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On a-locally closed sets

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Abstract. The aim of this paper is to introduce the notion of a-locally closed set by utilizing a-open sets defined by Ekici and to study some properties of this new notion. Also, some characterizations and many fundamental results regarding this new concept are obtained. Moreover, the relationships between the concepts defined within the scope of this study and some other types of locally closed sets in the literature have been revealed.

Keywords: a-locally closed, a-locally open, a-dense, a-submaximal, a-separated.

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1. Introduction

The elements of a topology defined as a family of sets consisting of some subsets of a non-empty set X and closed under finite intersection and any union operation are called open sets. The concept of open set has an important place in general topology and is one of the focal points of research for many mathematicians all over the world. The study of different versions of continuity, separation axioms, compactness, connectedness and other concepts defined with the help of special and general forms of the open set concept are important topics of study in general topology. Starting in 1963 with Levine's introduction of the notion of semiopen set [11], the process continued with Njastad's study of α-open set [14, 1965] and Ekici's study of e-open set [7, 2008]. These works of Levine, Njastad

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and Ekici inspired the work done today and since then, different types of open sets have been intensively studied.

In an attempt to generalize different ideas in topology, many mathematicians have focused on various forms of open sets such as α-open set, semi-open set, pre-open set, b-open set, β-open set, e-open set and e^*-open set. The weak and strong forms of these concepts have been studied by many researchers. These studies have evolved over time into the concept of local closed set [8] and some forms of this concept have been studied over time. For instance, Ravi et al. [15] have introduced the notion of $\ddot{g}$-locally closed set and study some of its fundamental properties. Furthermore, Ameen et al. [1] analysed the concept of locally closed set from different perspectives.

In this paper, we introduce the notion of a-locally closed set and discuss some of its basic properties from different perspectives. Also, we reveal the relationships between the notion of a-locally closed set and existing some types of locally closed set in the literature. Moreover, we define the notion of a-locally open set and obtain some of its characterizations. Finally, we give a product property of the concept of a-locally closed sets.

2. Preliminaries

Throughout this paper, unless otherwise stated the terms X and Y refer to topological spaces on which no separation axioms are imposed. For a subset A of X, $cl(A)$ and $int(A)$ stand for the closure of A and the interior of A in X, respectively. $O(X,x)$ stands for the family of all open subsets of X that contain x. A subset A is said to be regular open (resp. regular closed) if $A = int(cl(A))$ (resp. $A = cl(int(A))$) [16]. The δ-interior of a subset A of X is the union of all regular open sets of X contained in A and is denoted by δ-$int(A)$. The subset A of a space X is called δ-open if $A = \delta\text{-}int(A)$ [17], i.e., a set is δ-open if it is the union of some regular open sets.

Definition 2.1 A subset A of a space X is called:
a) semi-open [11] if $A \subseteq cl(int(A))$;
b) α-open [14] if $A \subseteq int(cl(int(A)))$;
c) a-open [4, 6] if $A \subseteq int(cl(\delta\text{-}int(A)))$;
d) b-open [2] if $A \subseteq cl(int(A)) \cup int(cl(A))$;
e) e-open [7] if $A \subseteq cl(\delta\text{-}int(A)) \cup int(\delta\text{-}cl(A))$;
f) feebly open [12] if there exists an open set U such that $U \subseteq A \subseteq scl(U)$, where $scl(U)$ denotes the semi-closure of U.

The family of all semiopen (resp. α-open, a-open, b-open, e-open, feebly open) sets in X is denoted by $SO(X)$ (resp. $\alpha O(X)$, $aO(X)$, $BO(X)$, $eO(X)$, $FO(X)$). The complement of a semi-open (resp. α-open, a-open, b-open, e-open, feebly open) set is said to be semi-closed (resp. α-closed, a-closed, b-closed, e-closed, feebly closed). The family of all semiclosed (resp. α-closed, a-closed, b-closed, e-closed, feebly closed) sets in X is denoted by $SC(X)$ (resp. $\alpha C(X)$, $aC(X)$, $BC(X)$, $eC(X)$, $FC(X)$).

Definition 2.2 The semi-closure (resp. a-closure) of a subset A of a space X is the smallest semi-closed (resp. a-closed) set containing A and denoted by $scl(A)$ (resp. a-$cl(A)$). Dually, the semi-interior (resp. a-interior) of a subset A of a space X is the largest semi-open (resp. a-open) set contained in A and denoted by $sint(A)$ (resp. a-$int(A)$).

Definition 2.3 A subset A of a space (X,τ) is called:
a) locally closed [3] if $A = U \cap V$, where U is open and V is closed in X;

b) α-locally closed [9] if $A = U \cap V$, where U is α-open and V is α-closed in X;
c) b-locally closed [13] if $A = U \cap V$, where U is b-open and V is b-closed in X;
d) e-locally closed [10] if $A = U \cap V$, where U is e-open and V is e-closed in X;
e) feebly locally closed [12] if $A = U \cap V$, where U is feebly open and V is feebly closed in X.

