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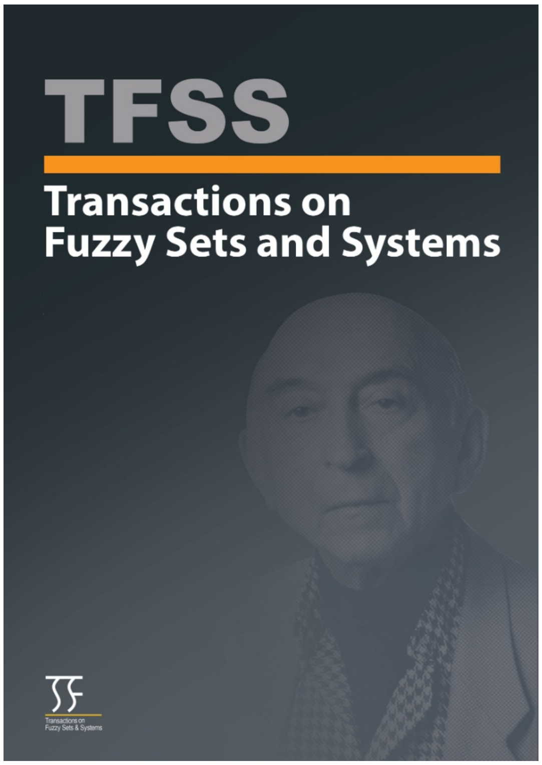

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Fixed Point Theorems in Orthogonal Intuitionistic Fuzzy b-metric Spaces with an Application to Fredholm Integral Equation

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Fixed Point Theorems in Orthogonal Intuitionistic Fuzzy b-metric Spaces with an Application to Fredholm Integral Equation

Fahim Uddin, **Muhammad Saeed**, **Khaleel Ahmed**, **Umar Ishtiaq**, **Salvatore Sessa***

Abstract. In this manuscript, the concept of an orthogonal intuitionistic fuzzy b-metric space is initiated as a generalization of an intuitionistic fuzzy b-metric space. We presented some fixed point results in this setting. For the validity of the obtained results, some non-trivial examples are given. In the last part, we established an application on the existence of a unique solution of a Fredholm-type integral equation.

AMS Subject Classification 2020: 47H10; 54H25
Keywords and Phrases: Orthogonal set, Intuitionistic fuzzy metric space, Unique solution, Integral equation.

1 Introduction

A publication showing there are solutions to differential equations established fixed-point theory in the second quarter of the eighteenth century (Joseph Liouville, 1837). This approach was further improved as a sequential approximation technique (Charles Emile Picard, 1890), and in the setting of complete normed space, it was generalized as a fixed-point theorem (Stefan Banach, 1922). It presents the a priori and a posteriori approximations for the convergence rate as well as a general way to actually determine the fixed point. Additionally, it ensures that a fixed point exists and is distinct. This information is helpful for studying metric spaces. Stefan Banach is acknowledged for developing fixed-point theory after that. Fixed-point theorems allow us to guarantee that the main problem has been resolved, as has the existence of a fixed point for a given function. In a large variety of scientific problems that are derive from many different branches of mathematics, the existence of a solution is equivalent to the existence of a fixed point for a suitable mapping.

In 1989, Bakhtin [1] established the notion of quasi-metric spaces and established some results for contraction mappings. In 1993, Czerwik [2] established the concept of b-metric spaces and discussed several fixed-point results. Eshaghi et al. [3] introduced the notion of orthogonal metric spaces and derived well-known Banach fixed point theorem. Uddin et al. [4] established orthogonal m-metric spaces and solve the integral equation. Eshaghi and Habibia [5] derived several fixed point results in the context of generalized orthogonal metric space. Senapati et al. [6] established some new fixed point theorems in the context of orthogonal metric spaces. In 1965, Zadeh [7] established the notion of fuzzy sets (FSs) to deal with those problems that do have not any clear boundaries.

