.com, ORCID: 0000-0002-0181-8969
Received: 14 June 2024; **Revised**: 14 August 2024; **Accepted**: 14 August 2024; **Available Online**: 25 August 2024; **Published Online**: 7 May 2025.

How to cite: Jun YB. Fuzzy filters of pre-ordered residuated systems. *Transactions on Fuzzy Sets and Systems.* 2025; 4(1): 30-41. DOI: https://doi.org/10.71602/tfss.2025.1129288

(i) $(X, \odot, 1)$ is a commutative monoid,

(ii) $(\forall \mathfrak{a} \in X)\ ((\mathfrak{a}, 1) \in R)$,

(iii) $(\forall \mathfrak{a}, \mathfrak{b}, \mathfrak{c} \in X)\ ((\mathfrak{a} \odot \mathfrak{b}, \mathfrak{c}) \in R) \Leftrightarrow (\mathfrak{a}, \mathfrak{b} \to \mathfrak{c}) \in R)$.

Let $\mathbb{X} := (X,\ \odot,\ \to,\ 1,\ R)$ be a residuated relational system. For every element y of X, consider the following two mappings:

$$f_y : X \to X,\ x \mapsto x \odot y \text{ and } g_y : X \to X,\ x \mapsto y \to x.$$

Proposition 2.2 ([1]). *Every residuated relational system* $\mathbb{X} := (X,\ \odot,\ \to,\ 1,\ R)$ *satisfies:*

$$(\forall x, y \in X)(g_x(y) = 1\ \Rightarrow\ (x, y) \in R). \tag{1}$$
$$(\forall x \in X)((x, g_1(1)) \in R). \tag{2}$$
$$(\forall x \in X)((1, g_x(1))) \in R). \tag{3}$$
$$(\forall x, y, z \in X)(g_x(y) = 1\ \Rightarrow\ (f_x(z), y) \in R). \tag{4}$$
$$(\forall x, y \in X)((x, g_y(1)) \in R). \tag{5}$$

Moreover, if R is reflexive, then

$$(\forall x \in X)((1, g_x(x)) \in R). \tag{6}$$
$$(\forall x, y \in X)(f_x((g_x(y)), y) \in R). \tag{7}$$
$$(\forall x, y \in X)((x, g_y(f_y(x))) \in R). \tag{8}$$
$$(\forall x, y \in X)((x, g_1(x)) \in R,\ (g_1(x), x) \in R). \tag{9}$$
$$(\forall x, y \in X)(x, g_{g_x(y)}(y)) \in R). \tag{10}$$

Also, if R is antisymmetric, then

$$(\forall x, y \in X)((x, y) \in R\ \Leftrightarrow\ g_x(y) = 1). \tag{11}$$
$$\textit{If } R \textit{ is also reflexive, then } (f_y(x), x) \in R \textit{ and } (f_y(x), y) \in R. \tag{12}$$

Recall that a binary relation "R" on a set X is said to be *pre-order* if it is reflexive and transitive. Note that the pre-order relation is sometimes called the quasi-order relation.

Definition 2.3 ([1, 2]). A residuated relational system $\mathbb{X} := (X,\ \odot,\ \to,\ 1,\ R)$ is called a *pre-ordered residuated system* if R is a pre-order relation on X.

The pre-ordered residuated system $\mathbb{X} := (X,\ \odot,\ \to,\ 1,\ R)$ will be denoted by $\mathbb{X} := (X,\ \odot,\ \to,\ 1,\ \lesssim)$.

Definition 2.4 ([2]). Let $\mathbb{X} := (X,\ \odot,\ \to,\ 1,\ \lesssim)$ be a pre-ordered residuated system. A subset F of X is called a *filter* of $\mathbb{X}$ if it satisfies:

$$(x, y \in X)(x \in F,\ x \lesssim y\ \Rightarrow\ y \in F), \tag{13}$$
$$(x, y \in X)(x \in F,\ g_x(y) \in F\ \Rightarrow\ y \in F). \tag{14}$$

A fuzzy set $\eth$ in a set X of the form

$$\eth(\mathfrak{b}) := \begin{cases} t \in (0, 1] & \text{if } \mathfrak{b} = \mathfrak{a}, \\ 0 & \text{if } \mathfrak{b} \neq \mathfrak{a}, \end{cases}$$

is said to be a *fuzzy point* with support $\mathfrak{a}$ and value t and is denoted by $\langle \mathfrak{a}_t \rangle$.

For a fuzzy set $\eth$ in a set X, we say that a fuzzy point $\langle \mathfrak{a}_t \rangle$ is

(i) *contained* in $\eth$, denoted by $\langle \mathfrak{a}_t \rangle \in \eth$, (see [6]) if $\eth(\mathfrak{a}) \geq t$.

(ii) *quasi-coincident* with $\eth$, denoted by $\langle \mathfrak{a}_t \rangle \, q \, \eth$, (see [6]) if $\eth(\mathfrak{a}) + t > 1$.

If a fuzzy point $\langle \mathfrak{a}_t \rangle$ is contained in $\eth$ or is quasi-coincident with $\eth$, we denote it $\langle \mathfrak{a}_t \rangle \in\!\vee q \; \eth$. If a fuzzy point $\langle \mathfrak{a}_t \rangle$ is contained in $\eth$ and is quasi-coincident with $\eth$, we denote it $\langle \mathfrak{a}_t \rangle \in\!\wedge q \, \eth$. If $\langle \mathfrak{a}_t \rangle \, \alpha \, \eth$ is not established for $\alpha \in \{\in, q, \in\!\vee q, \in\!\wedge q\}$, it is denoted by $\langle \mathfrak{a}_t \rangle \, \overline{\alpha} \, \eth$.

Given $t \in (0, 1]$ and a fuzzy set $\eth$ in a set X, consider the following sets

$$(\eth, t)_\in := \{\mathfrak{a} \in X \mid \langle \mathfrak{a}_t \rangle \in \eth\} \text{ and } (\eth, t)_q := \{\mathfrak{a} \in X \mid \langle \mathfrak{a}_t \rangle \, q \, \eth\}$$

which are called an $\in_t$*-set* and a q_t*-set* of $\eth$, respectively, in X. Also, we consider the set

$$(\eth, t)_{\in\vee q} := \{\mathfrak{a} \in X \mid \langle \mathfrak{a}_t \rangle \in\!\vee q \; \eth\}$$

which is called the $t(\in\!\vee q)$*-set* of $\eth$.

It is clear that $(\eth, t)_{\in\vee q} = (\eth, t)_\in \cup (\eth, t)_q$.

3 Fuzzy Filters

In what follows, let $\mathbb{X} := (X, \odot, \rightarrow, 1, \lesssim)$ denote a pre-ordered residuated system, and it will be simply written by $\mathbb{X}$ only.

First, we introduce a central concept that will be used throughout the paper.

Definition 3.1. A fuzzy set $\eth$ in X is called a *fuzzy filter* of $\mathbb{X}$ if its nonempty $\in_t$-set $(\eth, t)_\in$ is a filter of $\mathbb{X}$ for all $t \in (0, 1]$.