Lemma 2.4 [6] Let (X, τ) be a topological space. Then, the following hold:
a) $\emptyset, X \in aO(X)$,
b) If $A, B \in aO(X)$, then $A \cap B \in aO(X)$,
c) If $\mathcal{A} \subseteq aO(X)$, then $\bigcup \mathcal{A} \in aO(X)$.

Corollary 2.5 Let (X, τ) be a topological space. Then, the following hold:
a) $\emptyset, X \in aC(X)$,
b) If $A, B \in aC(X)$, then $A \cup B \in aC(X)$,
c) If $\mathcal{A} \subseteq aC(X)$, then $\bigcap \mathcal{A} \in aC(X)$.

Definition 2.6 A subset A of a space X is called dense if $cl(A) = X$. A space X is called submaximal if every dense subset of X is open in X.

Definition 2.7 [10] A subset A of a space X is called e-dense if $e\text{-}cl(A) = X$. A space X is called e-submaximal if every e-dense subset of X is e-open in X.

3. *a*-locally closed sets

Definition 3.1 A subset A of a topological space X is called a-locally closed if it is the intersection of an a-open and an a-closed set. The complement of an a-locally closed set is called a-locally open. The family of all a-locally closed sets (resp. a-locally open) in a space X will be denoted by $aLC(X)$ (resp. $aLO(X)$).

Theorem 3.2 Let X be a topological space. Then, the following hold.
a) $aO(X) \subseteq aLC(X)$,
b) $aC(X) \subseteq aLC(X)$.

Proof. (*a*) Let $A \in aO(X)$. We aim to show that $A \in aLC(X)$.

$$\left.\begin{array}{r} A \in aO(X) \\ (U := A)(V := X) \end{array}\right\} \Rightarrow (U \in aO(X))(V \in aC(X))(A = U \cap V) \Rightarrow A \in aLC(X).$$

(*b*) Let $A \in aC(X)$. We aim to show that $A \in aLC(X)$.

$$\left.\begin{array}{r} A \in aC(X) \\ (U := X)(V := A) \end{array}\right\} \Rightarrow (U \in aO(X))(V \in aC(X))(A = U \cap V) \Rightarrow A \in aLC(X).$$ ■

Theorem 3.3 Let X be a topological space. Then, $aLC(X) \subseteq eLC(X)$.

Proof. Let $A \in aLC(X)$. We aim to show that $A \in eLC(X)$.

$$\left.\begin{array}{r} A \in aLC(X) \Rightarrow (\exists U \in aO(X))(\exists V \in aC(X))(A = U \cap V) \\ (aO(X) \subseteq eO(X))(aC(X) \subseteq eC(X)) \end{array}\right\} \Rightarrow$$

$$\Rightarrow (\exists U \in eO(X))(\exists V \in eC(X))(A = U \cap V) \Rightarrow A \in eLC(X).$$ ■

Lemma 3.4 Let X be a topological space. Then, the following hold.
a) $aO(X) \subseteq FO(X)$,
b) $aC(X) \subseteq FC(X)$.

Proof. *a*) Let $A \in aO(X)$. We aim to show that $A \in FO(X)$.

$$\left.\begin{array}{r} A \in aO(X) \Rightarrow A \subseteq int(cl(\delta\text{-}int(A))) \\ U := \delta\text{-}int(A) \end{array}\right\} \Rightarrow$$

$\Rightarrow (U \in \tau)(U \subseteq A \subseteq int(cl(U)) \subseteq U \cup int(cl(U)) = scl(U)) \ \Rightarrow A \in FO(X).$
b) Let $A \in aC(X)$. We aim to show that $A \in FC(X)$.

$$A \in aC(X) \Rightarrow X \setminus A \in aO(X) \overset{(a)}{\Rightarrow} X \setminus A \in FO(X) \Rightarrow A \in FC(X).$$ ■

Theorem 3.5 Let X be a topological space. Then, the following statements hold:
a) $aLC(X) \subseteq FLC(X)$,
b) $aLC(X) \subseteq \alpha LC(X)$.

Proof. *a*) Let $A \in aLC(X)$. We aim to show that $A \in FLC(X)$.