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In 1960, Schweizer [8] introduced the notion of continuous t-norm and worked on statistical metric spaces. In 1975, the combination of metric spaces and FSs, named fuzzy metric spaces (FMSs), have been introduced by Kramosil and Michlek [9]. In 1994, George and Veeramani [10] modified the notion of FMSs and gave an interesting analysis of FMSs in 1997 in a research paper [11]. Deng [12] established the notion of fuzzy pseudo-metric spaces and proved neumours results in the existence and uniqueness of a solution. Shukla and Abbas [13] established the notion of fuzzy metric-like spaces as a generalization of FMSs. Hezarjaribi [14] established the notion of orthogonal FMSs as a generalization of FMSs. Ndban [15] established the concept of fuzzy b-metric spaces (FBMSs) and Jeved et al. [16] introduced fuzzy b-metric like spaces as a generalization of FBMSs. The authors [17, 18, 19, 20] derived several fixed points results under some circumstances in the context of FBMSs. In 2004, Park [21] introduced the notion of intuitionistic fuzzy metric spaces (IFMSs), in which he combined the notions of continuous t-norm, continuous t-conorm, FSs and metric space.

Rafi and Noorani [22], Sintunavarat and Kumam [23], Alaca et al. [24] and Mohamad [25] derived some fixed point results for contraction mappings in the context of IFMSs. Konwar [26] introduced the notion of intuitionistic fuzzy b-metric spaces (IFBMSs) as a generalization of IFMSs and derived fixed point results. Baleanu and Rezapour [27] and Sudsutad and Tariboon [28] worked on fractional differential equations. In this manuscript, we aim to toss the notion of orthogonal Intuitionistic fuzzy b-metric spaces (OIFBMSs) as a generalization of IFBMSs. We provide some related fixed point theorems, including non-trivial examples and an application. Some of the following notions are used throughout this paper, as CTN for a continuous t-norm, CTCN for a continuous t-conorm and FP for fixed point.

2 preliminaries

In this section, we will discuss some important definitions that support our main result.

Definition 2.1. [1] Suppose $\Xi \neq \phi$. Given a five tuple $(\Xi, G, H, *, \Delta)$ where $*$ is a CTN, Δ is a CTCN, $\theta \geq 1$ and G, H are FSs on $\Xi \times \Xi \times (0, \infty)$. If $(\Xi, G, H, *, \Delta)$ meets the below conditions for all $\mathrm{w}, \mathrm{k} \in \Xi$ and $\pi, \tau > 0$:

(B1) $G(\mathrm{w}, \mathrm{k}, \tau) + H(\mathrm{w}, \mathrm{k}, \tau) \leq 1$;

(B2) $G(\mathrm{w}, \mathrm{k}, \tau) > 0$;

(B3) $G(\mathrm{w}, \mathrm{k}, \tau) = 1 \Leftrightarrow \mathrm{w} = \mathrm{k}$;

(B4) $G(\mathrm{w}, \mathrm{k}, \tau) = G(\mathrm{k}, \mathrm{w}, \tau)$;

(B5) $G(\mathrm{w}, e, \theta(\tau + \pi)) \geq G(\mathrm{w}, \mathrm{k}, \tau) * G(\mathrm{k}, e, \Pi)$;

(B6) $G(\mathrm{w}, \mathrm{k}, \cdot)$ is a non decreasing function of R^+ and $lim_{\tau \to \infty} G(\mathrm{w}, \mathrm{k}, \tau) = 1$;

(B7) $H(\mathrm{w}, \mathrm{k}, \tau) > 0$;

(B8) $H(\mathrm{w}, \mathrm{k}, \tau) = 0 \Leftrightarrow \mathrm{w} = \mathrm{k}$;

(B9) $H(\mathrm{w}, \mathrm{k}, \tau) = H(\mathrm{k}, \mathrm{w}, \tau)$;

(B10) $H(\mathrm{w}, e, \theta(\tau + \pi)) \leq H(\mathrm{w}, \mathrm{k}, \tau) \Delta H(\mathrm{k}, e, \Pi)$;

(B11) $H(\mathrm{w}, \mathrm{k}, \cdot)$ is a non increasing function of R^+ and $lim_{\tau \to \infty} H(\mathrm{w}, \mathrm{k}, \tau) = 1$;

Then $(\Xi, G, H, *, \Delta)$ is an IFBMS.