Example 3.2. Let $X := (-\infty, 1] \subset \mathbb{R}$ (the set of real numbers). If we define two binary operations "$\odot$" and "$\rightarrow$" on X as follows:

$$x \odot y = \min\{x, y\} \text{ and } x \rightarrow y = \begin{cases} 1 & \text{if } x \leq y, \\ y & \text{if } x > y, \end{cases}$$

for all $x, y \in X$, then $\mathbb{X} := (X, \odot, \rightarrow, 1, \leq)$ is a pre-ordered residuated system (see [5]). Let $\eth$ be a fuzzy set in X given by

$$\eth : X \rightarrow [0, 1], \; x \mapsto \begin{cases} 0.78 & \text{if } x \in (0, 1], \\ 0.62 & \text{if } x \in (-3, 0], \\ 0.37 & \text{otherwise.} \end{cases}$$

Then $\eth$ is a fuzzy filter of $\mathbb{X} := (X, \odot, \rightarrow, 1, \leq)$.

Example 3.3. Let $X = \{b_1, b_2, b_3, b_4\}$ be a set and two binary operations "$\odot$" and "$\rightarrow$" on X are given as follows:

$\odot$	b_1	b_2	b_3	b_4
b_1	b_1	b_2	b_3	b_4
b_2	b_2	b_2	b_2	b_4
b_3	b_3	b_2	b_2	b_4
b_4	b_4	b_4	b_4	b_4

$\rightarrow$	b_1	b_2	b_3	b_4
b_1	b_1	b_2	b_3	b_4
b_2	b_1	b_1	b_1	b_1
b_3	b_1	b_2	b_1	b_4
b_4	b_1	b_2	b_3	b_1

We give a binary relation "$\lesssim$" as follows:

$$\lesssim := \{(b_1,b_1),(b_2,b_2),(b_3,b_3),(b_4,b_4),(b_4,b_1),(b_3,b_1),(b_2,b_1),(b_2,b_3),(b_2,b_4)\}.$$

Then $\mathbb{X} := (X, \odot, \rightarrow, b_1, \lesssim)$ is a pre-ordered residuated system (see [7]). Let $\eth$ be a fuzzy set in X given by

$$\eth : X \rightarrow [0,1],\ x \mapsto \begin{cases} \frac{1}{n} & \text{if } x = b_1, \\ \frac{1}{4n} & \text{if } x = b_2, \\ \frac{1}{2n} & \text{if } x = b_3, \\ \frac{1}{3n} & \text{if } x = b_4, \end{cases}$$

where n is a natural number. Then $\eth$ is a fuzzy filter of $\mathbb{X} := (X, \odot, \rightarrow, b_1, \leq)$.

We discuss the characterization of fuzzy filters.

Theorem 3.4. *A fuzzy set $\eth$ in X is a fuzzy filter of $\mathbb{X}$ if and only if it satisfies:*

$$(\forall x, y \in X)(x \lesssim y \ \Rightarrow\ \eth(x) \leq \eth(y)), \tag{15}$$

$$(\forall x, y \in X)(\eth(y) \geq \min\{\eth(x), \eth(g_x(y))\}). \tag{16}$$

Proof. Assume that $\eth$ is a fuzzy filter of $\mathbb{X}$. Then its nonempty $\in_t$-set $(\eth, t)_\in$ is a filter of $\mathbb{X}$ for all $t \in (0,1]$. If (15) is not valid, then there exists $\mathfrak{a}, \mathfrak{b} \in X$ such that $\mathfrak{a} \lesssim \mathfrak{b}$ and $\eth(\mathfrak{a}) > \eth(\mathfrak{b})$. Then $\mathfrak{a} \in (\eth, \eth(\mathfrak{a}))_\in$, but $\mathfrak{b} \notin (\eth, \eth(\mathfrak{a}))_\in$ which is a contradiction. Hence $\eth(x) \leq \eth(y)$ for all $x, y \in X$ with $x \lesssim y$. Suppose that (16) is false. Then $\eth(\mathfrak{b}) < \min\{\eth(\mathfrak{a}), \eth(g_\mathfrak{a}(\mathfrak{b}))\}$ for some $\mathfrak{a}, \mathfrak{b} \in X$. If we take $t := \min\{\eth(\mathfrak{a}), \eth(g_\mathfrak{a}(\mathfrak{b}))\}$, then $\mathfrak{a} \in (\eth, t)_\in$, $g_\mathfrak{a}(\mathfrak{b}) \in (\eth, t)_\in$ and $\mathfrak{b} \notin (\eth, t)_\in$. This is a contradiction, and thus $\eth(y) \geq \min\{\eth(x), \eth(g_x(y))\}$ for all $x, y \in X$.

Conversely, let $\eth$ be a fuzzy set in X that satisfies (15) and (16). Let $x, y \in X$. If $x \in (\eth, t)_\in$ and $x \lesssim y$, then $t \leq \eth(x) \leq \eth(y)$ by (15), i.e., $\langle y_t \rangle \in \eth$. Thus $y \in (\eth, t)_\in$. If $x \in (\eth, t)_\in$ and $g_x(y) \in (\eth, t)_\in$, then

$$\eth(y) \geq \min\{\eth(x), \eth(g_x(y))\} \geq t$$

by (16) and so $y \in (\eth, t)_\in$. Hence $(\eth, t)_\in$ is a filter of $\mathbb{X}$ for all $t \in (0,1]$, and therefore $\eth$ is a fuzzy filter of $\mathbb{X}$. □

Theorem 3.5. *In $\mathbb{X}$, a fuzzy set $\eth$ in X satisfies (15) if and only if the following assertion is valid.*

$$(\forall x, y, z \in X)(\forall t \in (0,1])(f_y(x) \in (\eth, t)_\in,\ x \lesssim g_y(z) \ \Rightarrow\ z \in (\eth, t)_\in). \tag{17}$$

Proof. Assume that $\eth$ satisfies (15) and let $x, y, z \in X$ be such that $x \lesssim g_y(z)$ and $f_y(x) \in (\eth, t)_\in$ for all $t \in (0,1]$. Then $f_y(x) \lesssim z$ by Definition 2.1(iii), and so $\eth(z) \geq \eth(f_y(x)) \geq t$ by (15). Hence $z \in (\eth, t)_\in$.