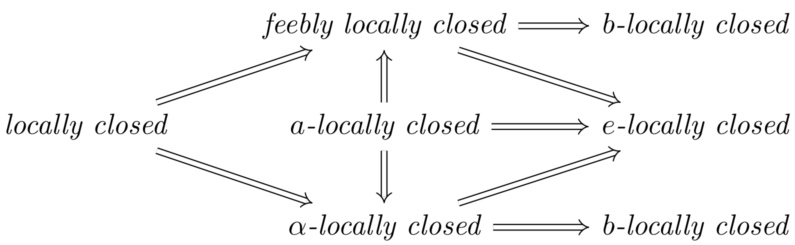
$$\left.\begin{array}{r} A \in aLC(X) \Rightarrow (\exists U \in aO(X))(\exists V \in aC(X))(A = U \cap V) \\ \text{Lemma 3.4} \end{array}\right\} \Rightarrow$$

$\Rightarrow (\exists U \in FO(X))(\exists V \in FC(X))(A = U \cap V) \ \Rightarrow A \in FLC(X).$
b) Let $A \in aLC(X)$. We aim to show that $A \in \alpha LC(X)$.

$$\left.\begin{array}{r} A \in aLC(X) \Rightarrow (\exists U \in aO(X))(\exists V \in aC(X))(A = U \cap V) \\ (aO(X) \subseteq \alpha O(X))(aC(X) \subseteq \alpha C(X)) \end{array}\right\} \Rightarrow$$

$\Rightarrow (\exists U \in \alpha O(X))(\exists V \in \alpha C(X))(A = U \cap V) \ \Rightarrow A \in \alpha LC(X).$ ■

Remark 1 *We have the following diagram from the previous definitions and results given above.*



The converses given above implications need not to be true as shown by the following examples.

***Example* 3.6** Let $X = \{a, b, c, d\}$ and $\tau = \{\emptyset, X, \{a\}, \{b\}, \{a, b\}, \{a, c, d\}\}$. Simple calculations show that $eLC(X) = \alpha LC(X) = FLC(X) = 2^X$ and $aLC(X) = \{\emptyset, X, \{b\}, \{a, c, d\}\}$. Then, it is clear that the set $\{a\}$ is feebly locally closed and so e-locally closed. Also, it is α-locally closed but it is not a-locally closed.

***Example* 3.7** Let $X = \{a, b, c, d\}$ and $\tau = \{\emptyset, X, \{a\}, \{b\}, \{a, b\}, \{a, c\}, \{a, b, c\}, \{a, b, d\}\}$. Simple calculations show that $LC(X) = 2^X$, $aLC(X) = \{\emptyset, X, \{b\}, \{d\}, \{a, c\}, \{b, d\}, \{a, b, c\}, \{a, c, d\}\}$. Then, it is clear that the set $\{a\}$ is locally closed but it is not a-locally closed.

***Example* 3.8** Let $X = \{a, b, c, d\}$ and $\tau = \{\emptyset, X, \{a\}, \{b\}, \{a, b\}\}$. Simple calculations show that $LC(X) = \{\emptyset, X, \{a\}, \{b\}, \{a, b\}, \{c, d\}, \{a, c, d\}, \{b, c, d\}\}$ and $aLC(X) = 2^X$. Then, it is clear that the set $\{c\}$ is an a-locally closed set but it is not locally closed.

Question 3.9 Are the notions b-locally closedness and e-locally closedness independent?

Theorem 3.10 Let A and B be two subsets of a space X. If A and B are a-locally closed, then so are their intersections.

Proof. Let $A, B \in aLC(X)$. We aim to show that $A \cap B \in aLC(X)$.

$$\left.\begin{array}{l} A \in aLC(X) \Rightarrow (\exists U_1 \in aO(X))(\exists V_1 \in aC(X))(A = U_1 \cap V_1) \\ B \in aLC(X) \Rightarrow (\exists U_2 \in aO(X))(\exists V_2 \in aC(X))(B = U_2 \cap V_2) \end{array}\right\} \overset{\text{Lemma 2.4}}{\Rightarrow}$$

$$\Rightarrow (U_1 \cap U_2 \in aO(X))(V_1 \cap V_2 \in aC(X))(A \cap B = (U_1 \cap U_2) \cap (V_1 \cap V_2))$$

$$\Rightarrow A \cap B \in aLC(X).$$ ■

Lemma 3.11 Let A and B be two subsets of a space X. Then, the following statements hold:
$a)$ If $A \in aO(X)$ and $B \in eO(X)$, then $A \cap B \in eO(X)$,
$b)$ If $A \in aC(X)$ and $B \in eC(X)$, then $A \cap B \in eC(X)$.

Proof. $a)$ Let $A \in aO(X)$ and $B \in eO(X)$. We aim to show that $A \cap B \in eO(X)$.

$$\left.\begin{array}{r} A \in aO(X) \Rightarrow A \subseteq int(cl(\delta\text{-}int(A))) \\ B \in eO(X) \Rightarrow B \subseteq int(\delta\text{-}cl(B)) \cup cl(\delta\text{-}int(B)) \end{array}\right\} \Rightarrow$$