Definition 2.2. Assume $\Xi \neq \phi$. Let $\perp \in \Xi \times \Xi$ be a binary relation. Suppose there exists $w_0 \in \Xi$ such that $w_0 \perp w$ or $w \perp w_0$ for all $w \in \Xi$. Thus, Ξ is known as orthogonal set (OS) and denoted by $(\Xi, \perp)$

Definition 2.3. Assume that $(\Xi, \perp)$ is an OS. A sequence $\{w_n\}$ for $n \in \mathbf{N}$ is known to be an O-sequence if $(\forall n, w_n \perp w_{n+1})$ or $(\forall n, w_{n+1} \perp w_n)$

3 Orthogonal Intuitionistic Fuzzy b-metric Spaces

Now, we establish the notion of OIFBMSs and derive several FP results with non-trivial examples.

Definition 3.1. $(\Xi, G, H, *, \Delta)$ is known to be an OIFBMS if Ξ is a (non empty) OS, $*$ is a CTN, Δ is a CTCN, and G, H are FSs on $\Xi \times \Xi \times (0, \infty)$ verifying the below conditions for a given real number $\theta \geq 1$:

$(B_\perp 1)$ $G(w, k, \tau) + H(w, k, \tau) \leq 1$ for all $w, k \in \Xi$, $\tau > 0$ such that $w \perp k$ and $k \perp w$;

$(B_\perp 2)$ $G(w, k, \tau) > 0$ for all $w, k \in \Xi$, $\tau > 0$ such that $w \perp k$ and $k \perp w$;

$(B_\perp 3)$ $G(w, k, \tau) = 1 \Leftrightarrow w = k$; for all $w, k \in \Xi$, $\tau > 0$ such that $w \perp k$ and $k \perp w$;

$(B_\perp 4)$ $G(w, k, \tau) = G(k, w, \tau)$ for all $w, k \in \Xi$, $\tau > 0$ such that $w \perp k$ and $k \perp w$;

$(B_\perp 5)$ $G(w, e, \theta(\tau + \pi)) \geq G(w, k, \tau) * G(k, e, \Pi)$ for all $w, k \in \Xi$, $\tau > 0$ such that $w \perp k$ and $k \perp w$;

$(B_\perp 6)$ $G(w, k, \cdot)$ is a non decreasing function of R^+ and $lim_{\tau \to \infty} G(w, k, \tau) = 1$ for all $w, k \in \Xi$, $\tau > 0$ such that $w \perp k$ and $k \perp w$;

$(B_\perp 7)$ $H(w, k, \tau) > 0$for all $w, k \in \Xi$, $\tau > 0$ such that $w \perp k$ and $k \perp w$;

$(B_\perp 8)$ $H(w, k, \tau) = 0 \Leftrightarrow w = k$ for all $w, k \in \Xi$, $\tau > 0$ such that $w \perp k$ and $k \perp w$;

$(B_\perp 9)$ $H(w, k, \tau) = H(k, w, \tau)$ for all $w, k \in \Xi$, $\tau > 0$ such that $w \perp k$ and $k \perp w$;

$(B_\perp 10)$ $H(w, e, \theta(\tau + \pi)) \leq H(w, k, \tau) \Delta H(k, e, \Pi)$ for all $w, k \in \Xi$, $\tau > 0$ such that $w \perp k$ and $k \perp w$;

$(B_\perp 11)$ $H(w, k, \cdot)$ is a non increasing function of R^+ and $\lim_{\tau \to \infty} H(w, k, \tau) = 1$ for all $w, k \in \Xi$, $\tau > 0$ such that $w \perp k$ and $k \perp w$;

Then $(\Xi, G, H, *, \Delta)$ is an IFBMS.