Conversely, let $\eth$ be a fuzzy set in X that satisfies (17). In the proof of Theorem 3.4, we can observe that $\eth$ satisfies (15) if and only if $\eth$ satisfies:

$$(\forall x, y \in X)(\forall t \in (0,1])(x \in (\eth, t)_\in,\ x \lesssim y \ \Rightarrow\ y \in (\eth, t)_\in).$$

Let $x, y \in X$ be such that $x \lesssim y$ and $x \in (\eth, t)_\in$ for all $t \in (0,1]$. Then $f_1(x) = x \in (\eth, t)_\in$ and $x \lesssim g_1(y)$. It follows from (17) that $y \in (\eth, t)_\in$. Therefore $\eth$ satisfies (15). □

Proposition 3.6. *In $\mathbb{X}$, if a fuzzy set $\eth$ in X satisfies (15), then*

$$(\forall x \in X)(\forall t \in (0,1])(x \in (\eth, t)_\in \ \Leftrightarrow\ g_1(x) \in (\eth, t)_\in), \tag{18}$$

or equivalently,

$$(\forall x \in X)(\forall t \in (0,1])(\eth(x) \geq t \ \Leftrightarrow\ \eth(g_1(x)) \geq t). \tag{19}$$

Proof. Let $\eth$ be a fuzzy set in X that satisfies (15). Then by the proof process of Theorem 3.4, we know the following:

$$x \in (\eth, t)_{\in},\ x \lesssim y \ \Rightarrow\ y \in (\eth, t)_{\in}$$

for all $x, y \in X$ and $t \in (0, 1]$. Using (9) leads to $x \lesssim g_1(x)$ and $g_1(x) \lesssim x$. It follows that if $x \in (\eth, t)_{\in}$ (resp. $g_1(x) \in (\eth, t)_{\in}$), then $g_1(x) \in (\eth, t)_{\in}$ (resp., $x \in (\eth, t)_{\in}$). Hence (18) is valid. □

Corollary 3.7. *Every fuzzy filter* $\eth$ *of* $\mathbb{X}$ *satisfies the condition* (18).

Lemma 3.8 ([1])**.** *Every pre-ordered residuated system* $\mathbb{X} := (X, \odot, \to, 1, \lesssim)$ *satisfies:*

$$(\forall x, y \in X)(f_y(x) \lesssim x,\ f_y(x) \lesssim y). \tag{20}$$

Proposition 3.9. *In* $\mathbb{X}$*, if a fuzzy set* $\eth$ *in* X *satisfies* (15)*, then*

$$(\forall t \in (0, 1])((\eth, t)_{\in} \neq \emptyset \ \Rightarrow\ 1 \in (\eth, t)_{\in}). \tag{21}$$
$$(\forall x, y \in X)(\forall t \in (0, 1])(f_y(x) \in (\eth, t)_{\in} \ \Rightarrow\ x \in (\eth, t)_{\in},\ y \in (\eth, t)_{\in}). \tag{22}$$
$$(\forall x, y \in X)(\forall t \in (0, 1])(x, y \in (\eth, t)_{\in},\ x \lesssim y \ \Rightarrow\ g_x(y) \in (\eth, t)_{\in}). \tag{23}$$
$$(\forall x, y \in X)(\forall t \in (0, 1])(1 \in (\eth, t)_{\in},\ x \lesssim y \ \Rightarrow\ g_x(y) \in (\eth, t)_{\in}). \tag{24}$$

Proof. Assume that $(\eth, t)_{\in} \neq \emptyset$ for all $t \in (0, 1]$, and let $x \in (\eth, t)_{\in}$. Since $x \lesssim 1$ by Definition 2.1(ii), it follows from (15) that $\eth(1) \geq \eth(x) \geq t$. Hence $1 \in (\eth, t)_{\in}$. Let $x, y \in X$ and $t \in (0, 1]$ be such that $f_y(x) \in (\eth, t)_{\in}$. Then $\eth(f_y(x)) \geq t$. Since $f_y(x) \lesssim x$ and $f_y(x) \lesssim y$ by (20), it follows from (15) that $\eth(x) \geq \eth(f_y(x)) \geq t$ and $\eth(y) \geq \eth(f_y(x)) \geq t$, that is, $\langle x_t \rangle \in \eth$ and $\langle y_t \rangle \in \eth$. Hence $x \in (\eth, t)_{\in}$ and $y \in (\eth, t)_{\in}$, and so (22) is valid. Let $x, y \in X$ and $t \in (0, 1]$ be such that $x, y \in (\eth, t)_{\in}$ and $x \lesssim y$. Since $f_x(x) \lesssim x$ by (20), we have $f_x(x) \lesssim y$ by the transitivity of $\lesssim$. Thus $x \lesssim g_x(y)$ by Definition 2.1(iii), which implies from (15) that $\eth(g_x(y)) \geq \eth(x) \geq t$. Hence $g_x(y) \in (\eth, t)_{\in}$. Suppose that $1 \in (\eth, t)_{\in}$ for all $t \in (0, 1]$ and let $x, y \in X$ be such that $x \lesssim y$. Then $f_x(1) = x \lesssim y$, and so $1 \lesssim g_x(y)$ by Definition 2.1(iii). Using (15) leads to $\eth(g_x(y)) \geq \eth(1) \geq t$, and so $g_x(y) \in (\eth, t)_{\in}$. □

Corollary 3.10. *Every fuzzy filter* $\eth$ *of* $\mathbb{X}$ *satisfies the four conditions* (21)*,* (22)*,* (23) *and* (24)*.*

Theorem 3.11. *For every nonempty subset* F *of* X*, consider a fuzzy set* $\eth_F$ *in* X *which is defined by*

$$\eth_F : X \to [0, 1],\ x \mapsto \begin{cases} s_1 & \text{if } x \in F, \\ s_2 & \text{otherwise} \end{cases}$$

where $s_1 > s_2$ *in* $[0, 1]$*. Then* $\eth_F$ *is a fuzzy filter of* $\mathbb{X}$ *if and only if* F *is a filter of* $\mathbb{X}$*.*

Proof. Assume that $\eth_F$ is a fuzzy filter of $\mathbb{X}$. Let $x, y \in X$. If $x \in F$ and $x \lesssim y$, then $\eth_F(y) \geq \eth_F(x) = s_1$ by (15), and so $\eth_F(y) = s_1$. Thus $y \in F$. If $x \in F$ and $g_x(y) \in F$, then $\eth_F(x) = s_1$ and $\eth_F(g_x(y)) = s_1$. Using (16) leads to $\eth_F(y) \geq \min\{\eth_F(x), \eth_F(g_x(y))\} = s_1$, and so $\eth_F(y) = s_1$. Thus $y \in F$. Therefore F is a filter of $\mathbb{X}$.

Conversely, suppose that F is a filter of $\mathbb{X}$. For every $x, y \in X$ with $x \lesssim y$, if $x \in F$, then $y \in F$ and so $\eth_F(y) = s_1 = \eth_F(x)$. If $x \notin F$, then $\eth_F(x) = s_2 < \eth_F(y)$. Let $x, y \in X$. If $x \in F$ and $g_x(y) \in F$, then $y \in F$ and thus $\eth_F(y) = s_1 = \min\{\eth_F(x), g_x(y)\}$. If $x \notin F$ or $g_x(y) \notin F$, then $\eth_F(x) = s_2$ or $\eth_F(g_x(y)) = s_2$. Hence $\eth_F(y) \geq s_2 = \min\{\eth_F(x), \eth_F(g_x(y))\}$. Therefore $\eth_F$ is a fuzzy filter of $\mathbb{X}$ by Theorem 3.4 □

Let $\eth$ be a non-constant fuzzy set in X and we construct the next set called *positive set*.

$$X_0 := \{x \in X \mid \eth(x) \neq 0\}. \tag{25}$$

It is clear that $X_0 \neq \emptyset$. We explore conditions for the positive set of $\eth$ to be a filter.