$$\begin{aligned} \Rightarrow A \cap B &\subseteq [int(cl(\delta\text{-}int(A))) \cap int(\delta\text{-}cl(B))] \cup [int(cl(\delta\text{-}int(A))) \cap cl(\delta\text{-}int(B))] \\ &\subseteq int[cl(\delta\text{-}int(A)) \cap int(\delta\text{-}cl(B))] \cup cl[int(cl(\delta\text{-}int(A))) \cap \delta\text{-}int(B)] \\ &\subseteq int(cl[int(\delta\text{-}cl(B)) \cap \delta\text{-}int(A)]) \cup cl[\delta\text{-}int(\delta\text{-}cl(\delta\text{-}int(A))) \cap \delta\text{-}int(\delta\text{-}int(B))] \\ &\subseteq int(cl[\delta\text{-}int(\delta\text{-}cl(B)) \cap \delta\text{-}int(\delta\text{-}int(A))]) \cup cl[\delta\text{-}int(\delta\text{-}cl(\delta\text{-}int(A))) \cap \delta\text{-}int(B)] \\ &\subseteq int(cl[\delta\text{-}int[\delta\text{-}cl(B) \cap \delta\text{-}int(A)]]) \cup cl(\delta\text{-}int[\delta\text{-}cl(\delta\text{-}int(A) \cap \delta\text{-}int(B))]) \\ &\subseteq int(cl[\delta\text{-}int[\delta\text{-}cl(\delta\text{-}int(A) \cap B)]]) \cup \delta\text{-}cl(\delta\text{-}int(\delta\text{-}cl(\delta\text{-}int(A \cap B)))) \\ &\subseteq \delta\text{-}int(\delta\text{-}cl(\delta\text{-}int(\delta\text{-}cl(A \cap B)))) \cup \delta\text{-}cl(\delta\text{-}int(A \cap B)) \\ &= \delta\text{-}int(\delta\text{-}cl(A \cap B)) \cup cl(\delta\text{-}int(A \cap B)) \\ &= int(\delta\text{-}cl(A \cap B)) \cap cl(\delta\text{-}int(A \cap B)) \end{aligned}$$

This means $A \cap B \in eO(X)$.
$b)$ Let $A \in aC(X)$ and $B \in eC(X)$. We aim to show that $A \cap B \in eO(X)$.

$$\left.\begin{array}{l} A \in aC(X) \Rightarrow X \setminus A \in aO(X) \\ B \in eC(X) \Rightarrow X \setminus B \in eO(X) \end{array}\right\} \overset{(a)}{\Rightarrow} X \setminus (A \cup B) = (X \setminus A) \cap (X \setminus B) \in eO(X)$$

$$\Rightarrow A \cup B \in eC(X).$$ ■

Theorem 3.12 Let A and B be two subsets of a space X. If $A \in aLC(X)$ and $B \in eLC(X)$, then $A \cap B \in eLC(X)$.

Proof. Let $A \in aLC(X)$ and $B \in eLC(X)$. We aim to show that $A \cap B \in eLC(X)$.

$$\left.\begin{array}{l} A \in aLC(X) \Rightarrow (\exists U_1 \in aO(X))(\exists V_1 \in aC(X))(A = U_1 \cap V_1) \\ B \in eLC(X) \Rightarrow (\exists U_2 \in eO(X))(\exists V_2 \in eC(X))(B = U_2 \cap V_2) \end{array}\right\} \overset{\text{Lemma 3.11}}{\Rightarrow}$$

$$\left.\begin{array}{r} \Rightarrow (U_1 \cap U_2 \in eO(X))(V_1 \cap V_2 \in eC(X))(A \cap B = (U_1 \cap U_2) \cap (V_1 \cap V_2)) \\ (U := U_1 \cap U_2)(V := V_1 \cap V_2) \end{array}\right\} \Rightarrow$$

$$\Rightarrow (U \in eO(X))(V \in eC(X))(A \cap B = U \cap V) \ \Rightarrow A \cap B \in eLC(X). \quad \blacksquare$$

Theorem 3.13 Let A be a subset of a space X. If A is an a-locally closed in X, then there exists an a-closed set F in X such that $A \cap F = \emptyset$.

Proof. Let $A \in aLC(X)$. Our aim is to show that there exists an a-closed sets F in X such that $A \cap F = \emptyset$.

$$\left.\begin{array}{r} A \in aLC(X) \Rightarrow (\exists U \in aO(X))(\exists V \in aC(X))(A = U \cap V) \\ F := V \setminus U \end{array}\right\} \Rightarrow$$

$$\Rightarrow (F \in aC(X))(A \cap F = \emptyset). \quad \blacksquare$$

Remark 2 *As seen in the example below, the converse of the conditional statement given in Theorem 3.13 need not always to be true.*

***Example* 3.14** Let $X = \{a, b, c, d\}$, $\tau = \{\emptyset, X, \{a\}, \{b\}, \{a, b\}, \{a, c, d\}\}$ and $A = \{a\}$. Simple calculations show that $aO(X) = aC(X) = aLC(X) = \{\emptyset, X, \{b\}, \{a, c, d\}\}$. Then, it is clear that the set $F = \{b\} \in aC(X)$ and $A \cap F = \{a\} \cap \{b\} = \emptyset$ but A is not an a-locally closed in X.