Theorem 3.12. *If $\eth$ is a non-constant fuzzy filter of $\mathbb{X}$, then its positive set is a filter of $\mathbb{X}$.*

Proof. Let $\eth$ be a non-constant fuzzy filter of $\mathbb{X}$. Let $x, y \in X$. If $x \in X_0$ and $x \lesssim y$, then $\eth(y) \geq \eth(x) \neq 0$ by (15), and so $y \in X_0$. If $x \in X_0$ and $g_x(y) \in X_0$, then $\eth(y) \geq \min\{\eth(x), \eth(g_x(y))\} \neq 0$ by (16). Hence $y \in X_0$, and therefore X_0 is a filter of $\mathbb{X}$. □

In the following example, we can see that the converse of Theorem 3.12 is not true in general.

Example 3.13. Consider the pre-ordered residuated system $\mathbb{X} := (X, \odot, \rightarrow, b_1, \lesssim)$ in Example 3.3. Let $\eth$ be a fuzzy set in X given by

$$\eth : X \rightarrow [0,1],\ x \mapsto \begin{cases} \frac{0.2}{k} & \text{if } x = b_1, \\ \frac{0}{k} & \text{if } x = b_2, \\ \frac{0.8}{k} & \text{if } x = b_3, \\ \frac{0.6}{k} & \text{if } x = b_4, \end{cases}$$

where k is a natural number. Then $X_0 = \{b_1, b_3, b_4\}$ is filter of $\mathbb{X}$. If we take $t := \frac{0.5}{k}$, then $(\eth, t)_\in = \{b_3, b_4\}$. We can observe that $b_4 \lesssim b_1$ and $b_1 \notin (\eth, t)_\in$. Hence $\eth$ is not a fuzzy filter of $\mathbb{X}$.

Theorem 3.14. *If a non-constant fuzzy set $\eth$ in X satisfies the following conditions:*

$$(\forall x, y \in X)(\forall t \in (0,1])(x \in (\eth, t)_\in,\ x \lesssim y \ \Rightarrow\ y \in (\eth, t)_q), \tag{26}$$

$$(\forall x, y \in X)(\forall t \in (0,1])(x \in (\eth, t)_\in,\ g_x(y) \in (\eth, t)_\in \ \Rightarrow\ y \in (\eth, t)_q), \tag{27}$$

then the positive set of $\eth$ is a filter of $\mathbb{X}$.

Proof. Let $x, y \in X$ be such that $x \in X_0$ and $x \lesssim y$. Since $x \in (\eth, \eth(x))_\in$, it follows from (26) that $y \in (\eth, \eth(x))_q$. If $y \notin X_0$, then $\eth(y) = 0$ and so $\langle y_{\eth(x)} \rangle\, \overline{q}\, \eth$, i.e., $y \notin (\eth, \eth(x))_q$. This is a contradiction, and thus $y \in X_0$. Let $x \in X_0$ and $g_x(y) \in X_0$. If we take $t := \min\{\eth(x), \eth(g_x(y))\}$, then $x \in (\eth, t)_\in$ and $g_x(y) \in (\eth, t)_\in$. Using (27) leads to $y \in (\eth, t)_q$. Hence $\eth(y) + t > 1$, and so $\eth(y) \neq 0$, i.e., $y \in X_0$. Therefore X_0 is a filter of $\mathbb{X}$. □

Theorem 3.15. *If a non-constant fuzzy set $\eth$ in X satisfies the following conditions:*

$$(\forall x, y \in X)(\forall t \in (0,1])(x \in (\eth, t)_q,\ x \lesssim y \ \Rightarrow\ y \in (\eth, t)_\in), \tag{28}$$

$$(\forall x, y \in X)(\forall t \in (0,1])(x \in (\eth, t)_q,\ g_x(y) \in (\eth, t)_q \ \Rightarrow\ y \in (\eth, t)_\in), \tag{29}$$

then the positive set of $\eth$ is a filter of $\mathbb{X}$.

Proof. Let $x, y \in X$ be such that $x \in X_0$ and $x \lesssim y$. Then $\eth(x) \neq 0$, and so $\eth(x) + 1 > 1$, i.e., $x \in (\eth, 1)_q$. Thus $y \in (\eth, 1)_\in$ by (28), which shows that $y \in X_0$. Let $x \in X_0$ and $g_x(y) \in X_0$. Then $\eth(x) \neq 0 \neq \eth(g_x(y))$, and hence $\eth(x) + 1 > 1$ and $\eth(g_x(y)) + 1 > 1$, that is, $x \in (\eth, 1)_q$ and $g_x(y) \in (\eth, 1)_q$. It follows from (29) that $y \in (\eth, 1)_\in$. Thus $y \in X_0$ and therefore X_0 is a filter of $\mathbb{X}$. □

Theorem 3.16. *If a non-constant fuzzy set $\eth$ in X satisfies the following conditions:*

$$(\forall x, y \in X)(\forall t \in (0,1])(x \in (\eth, t)_q,\ x \lesssim y \ \Rightarrow\ y \in (\eth, t)_q), \tag{30}$$

$$(\forall x, y \in X)(\forall t \in (0,1])(x \in (\eth, t)_q,\ g_x(y) \in (\eth, t)_q \ \Rightarrow\ y \in (\eth, t)_q), \tag{31}$$

then the positive set of $\eth$ is a filter of $\mathbb{X}$.

Proof. Let $x, y \in X$ be such that $x \in X_0$ and $x \lesssim y$. Then $\eth(x) \neq 0$, and so $\eth(x) + 1 > 1$, i.e., $x \in (\eth, 1)_q$. Thus $y \in (\eth, 1)_q$ by (30), which implies $\eth(y) + 1 > 1$. Hence $\eth(y) \neq 0$, and so $y \in X_0$. Let $x \in X_0$ and $g_x(y) \in X_0$. Then $\eth(x) \neq 0 \neq \eth(g_x(y))$, and hence $\eth(x) + 1 > 1$ and $\eth(g_x(y)) + 1 > 1$, that is, $x \in (\eth, 1)_q$ and $g_x(y) \in (\eth, 1)_q$. Using (31) induces $y \in (\eth, 1)_q$. If $y \notin X_0$, then $\eth(y) = 0$ and thus $y \notin (\eth, 1)_q$ which is a contradiction. Hence $y \in X_0$ and therefore X_0 is a filter of $\mathbb{X}$. □

We establish conditions for the q_t-set $(\eth, t)_q$ to be a filter of $\mathbb{X}$.

Theorem 3.17. *If $\eth$ is a fuzzy filter of $\mathbb{X}$, then its nonempty q_t-set is a filter of $\mathbb{X}$ for all $t \in (0, 1]$.*

Proof. Assume that $(\eth, t)_q \neq \emptyset$ for all $t \in (0, 1]$. Let $x, y \in X$ be such that $x \in (\eth, t)_q$ and $x \lesssim y$. Then $\eth(x) + t > 1$ and so $\eth(y) \geq \eth(x) > 1 - t$ by (15). Hence $y \in (\eth, t)_q$. Let $x \in (\eth, t)_q$ and $g_x(y) \in (\eth, t)_q$. Then $\eth(x) + t > 1$ and $\eth(g_x(y)) + t > 1$. It follows from (16) that $\eth(y) \geq \min\{\eth(x), \eth(g_x(y))\} > 1 - t$. Thus $y \in (\eth, t)_q$, and therefore $(\eth, t)_q$ is a filter of $\mathbb{X}$ for all $t \in (0, 1]$. □

Proposition 3.18. *Given a fuzzy set $\eth$ in X, if its q_t-set is a filter of $\mathbb{X}$ for all $t \leq 0.5$, then the following conditions are established.*

$$(\forall x, y \in X)(\forall t \in (0, 0.5])(x \in (\eth, t)_q,\, x \lesssim y \;\Rightarrow\; y \in (\eth, t)_\in), \tag{32}$$

$$(\forall x, y \in X)(\forall t_1, t_2 \in (0, 0.5]) \left(\begin{array}{c} x \in (\eth, t_1)_q,\, g_x(y) \in (\eth, t_2)_q \\ \Rightarrow\; y \in (\eth, \max\{t_1, t_2\})_\in \end{array} \right). \tag{33}$$

Proof. Let $t \in (0, 0.5]$ and suppose that $(\eth, t)_q$ is a filter of $\mathbb{X}$. Let $x, y \in X$ be such that $x \in (\eth, t)_q$ and $x \lesssim y$. Then $y \in (\eth, t)_q$ by (13), and so $\eth(y) > 1 - t \geq t$. Hence $y \in (\eth, t)_\in$. Let $x, y \in X$ and $t_1, t_2 \in (0, 0.5]$ be such that $x \in (\eth, t_1)_q$ and $g_x(y) \in (\eth, t_2)_q$. Then $x \in (\eth, \max\{t_1, t_2\})_q$ and $g_x(y) \in (\eth, \max\{t_1, t_2\})_q$. It follows from (14) that $y \in (\eth, \max\{t_1, t_2\})_q$. Hence

$$\eth(y) > 1 - \max\{t_1, t_2\} \geq \max\{t_1, t_2\},$$

and so $y \in (\eth, \max\{t_1, t_2\})_\in$. □

Proposition 3.19. *Given a fuzzy set $\eth$ in X, if its q_t-set is a filter of $\mathbb{X}$ for all $t \geq 0.5$, then the following conditions are established.*

$$(\forall x, y \in X)(\forall t \in (0.5, 1])(x \in (\eth, t)_\in,\, x \lesssim y \;\Rightarrow\; y \in (\eth, t)_q), \tag{34}$$

$$(\forall x, y \in X)(\forall t_1, t_2 \in (0.5, 1]) \left(\begin{array}{c} x \in (\eth, t_1)_\in,\, g_x(y) \in (\eth, t_2)_\in \\ \Rightarrow\; y \in (\eth, \max\{t_1, t_2\})_q \end{array} \right). \tag{35}$$

Proof. Let $t \in (0.5, 1]$ and suppose that $(\eth, t)_q$ is a filter of $\mathbb{X}$. Let $x, y \in X$. If $x \in (\eth, t)_\in$ and $x \lesssim y$, then $\eth(x) \geq t > 1 - t$, i.e., $x \in (\eth, t)_q$. Hence $y \in (\eth, t)_q$ by (13). If $x \in (\eth, t_1)_\in$ and $g_x(y) \in (\eth, t_2)_\in$, then $\eth(x) \geq t_1 > 1 - t_1 \geq 1 - \max\{t_1, t_2\}$ and $\eth(g_x(y)) \geq t_2 > 1 - t_2 \geq 1 - \max\{t_1, t_2\}$, that is, $x \in (\eth, \max\{t_1, t_2\})_q$ and $g_x(y) \in (\eth, \max\{t_1, t_2\})_q$. It follows from (14) that $y \in (\eth, \max\{t_1, t_2\})_q$. □

Corollary 3.20. *Every fuzzy filter $\eth$ of $\mathbb{X}$ satisfies (32), (33), (34) and (35).*

Theorem 3.21. *If a fuzzy set $\eth$ in X satisfies the following conditions:*

$$(\forall x, y \in X)(\forall t \in (0.5, 1])(x \in (\eth, t)_q,\, x \lesssim y \;\Rightarrow\; y \in (\eth, t)_{\in\vee q}), \tag{36}$$

$$(\forall x, y \in X)(\forall t \in (0.5, 1])(x \in (\eth, t)_q,\, g_x(y) \in (\eth, t)_q \;\Rightarrow\; y \in (\eth, t)_{\in\vee q}), \tag{37}$$

then the nonempty q_t-set is a filter of $\mathbb{X}$ for all $t \in (0.5, 1]$.

Proof. Let $x, y \in X$ and $t \in (0.5, 1]$, and assume that $(\eth, t)_q$ is nonempty. If $x \in (\eth, t)_q$ and $x \lesssim y$, then $y \in (\eth, t)_{\in\vee q}$ by (36). It follows that $y \in (\eth, t)_\in$ or $y \in (\eth, t)_q$. If $y \in (\eth, t)_\in$, then $\eth(y) \geq t > 1 - t$ and so $y \in (\eth, t)_q$. Let $x \in (\eth, t)_q$ and $g_x(y) \in (\eth, t)_q$. Then $y \in (\eth, t)_{\in\vee q}$ by (37), and thus $y \in (\eth, t)_\in$ or $y \in (\eth, t)_q$. If $y \in (\eth, t)_\in$, then $\eth(y) \geq t > 1 - t$ and so $y \in (\eth, t)_q$. Consequently, $(\eth, t)_q$ is a filter of $\mathbb{X}$. □

Proposition 3.22. *Given a filter F of $\mathbb{X}$, if we define a fuzzy set ∂_F in X as follows:*

$$\partial_F : X \to [0,1],\ x \mapsto \begin{cases} s_1 & \text{if } x \in F, \\ s_2 & \text{otherwise} \end{cases}$$

where $s_1 \geq 0.5 > s_2 = 0$, then the following assertions hold.

$$(\forall x, y \in X)(\forall t \in (0,1])(x \in (\partial_F, t)_q,\ x \lesssim y \ \Rightarrow\ y \in (\partial_F, t)_{\in\vee q}). \tag{38}$$

$$(\forall x, y \in X)(\forall t_1, t_2 \in (0,1]) \left(\begin{array}{c} x \in (\partial_F, t_1)_q,\ g_x(y) \in (\partial_F, t_2)_q \\ \Rightarrow\ y \in (\partial_F, \min\{t_1, t_2\})_{\in\vee q}. \end{array} \right). \tag{39}$$