Theorem 3.15 Let P and Q be two subsets of a space X such that $P \in aO(X)$ and $Q \in aC(X)$. Then, there exist an a-open set E and an a-closed F such that $P \cap Q \subseteq F$ and $E \subseteq P \cup Q$.

Proof. Let $P \in aO(X)$ and $Q \in aC(X)$. Our aim is to show that there exist an a-open set E and an a-closed sets F such that $P \cap Q \subseteq F$ and $E \subseteq P \cup Q$.

$$\left.\begin{array}{r} (P \in aO(X))(Q \in aC(X)) \\ (E := P \cup a\text{-}int(Q))(F := Q \cap a\text{-}cl(P)) \end{array}\right\} \Rightarrow$$

$$\Rightarrow (E \in aO(X))(F \in aC(X))(P \cap Q \subseteq F)(E \subseteq P \cup Q). \quad \blacksquare$$

Theorem 3.16 Let A and B be two subsets of a space X. If $A, B \in aLC(X)$, then $A \cap B \in aLC(X)$.

Proof. Let $A, B \in aLC(X)$. Our aim is to show that $A \cap B \in aLC(X)$.

$$\left.\begin{array}{l} A \in aLC(X) \Rightarrow (\exists U_1 \in aO(X))(\exists V_1 \in aC(X))(A = U_1 \cap V_1) \\ B \in aLC(X) \Rightarrow (\exists U_2 \in aO(X))(\exists V_2 \in aC(X))(B = U_2 \cap V_2) \end{array}\right\} \overset{\text{Lemma 2.4}}{\Rightarrow}$$

$$\Rightarrow (U_1 \cap U_2 \in aO(X))(V_1 \cap V_2 \in aC(X))(A \cap B = (U_1 \cap V_1) \cap (U_2 \cap V_2))$$

$$\Rightarrow (U_1 \cap U_2 \in aO(X))(V_1 \cap V_2 \in aC(X))(A \cap B = (U_1 \cap U_2) \cap (V_1 \cap V_2))$$

$$\Rightarrow A \cap B \in aLC(X). \quad \blacksquare$$

Definition 3.17 A space X is called an a-space if $\tau = \tau^a$, where $\tau^a = aO(X)$.

***Example* 3.18** Let $X = \{a, b, c, d\}$ and $\tau = \{\emptyset, X, \{a\}, \{b, c, d\}\}$. Simple calculations show that $\tau^a = \{\emptyset, X, \{a\}, \{b, c, d\}\}$. This means that (X, τ) is an a-space since $\tau = \tau^a$.

Theorem 3.19 Let A be a subset of a space X. Then, the following statements are equivalent:
a) A is a-locally closed;
b) $A = P \cap a\text{-}cl(A)$ for some a-open set P;
c) $a\text{-}cl(A) \setminus A$ is a-closed;
d) $A \cup (X \setminus a\text{-}cl(A))$ is a-open;
e) $A \subseteq a\text{-}int(A \cup (X \setminus a\text{-}cl(A)))$.

Proof. $(a) \Rightarrow (b)$: Let $A \in aLC(X)$.

$$A \in aLC(X) \Rightarrow (\exists P \in aO(X))(\exists Q \in aC(X))(A = P \cap Q)$$
$$\Rightarrow (\exists P \in aO(X))(\exists Q \in aC(X))(A \subseteq Q)(A = P \cap Q)$$
$$\Rightarrow (\exists P \in aO(X))(a\text{-}cl(A) \subseteq a\text{-}cl(Q) = Q)(A = P \cap Q)$$
$$\Rightarrow (\exists P \in aO(X))(A = A \cap a\text{-}cl(A) = (P \cap Q) \cap a\text{-}cl(A) = P \cap a\text{-}cl(A) \subseteq P \cap Q = A)$$
$$\Rightarrow (\exists P \in aO(X))(A = P \cap a\text{-}cl(A)).$$

$(b) \Rightarrow (c)$: Let $A \subseteq X$.

$$\left.\begin{array}{r} A \subseteq X \\ \text{Hypothesis} \end{array}\right\} \Rightarrow (\exists P \in aO(X))(a\text{-}cl(A) \setminus A = a\text{-}cl(A) \setminus (P \cap a\text{-}cl(A)))$$
$$\Rightarrow (X \setminus P \in aC(X))(a\text{-}cl(A) \setminus A = a\text{-}cl(A) \setminus P = a\text{-}cl(A) \cap (X \setminus P))$$
$$\Rightarrow a\text{-}cl(A) \setminus A \in aC(X).$$

$(c) \Rightarrow (d)$: Let $A \subseteq X$.

$$\left.\begin{array}{r} A \subseteq X \Rightarrow A \cup (X \setminus a\text{-}cl(A)) = X \setminus (a\text{-}cl(A) \setminus A) \\ \text{Hypothesis} \end{array}\right\} \Rightarrow A \cup (X \setminus a\text{-}cl(A)) \in aO(X).$$

$(d) \Rightarrow (e)$: Let $A \subseteq X$.

$$\left.\begin{array}{r} A \subseteq X \\ \text{Hypothesis} \end{array}\right\} \Rightarrow A \cup (X \setminus a\text{-}cl(A)) \in aO(X)$$
$$\left.\begin{array}{r} \Rightarrow A \cup (X \setminus a\text{-}cl(A)) = a\text{-}int(A \cup (X \setminus a\text{-}cl(A))) \\ A \subseteq X \Rightarrow A \subseteq A \cup (X \setminus a\text{-}cl(A)) \end{array}\right\} \Rightarrow A \subseteq a\text{-}int(A \cup (X \setminus a\text{-}cl(A))).$$

$(e) \Rightarrow (a)$: Let $A \subseteq X$.

$$\left.\begin{array}{r} A \subseteq X \\ \text{Hypothesis} \end{array}\right\} \Rightarrow A \subseteq a\text{-}int(A \cup (X \setminus a\text{-}cl(A)))$$
$$\begin{aligned} \Rightarrow A = A \cap a\text{-}cl(A) &\subseteq a\text{-}int(A \cup (X \setminus a\text{-}cl(A))) \cap a\text{-}cl(A) \\ &\subseteq [A \cup (X \setminus a\text{-}cl(A))] \cap a\text{-}cl(A) \\ &= [A \cap a\text{-}cl(A)] \cup [(X \setminus a\text{-}cl(A)) \cap a\text{-}cl(A)] \\ &= A \cup \emptyset \\ &= A \end{aligned}$$
$$\left.\begin{array}{r} \Rightarrow A = a\text{-}int(A \cup (X \setminus a\text{-}cl(A))) \cap a\text{-}cl(A) \\ (U := a\text{-}int(A \cup (X \setminus a\text{-}cl(A))))(V := a\text{-}cl(A)) \end{array}\right\} \Rightarrow$$
$$\Rightarrow (U \in aO(X))(V \in aC(X))(A = U \cap V) \ \Rightarrow A \in aLC(X).$$ ■

Corollary 3.20 Let A be a subset of a space X. Then, the following statements are equivalent:
a) A is a-locally open;
b) $A = Q \cup a\text{-}int(A)$ for some a-closed set Q;
c) $(\setminus A) \cup a\text{-}int(A)$ is a-closed;
d) $A \cap (X \setminus a\text{-}int(A))$ is a-closed;
e) $A \supseteq a\text{-}cl(A \cap (X \setminus a\text{-}int(A)))$.

Theorem 3.21 If $W \subseteq H \subseteq X$ and $H \in aLC(X)$, then there exists an a-locally closed set K such that $W \subseteq K \subseteq H$.

Proof. Let $W \subseteq H \subseteq X$ and $H \in aLC(X)$. Our aim is to show that there exist an a-locally closed set K such that $W \subseteq K \subseteq H$.

$$\left.\begin{array}{r} H \in aLC(X) \overset{\text{Theorem 3.19}}{\Rightarrow} (\exists P \in aO(X))(H = P \cap a\text{-}cl(H)) \\ W \subseteq H \end{array}\right\} \Rightarrow$$

$$\Rightarrow (W \subseteq P)(P \in aO(X))(P \cap a\text{-}cl(W) \subseteq P \cap a\text{-}cl(H) = H)$$

$$\left.\begin{array}{r} \Rightarrow (P \in aO(X))(W \subseteq P \cap a\text{-}cl(W) \subseteq P \cap a\text{-}cl(H) = H) \\ K := P \cap a\text{-}cl(W) \end{array}\right\} \Rightarrow$$

$$\Rightarrow (K \in aLC(X))(W \subseteq K \subseteq H).$$ ■

Definition 3.22 A subset A of a space X is called a-dense if $a\text{-}cl(A) = X$. A space X is called a-submaximal if every a-dense subset of X is a-open in X.

Theorem 3.23 A topological space X is a-submaximal iff $aLC(X) = 2^X$.

Proof. $(\Rightarrow)$: Let X be an a-submaximal space. We have always $aLC(X) \subseteq 2^X \ldots (1)$ Now, let $A \in 2^X$.

$$\left.\begin{array}{r} A \in 2^X \\ B := A \cup (X \setminus a\text{-}cl(A)) \end{array}\right\} \Rightarrow$$

$$\Rightarrow X \supseteq a\text{-}cl(B) = a\text{-}cl(A \cup (X \setminus a\text{-}cl(A))) \supseteq a\text{-}cl(A) \cup a\text{-}cl(X \setminus a\text{-}cl(A)) = X$$

$$\left.\begin{array}{r} \Rightarrow a\text{-}cl(B) = X \\ (X,\tau) \text{ is } a\text{-submaximal} \end{array}\right\} \Rightarrow B = A \cup (X \setminus a\text{-}cl(A)) \in aO(X)$$

$$\overset{\text{Theorem 3.19}(d)}{\Rightarrow} A \in aLC(X)$$

Therefore, $2^X \subseteq aLC(X) \ldots (2)$

$(1),(2) \Rightarrow aLC(X) = 2^X$.
$(\Leftarrow)$: Let $a\text{-}cl(A) = X$. Our aim is to show that $A \in aO(X)$.

$$\left.\begin{array}{r} a\text{-}cl(A) = X \Rightarrow A = A \cup (X \setminus a\text{-}cl(A)) \\ aLC(X) = 2^X \end{array}\right\} \overset{\text{Theorem 3.19}(d)}{\Rightarrow} A \in aO(X).$$ ■

Definition 3.24 A space (X,τ) is called an e-space if $\tau = \tau^e$, where $\tau^e = eO(X)$.