Proof. Let $x, y \in X$ and $t \in (0, 1]$ be such that $x \in (\partial_F, t)_q$ and $x \lesssim y$. Then $\partial_F(x) + t > 1$. If $x \notin F$, then $\partial_F(x) = s_2 = 0$ and so $t > 1$ a contradiction. Thus $x \in F$ and hence $y \in F$ since F is a filter of $\mathbb{X}$. Hence $\partial_F(y) = s_1 \geq 0.5$. If $t \leq 0.5$, then $\partial_F(y) \geq 0.5 \geq t$ and so $y \in (\partial_F, t)_\in$. If $t > 0.5$, then $\partial_F(y) + t > 0.5 + 0,5 = 1$ which means $y \in (\partial_F, t)_q$. Thus $y \in (\partial_F, t)_{\in\vee q}$. Let $x, y \in X$ and $t_1, t_2 \in (0, 1]$ be such that $x \in (\partial_F, t_1)_q$ and $g_x(y) \in (\partial_F, t_2)_q$, that is, $\partial_F(x) + t_1 > 1$ and $\partial_F(g_x(y)) + t_2 > 1$. If $x \notin F$ or $g_x(y) \notin F$, then $\partial_F(x) = s_2 = 0$ or $\partial_F(g_x(y)) = s_2 = 0$. Hence $t_1 > 1$ or $t_2 > 1$ which is a contradiction. Hence $x \in F$ and $g_x(y) \in F$. Since F is a filter of $\mathbb{X}$, we have $y \in X$ and thus $\eth_F(y) = s_1 \geq 0.5$. If $t_1 \leq 0.5$ or $t_2 \leq 0.5$, then $\eth_F(y) \geq 0.5 \geq \min\{t_1, t_2\}$. Thus $y \in (\partial_F, \min\{t_1, t_2\})_\in$. If $t_1 > 0.5$ and $t_2 > 0.5$, then $\partial_F(y) + \min\{t_1, t_2\} > 0.5 + 0.5 = 1$, i.e., $y \in (\partial_F, \min\{t_1, t_2\})_q$. Therefore, $y \in (\partial_F, \min\{t_1, t_2\})_{\in\vee q}$. □

Theorem 3.23. *If ∂_F is the fuzzy set in X which is described in Proposition 3.22, then its q_t-set $(\partial_F, t)_q$ is a filter of $\mathbb{X}$ for all $t \in (0.5, 1]$*

Proof. Let $x, y \in X$ and $t \in (0.5, 1]$. If $x \in (\partial_F, t)_q$ and $x \lesssim y$, then $\partial_F(x) + t > 1$, and so $x \in F$ because if not, then $\partial_F(x) = s_2 = 0$ and thus $t > 1$ a contradiction. Since F is a filter of $\mathbb{X}$, we get $y \in F$. So $\partial_F(y) = s_1 \geq 0.5$. Since $t \in (0.5, 1]$, it follows that $\partial_F(y) + t = s_1 + t > 0.5 + 0.5 = 1$, i.e., $y \in (\partial_F, t)_q$. Suppose that $x \in (\partial_F, t)_q$ and $g_x(y) \in (\partial_F, t)_q$. Then $\partial_F(x) + t > 1$ and $\partial_F(g_x(y)) + t > 1$. If $x \notin F$ or $g_x(y) \notin F$, then $\partial_F(x) = s_2 = 0$ or $\partial_F(g_x(y)) = s_2 = 0$. Hence $t = \partial_F(x) + t > 1$ or $t = \partial_F(g_x(y)) + t > 1$, a contradiction. Thus $x \in F$ and $g_x(y) \in F$, which induces $y \in F$. So $\partial_F(y) = s_1 \geq 0.5$. Since $t \in (0.5, 1]$, it follows that $\partial_F(y) + t = s_1 + t > 0.5 + 0.5 = 1$, i.e., $y \in (\partial_F, t)_q$. Therefore $(\partial_F, t)_q$ is a filter of $\mathbb{X}$. □

Definition 3.24. A fuzzy set $\eth$ in X is called a $(0.5, 1]$-*fuzzy filter* of $\mathbb{X}$ if its nonempty $\in_t$-set $(\eth, t)_\in$ is a filter of $\mathbb{X}$ for all $t \in (0.5, 1]$.

Example 3.25. Consider the pre-ordered residuated system $\mathbb{X} := (X, \odot, \to, b_1, \lesssim)$ in Example 3.3. Let ∂ be a fuzzy set in X given by

$$\partial : X \to [0,1],\ x \mapsto \begin{cases} 0.9 & \text{if } x = b_1, \\ 0.4 & \text{if } x = b_2, \\ 0.7 & \text{if } x = b_3, \\ 0.3 & \text{if } x = b_4. \end{cases}$$

Then ∂ is a $(0.5, 1]$-fuzzy filter of $\mathbb{X}$.

It is clear that every fuzzy filter is a $(0.5, 1]$-fuzzy filter. But the converse may not be true. In fact, the $(0.5, 1]$-fuzzy filter ∂ in Example 3.25 is not a fuzzy filter of $\mathbb{X}$ since $b_2 \lesssim b_4$ and $\partial(b_2) = 0.4 \nleq 0.3 = \partial(b_4)$.

We now discuss the characterization of $(0.5, 1]$-fuzzy filters.

Theorem 3.26. *A fuzzy set* $\eth$ *in* X *is a* $(0.5, 1]$*-fuzzy filter of* $\mathbb{X}$ *if and only if it satisfies:*

$$(\forall x, y \in X)(x \lesssim y \ \Rightarrow\ \eth(x) \leq \max\{\eth(y), 0.5\}), \tag{40}$$

$$(\forall x, y \in X)(\max\{\eth(y), 0.5\} \geq \min\{\eth(x), \eth(g_x(y))\}). \tag{41}$$

Proof. Assume that $\eth$ in X is a $(0.5, 1]$-fuzzy filter of $\mathbb{X}$. Then the nonempty $\in_t$-set $(\eth, t)_\in$ is a filter of $\mathbb{X}$ for all $t \in (0.5, 1]$. If the condition (40) is not valid, then there exists $\mathfrak{a}, \mathfrak{b} \in X$ such that $\mathfrak{a} \lesssim \mathfrak{b}$ and $\eth(\mathfrak{a}) > \max\{\eth(\mathfrak{b}), 0.5\}$. Hence $t := \eth(\mathfrak{a}) \in (0.5, 1]$ and $\mathfrak{a} \in (\eth, t)_\in$. But $\mathfrak{b} \notin (\eth, t)_\in$, a contradiction. Hence $\eth(x) \leq \max\{\eth(y), 0.5\}$ for all $x, y \in X$ with $x \lesssim y$. Suppose that the condition (41) is not establish. Then $\max\{\eth(\mathfrak{b}), 0.5\} < \min\{\eth(\mathfrak{a}), \eth(g_\mathfrak{a}(\mathfrak{b}))\}$ for some $\mathfrak{a}, \mathfrak{b} \in X$. If we take $s := \min\{\eth(\mathfrak{a}), \eth(g_\mathfrak{a}(\mathfrak{b}))\}$, then $s \in (0.5, 1]$, $\mathfrak{a} \in (\eth, s)_\in$ and $g_\mathfrak{a}(\mathfrak{b}) \in (\eth, s)_\in$. But $\max\{\eth(\mathfrak{b}), 0.5\} < s$ leads to $\mathfrak{b} \notin (\eth, s)_\in$, which is a contradiction. Therefore $\max\{\eth(y), 0.5\} \geq \min\{\eth(x), \eth(g_x(y))\}$ for all $x, y \in X$.