Remark 3 *If (X,τ) is regular and e-space, then the notions submaximal, a-submaximal, e-submaximal coincides with one another.*

Proof. $(a) \Rightarrow (b)$: Let $A \subseteq X$ and $a\text{-}cl(A) = X$.

$$\left.\begin{array}{r} A \subseteq X \\ (X,\tau) \text{ is regular} \end{array}\right\} \Rightarrow \left.\begin{array}{r} \alpha\text{-}cl(A) = a\text{-}cl(A) \subseteq cl(A) \\ a\text{-}cl(A) = X \end{array}\right\} \Rightarrow cl(A) = X$$

$$\left.\begin{array}{r} \Rightarrow cl(A) = X \\ (X,\tau) \text{ is submaximal} \end{array}\right\} \Rightarrow \left.\begin{array}{r} A \in \tau \\ (X,\tau) \text{ is regular} \end{array}\right\} \Rightarrow A \in \delta O(X) \subseteq aO(X) \quad \Rightarrow A \in aO(X).$$

$(b) \Rightarrow (c)$: Let $A \subseteq X$ and $e\text{-}cl(A) = X$.

$$\left.\begin{array}{r} A \subseteq X \Rightarrow e\text{-}cl(A) \subseteq a\text{-}cl(A) \\ e\text{-}cl(A) = X \end{array}\right\} \Rightarrow \left.\begin{array}{r} a\text{-}cl(A) = X \\ (X,\tau) \text{ is } a\text{-submaximal} \end{array}\right\} \Rightarrow A \in aO(X) \subseteq eO(X)$$

$\Rightarrow A \in eO(X)$.

$(c) \Rightarrow (a)$: Let $A \subseteq X$ and $cl(A) = X$.

$$\left.\begin{array}{r} A \subseteq X \\ (X,\tau) \text{ is } e\text{-space} \end{array}\right\} \Rightarrow \left.\begin{array}{r} e\text{-}cl(A) = cl(A) \\ cl(A) = X \end{array}\right\} \Rightarrow e\text{-}cl(A) = X$$

$$\left.\begin{array}{r} \Rightarrow e\text{-}cl(A) = X \\ (X,\tau) \text{ is } e\text{-submaximal} \end{array}\right\} \Rightarrow \left.\begin{array}{r} A \in eO(X) \\ (X,\tau) \text{ is } e\text{-space} \end{array}\right\} \Rightarrow A \in \tau.$$

■

Definition 3.25 Let A and B be two subsets of a space X. Then, A and B are said to be a-separated if $A \cap a\text{-}cl(B) = \emptyset$ and $B \cap a\text{-}cl(A) = \emptyset$.

Theorem 3.26 Let A and B be two a-locally closed sets in a space X. If A and B are a-separated, then $A \cup B \in aLC(X)$.

Proof. Let A and B be a-separated and let A and B two a-locally closed sets. We aim to show that $A \cup B \in aLC(X)$.

$$\left.\begin{array}{r} A \in aLC(X) \Rightarrow (\exists P \in aO(X))(A = P \cap a\text{-}cl(A)) \\ B \in aLC(X) \Rightarrow (\exists Q \in aO(X))(B = Q \cap a\text{-}cl(B)) \\ (U := P \cap (X \setminus a\text{-}cl(B)))(V := Q \cap (X \setminus a\text{-}cl(A))) \end{array}\right\} \Rightarrow$$

$\Rightarrow (U, V \in aO(X))(U \cap a\text{-}cl(A) = A)(V \cap a\text{-}cl(B) = B)$
$((U \cup V) \cap a\text{-}cl(A \cup B) = (U \cup V) \cap a\text{-}cl(A) \cup a\text{-}cl(B))$

$\Rightarrow (U, V \in aO(X))(U \cap a\text{-}cl(A) = A)(V \cap a\text{-}cl(B) = B)$
$((U \cup V) \cap a\text{-}cl(A \cup B) = (U \cap a\text{-}cl(A)) \cup (U \cap a\text{-}cl(B)) \cup (V \cap a\text{-}cl(A)) \cup (V \cap a\text{-}cl(B))$