Conversely, let $\eth$ be a fuzzy set in X that satisfies two conditions (40) and (41). Let $t \in (0.5, 1]$ be such that $(\eth, t)_\in \neq \emptyset$. If $x \in (\eth, t)_\in$ and $x \lesssim y$, then $\max\{\eth(y), 0.5\} \geq \eth(x) \geq t > 0.5$ by (40). Thus $\eth(y) \geq t$, i.e., $y \in (\eth, t)_\in$. If $x \in (\eth, t)_\in$ and $g_x(y) \in (\eth, t)_\in$, then $\eth(x) \geq t$ and $\eth(g_x(y)) \geq t$. It follows from (41) that $\max\{\eth(y), 0.5\} \geq \min\{\eth(x), \eth(g_x(y))\} \geq t$. Since $t > 0.5$, we get $\eth(y) \geq t$ and so $y \in (\eth, t)_\in$. Therefore $\eth$ is a $(0.5, 1]$-fuzzy filter of $\mathbb{X}$. □

Corollary 3.27. *Every fuzzy filter* $\eth$ *of* $\mathbb{X}$ *satisfies the two conditions* (40) *and* (41).

Theorem 3.28. *If* $\eth$ *is a* $(0.5, 1]$*-fuzzy filter of* $\mathbb{X}$*, then its nonempty* q_t*-set is a filter of* $\mathbb{X}$ *for all* $t \in (0, 0.5)$.

Proof. Assume that $\eth$ is a $(0.5, 1]$-fuzzy filter of $\mathbb{X}$. Let $t \in (0, 0.5)$ be such that $(\eth, t)_q \neq \emptyset$. If $x \in (\eth, t)_q$ and $x \lesssim y$, then

$$\max\{\eth(y), 0.5\} \geq \eth(x) > 1 - t > 0.5$$

by (40), and so $\eth(y) > 1 - t$. Hence $y \in (\eth, t)_q$. Let $x, y \in X$ be such that $x \in (\eth, t)_q$ and $g_x(y) \in (\eth, t)_q$. Then $\eth(x) > 1 - t$ and $\eth(g_x(y)) > 1 - t$. It follows from (41) that

$$\max\{\eth(y), 0.5\} \geq \min\{\eth(x), \eth(g_x(y))\} > 1 - t > 0.5.$$

Hence $\eth(y) > 1 - t$ and thus $y \in (\eth, t)_q$. Therefore $(\eth, t)_q$ is a filter of $\mathbb{X}$. □

Now let's think about a more generalized form of Definition 3.1 and Definition 3.24.

Let $\eth$ be a fuzzy set in X. Then the $\in_t$-set $(\eth, t)_\in$ is a filter of $\mathbb{X}$ for some $t \in (0, 1]$, but can not be a filter of $\mathbb{X}$ for other $t \in (0, 1]$. Let

$$J_X := \{t \in (0, 1] \mid (\eth, t)_\in \text{ is a filter of } \mathbb{X}\}.$$

If $J_X = (0, 1]$, then $\eth$ is a fuzzy filter of $\mathbb{X}$. If $J_X = (0.5, 1]$, then $\eth$ is a $(0.5, 1]$-fuzzy filter of $\mathbb{X}$. However, in general, the question arises as to what the form of the fuzzy filter is if J_X is a non-empty subset of $(0, 1]$, for example $J_X = (0, 0.5]$ or $J_X = (\delta, \varepsilon]$ for $\delta, \varepsilon \in (0, 1]$ with $\delta < \varepsilon$. Based on this question, we consider the following definition.

Definition 3.29. Let $\delta < \varepsilon$ in $[0, 1]$. A fuzzy set $\eth$ in X is called a *fuzzy filter with thresholds* δ and ε (briefly, $(\delta, \varepsilon]$*-fuzzy filter*) of $\mathbb{X}$ if its nonempty $\in_t$-set $(\eth, t)_\in$ is a filter of $\mathbb{X}$ for all $t \in (\delta, \varepsilon]$.

It is clear that if a fuzzy set $\eth$ in X satisfies $\eth(x) \leq \delta < \varepsilon$ for all $x \in X$, then $\eth$ is a $(\delta, \varepsilon]$-fuzzy filter of $\mathbb{X}$, and every fuzzy filter is a $(\delta, \varepsilon]$-fuzzy filter for every $\delta, \varepsilon \in (0, 1]$ with $\delta < \varepsilon$.

Example 3.30. Consider the pre-ordered residuated system $\mathbb{X} := (X, \odot, \rightarrow, b_1, \lesssim)$ in Example 3.3. Let $\eth$ be a fuzzy set in X given by

$$\eth : X \to [0,1],\ x \mapsto \begin{cases} 0.6 & \text{if } x = b_1, \\ 0.3 & \text{if } x = b_2, \\ 0.8 & \text{if } x = b_3, \\ 0.5 & \text{if } x = b_4. \end{cases}$$

Then $\eth$ is a $(0.27, 0.58]$-fuzzy filter of $\mathbb{X}$ since

$$(\eth, t)_\in = \begin{cases} \{b_1, b_3\} & \text{if } 0.5 < t \le 0.58, \\ \{b_1, b_3, b_4\} & \text{if } 0.3 < t \le 0.5, \\ X & \text{if } 0.27 < t \le 0.3, \end{cases}$$

is a filter of $\mathbb{X}$ for all $t \in (0.27, 0.58]$. But it is not a $(0.27, 0.65]$-fuzzy filter of $\mathbb{X}$ because if we take $t := 0.63 \in (0.27, 0.65]$, then $(\eth, t)_\in = \{b_3\}$ is not a filter of $\mathbb{X}$.

It is obvious that if $(\delta, \varepsilon_1] \subseteq (\delta, \varepsilon_2]$, then every $(\delta, \varepsilon_2]$-fuzzy filter is a $(\delta, \varepsilon_1]$-fuzzy filter, but the converse may not be true as shown in Example 3.30.

Theorem 3.31. *A fuzzy set $\eth$ in X is a $(\delta, \varepsilon]$-fuzzy filter of $\mathbb{X}$ if and only if the following conditions hold.*

$$(\forall x, y \in X)(x \lesssim y \ \Rightarrow\ \min\{\eth(x), \varepsilon\} \le \max\{\eth(y), \delta\}), \tag{42}$$

$$(\forall x, y \in X)(\max\{\eth(y), \delta\} \ge \min\{\eth(x), \eth(g_x(y)), \varepsilon\}). \tag{43}$$

Proof. Suppose that $\eth$ is a $(\delta, \varepsilon]$-fuzzy filter of $\mathbb{X}$. Let $x \lesssim y$ in $\mathbb{X}$. If $\min\{\eth(x), \varepsilon\} > \max\{\eth(y), \delta\}$, then there exists $t \in (0, 1]$ such that

$$\min\{\eth(x), \varepsilon\} \ge t > \max\{\eth(y), \delta\}.$$

Then $\eth(y) < t$ and $\eth(x) \ge t$, that is, $y \notin (\eth, t)_\in$ and $x \in (\eth, t)_\in$, and $t \in (\delta, \varepsilon]$. This is a contradiction, and so $\min\{\eth(x), \varepsilon\} \le \max\{\eth(y), \delta\}$. If (43) is not established, then

$$\max\{\eth(\mathfrak{b}), \delta\} < t \le \min\{\eth(\mathfrak{a}), \eth(g_{\mathfrak{a}}(\mathfrak{b})), \varepsilon\}$$

for some $\mathfrak{a}, \mathfrak{b} \in X$ and $t \in (0, 1]$. It follows that $t \in (\delta, \varepsilon]$, $\mathfrak{b} \notin (\eth, t)_\in$, $\mathfrak{a} \in (\eth, t)_\in$ and $g_{\mathfrak{a}}(\mathfrak{b}) \in (\eth, t)_\in$. This is a contradiction, and thus $\eth$ satisfies the condition (43).

Conversely, we assume that $\eth$ satisfies the two conditions (42) and (43). Let $x, y \in X$ and $t \in (\delta, \varepsilon]$. If $x \lesssim y$ and $x \in (\eth, t)_\in$, then

$$\max\{\eth(y), \delta\} \ge \min\{\eth(x), \varepsilon\} \ge t > \delta$$

by (42). Hence $\eth(y) \ge t$, i.e., $y \in (\eth, t)_\in$. If $x \in (\eth, t)_\in$ and $g_x(y) \in (\eth, t)_\in$, then

$$\max\{\eth(y), \delta\} \ge \min\{\eth(x), \eth(g_x(y)), \varepsilon\} \ge t > \delta$$

by (43), and so $\eth(y) \ge t$, i.e., $y \in (\eth, t)_\in$. Consequently, $(\eth, t)_\in$ is a filter of $\mathbb{X}$ for all $t \in (\delta, \varepsilon]$. Therefore $\eth$ is a $(\delta, \varepsilon]$-fuzzy filter of $\mathbb{X}$. □

Given a fuzzy set $\eth$ in X, we say the set

$$(\eth, t)^*_q := \{x \in X \mid \eth(x) + t \ge 1\}$$

is an *extended q_t-set* of $\eth$.

Theorem 3.32. *If $\eth$ is a $(\delta, \varepsilon]$-fuzzy filter of $\mathbb{X}$ and $\delta < 0.5$, then its nonempty extended q_t-set is a filter of $\mathbb{X}$ for all $t \in (0, \delta] \cap [1-\varepsilon, 1]$.*

Proof. Assume that $\eth$ is a $(\delta, \varepsilon]$-fuzzy filter of $\mathbb{X}$ and $\delta < 0.5$. Let $t \in (0, \delta] \cap [1-\varepsilon, 1]$ be such that $(\eth, t)^*_q \neq \emptyset$. If $x \in (\eth, t)^*_q$ and $x \lesssim y$, then

$$\max\{\eth(y), \delta\} \geq \min\{\eth(x), \varepsilon\} \geq \min\{1-t, \varepsilon\} = 1-t \geq 1-\delta > \delta$$

by (42). Hence $\eth(y) \geq 1-t$, that is, $y \in (\eth, t)^*_q$. Let $x, y \in X$ be such that $x \in (\eth, t)^*_q$ and $g_x(y) \in (\eth, t)^*_q$. Then $\eth(x) + t \geq 1$ and $\eth(g_x(y)) + t \geq 1$. It follows from (43) that

$$\max\{\eth(y), \delta\} \geq \min\{\eth(x), \eth(g_x(y)), \varepsilon\} \geq \min\{1-t, \varepsilon\} = 1-t \geq 1-\delta > \delta.$$

Thus $\eth(y) \geq 1-t$, that is, $y \in (\eth, t)^*_q$. Consequently, $(\eth, t)^*_q$ is a filter of $\mathbb{X}$. □

4 Conclusion

As a mathematical structure, a residuated relational system has been introduced by S. Bonzio and I. Chajda in 2018, and it combines elements of algebra, order theory, and relational calculus. They also extended the residuated relational system by introducing pre-ordered residuated systems using pre-order relation, and further studied the various properties involved. D. A. Romano [2, 3, 4, 5] introduced and analyzed the concept of (weak implicative, shipt, implicative, comparative) filters in pre-ordered residuated systems. With the purpose of this paper in the study of filters in pre-ordered residuated systems using the concept of fuzzy points, we introduced fuzzy filter and identified various properties. Based on the ideas of this paper and the results obtained, we will study various fuzzy versions for different types of filters, for example, (weak implicative, shipt, implicative, comparative) filters, in the future.

Acknowledgements: The author appreciates the valuable suggestions of anonymous reviewers.

Conflict of Interest: The author declares no conflict of interest.

References

[1] Bonzio S, Chajda I. Residuated relational systems. *Asian-European Journal of Mathematics*. 2018; 11(02): 1850024. DOI: https://doi.org/10.1142/S1793557118500249

[2] Romano DA. Filters in residuated relational system ordered under quasi-order. *Bulletin of the International Mathematical Virtual Institute*. 2020; 10(3): 529-534. DOI: http://doi.org/10.7251/BIMVI2003529R

[3] Romano DA. Comparative filters in quasi-ordered residuated system. *Bulletin of the International Mathematical Virtual Institute*. 2021; 11(1): 177-184. DOI: http://doi.org/10.7251/BIMVI2101177R

[4] Romano DA. Implicative filters in quasi-ordered residuated systems. *Proyecciones Journal of Mathematics*. 2021; 40(2): 417-424. DOI: http://dx.doi.org/10.22199/issn.0717-6279-2021-02-0025

[5] Romano DA. Shift filter of quasi-ordered residuated systems. *Communications in Advanced Mathematical Sciences*. 2022; 5(3): 124-130. DOI: http://doi.org/10.33434/cams.1089222

[6] Pao-Ming P, Ying-Ming L. Fuzzy topology I, Neighborhood structure of a fuzzy pointand Moore-Smith convergence. *Journal of Mathematical Analysis and Applications*. 1980; 76(2): 571-599. DOI: https://doi.org/10.1016/0022-247X(80)90048-7

[7] Romano DA. Weak implicative filters in quasi-ordered residuated systems. *Proyecciones Journal of Mathematics*. 2021; 40(3): 797-804. DOI: http://dx.doi.org/10.22199/issn.0717-6279-4332

Young Bae Jun
Department of Mathematics Education
Gyeongsang National University
Jinju 52828, Korea
E-mail: skywine@gmail.com