$\Rightarrow (U, V \in aO(X))(U \cap a\text{-}cl(A) = A)(V \cap a\text{-}cl(B) = B)$
$((U \cup V) \cap a\text{-}cl(A \cup B) = (\underbrace{U \cap a\text{-}cl(A)}_{A}) \cup (\underbrace{U \cap a\text{-}cl(B)}_{\emptyset}) \cup (\underbrace{V \cap a\text{-}cl(A)}_{\emptyset}) \cup (\underbrace{V \cap a\text{-}cl(B)}_{B})$

$\Rightarrow (U \cup V \in aO(X))(a\text{-}cl(A \cup B) \in aC(X))((U \cup V) \cap a\text{-}cl(A \cup B) = A \cup B)$

$\Rightarrow A \cup B \in aLC(X)$. ■

Lemma 3.27 Let X and Y be two topological spaces and $A \subseteq X$ and $B \subseteq Y$.

$a)$ If $A \in aO(X)$ and $B \in aO(Y)$, then $A \times B \in aO(X \times Y)$,

b) If $A \in aC(X)$ and $B \in aC(Y)$, then $A \times B \in aC(X \times Y)$.

Proof. a) Let $A \in aO(X)$ and $B \in aO(Y)$. We aim to show that $A \times B$ is a-open in $X \times Y$.

$$\left.\begin{array}{l} A \in aO(X) \Rightarrow A \subseteq int(cl(\delta\text{-}int(A))) \\ B \in aO(Y) \Rightarrow B \subseteq int(cl(\delta\text{-}int(B))) \end{array}\right\} \Rightarrow$$

$$\begin{aligned} \Rightarrow A \times B &\subseteq int(cl(\delta\text{-}int(A))) \times int(cl(\delta\text{-}int(B))) \\ &= int[cl(\delta\text{-}int(A)) \times cl(\delta\text{-}int(B))] \\ &= int(cl[\delta\text{-}int(A) \times \delta\text{-}int(B)]) \\ &= int(cl(\delta\text{-}int(A \times B))) \end{aligned}$$

Then, we have $A \times B \in aO(X \times Y)$.
b) Let $A \in aC(X)$ and $B \in aC(Y)$. We aim to show that $A \times B$ is a-closed in $X \times Y$.

$$\left.\begin{array}{l} A \in aC(X) \Rightarrow A \supseteq cl(int(\delta\text{-}cl(A))) \\ B \in aC(Y) \Rightarrow B \supseteq cl(int(\delta\text{-}cl(B))) \end{array}\right\} \Rightarrow$$

$$\begin{aligned} \Rightarrow A \times B &\supseteq cl(int(\delta\text{-}cl(A))) \times cl(int(\delta\text{-}cl(B))) \\ &= cl[int(\delta\text{-}cl(A)) \times int(\delta\text{-}cl(B))] \\ &= cl(int[\delta\text{-}cl(A) \times \delta\text{-}cl(B)]) \\ &= cl(int(\delta\text{-}cl(A \times B))) \end{aligned}$$

Then, we have $A \times B \in aC(X \times Y)$. ■

Theorem 3.28 Let X and Y be two topological spaces and $A \subseteq X$ and $B \subseteq Y$. If $A \in aLC(X)$ and $B \in aLC(Y)$, then $A \times B \in aLC(X \times Y)$.

Proof. Let $A \in aLC(X)$ and $B \in aLC(Y)$. Our aim is to show that $A \times B$ is a-locally closed in $X \times Y$.

$$\left.\begin{array}{l} A \in aLC(X) \Rightarrow (\exists U_1 \in aO(X))(\exists V_1 \in aC(X))(A = U_1 \cap V_1) \\ B \in aLC(X) \Rightarrow (\exists U_2 \in aO(X))(\exists V_2 \in aC(X))(B = U_2 \cap V_2) \end{array}\right\} \overset{\text{Lemma 3.27}}{\Rightarrow}$$

$$\left.\begin{array}{r} \Rightarrow (U_1 \times U_2 \in aO(X))(V_2 \times V_2 \in aC(X))(A \times B = (U_1 \cap V_1) \times (U_2 \cap V_2)) \\ (U_1 \cap V_1) \times (U_2 \cap V_2) = (U_1 \times U_2) \cap (V_1 \times V_2) \end{array}\right\} \Rightarrow$$

$$\Rightarrow (U_1 \times U_2 \in aO(X))(V_2 \times V_2 \in aC(X))(A \times B = (U_1 \times U_2) \cap (V_1 \times V_2))$$

$$\Rightarrow A \times B \in aLC(X \times Y).$$ ■

4. Conclusion

In this article, we defined a new type of set, called a-locally closed, by utilizing the notion of a-open and a-closed sets and investigated their fundamental properties. Also, we obtained some characterizations of this new notion. Moreover, we compared the class of sets with the existing ones in the literature. Furthermore, we proved some relationships between this new notion and the other notions that existed in the literature and we also gave several examples. We hope that this paper will stimulate further research on the notion of locally closedness.

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