Example 3.2. Let $\Xi = R$ and define $\sigma * \theta = \sigma\theta$, $\sigma \Delta \theta = \min\{\sigma, \theta\}$ and $\perp$ by $w \perp k$ iff $w + k \geq 0$. Let

$$G(w, k, \tau) = \begin{cases} 1 \text{ if } w = k, \\ \frac{\tau}{\tau + \max\{w, k\}^\alpha} \text{ otherwise.} \end{cases} \tag{1}$$

and

$$H(w, k, \tau) = \begin{cases} 0 \text{ if } w = k, \\ \frac{\max\{w, k\}^\alpha}{\tau + \max\{w, k\}^\alpha} \text{ otherwise.} \end{cases} \tag{2}$$

for all $w, k \in \Xi, \tau > 0$ with α belong to odd natural numbers.
Proof. $(B_\perp 1) - (B_\perp 3)$, $(B_\perp 5) - (B_\perp 9)$ and $(B_\perp 11)$ are obvious. Here, we prove $(B_\perp 4)$ and $(B_\perp 10)$. $(B_\perp 4)$: for a random number $\theta \geq 1$, one writes

$$\max\{w, e\}^\alpha \leq \theta[\max\{w, k\}^\alpha + \max\{k, e\}^\alpha]$$

Thus,

$$\tau\pi \max\{\mathrm{w}, e\}^\alpha \leq \theta(\tau+\pi)\pi \max\{\mathrm{w}, \mathrm{k}\}^\alpha + \theta(\tau+\pi)\tau \max\{\mathrm{k}, e\}^\alpha.$$

Consequently,

$$\tau\pi \max\{\mathrm{w}, e\}^\alpha \leq \theta(\tau+\pi)\pi \max\{\mathrm{w}, \mathrm{k}\}^\alpha + \theta(\tau+\pi)\tau \max\{\mathrm{k}, e\}^\alpha + \theta(\tau+\pi) \max\{\mathrm{k}, e\}^\alpha.$$

Thus,

$$\tau\pi \max\{\mathrm{w}, e\}^\alpha \leq \theta(\tau+\pi)[\pi \max\{\mathrm{w}, \mathrm{k}\}^\alpha + \tau \max\{\mathrm{k}, e\}^\alpha + \max\{\mathrm{w}, \mathrm{k}\}^\alpha \max\{\mathrm{k}, e\}^\alpha].$$

one write

$$\theta(\tau+\pi)\tau\pi + \tau\pi \max\{\mathrm{w}, e\}^\alpha \leq \theta(\tau+\pi)\tau\pi + \theta(\tau+\pi)[\pi \max\{\mathrm{w}, \mathrm{k}\}^\alpha + \tau \max\{\mathrm{k}, e\}^\alpha + \max\{\mathrm{w}, \mathrm{k}\}^\alpha \max\{\mathrm{k}, e\}^\alpha].$$

Therefore,

$$\theta(\tau+\pi)\tau\pi + \tau\pi \max\{\mathrm{w}, e\}^\alpha \leq \theta(\tau+\pi)[\tau\pi + \pi \max\{\mathrm{w}, \mathrm{k}\}^\alpha + \tau \max\{\mathrm{k}, e\}^\alpha + \max\{\mathrm{w}, \mathrm{k}\}^\alpha \max\{\mathrm{k}, e\}^\alpha].$$

That is,

$$\tau\pi[\theta(\tau+\pi) + \max\{\mathrm{w}, e\}^\alpha] \leq \theta(\tau+\pi)[\tau + \max\{\mathrm{w}, \mathrm{k}\}^\alpha][\pi + \max\{\mathrm{k}, e\}^\alpha]$$

Hence,

$$\frac{\theta(\tau+\pi)}{\theta(\tau+\pi) + \max\{\mathrm{w}, e\}^\alpha} \geq \frac{\tau\pi}{[\tau + \max\{\mathrm{w}, \mathrm{k}\}^\alpha][\pi + \max\{\mathrm{k}, e\}^\alpha]}.$$

$$\frac{\theta(\tau+\pi)}{\theta(\tau+\pi) + \max\{\mathrm{w}, e\}^\alpha} \geq \frac{\tau}{\tau + \max\{\mathrm{w}, \mathrm{k}\}^\alpha}.$$

That is,

$$G(\mathrm{w}, e, \theta(\tau+\pi)) \geq G(\mathrm{w}, \mathrm{k}\tau) * G(\mathrm{k}, e, \pi).$$

$(B_\perp 10)$: One writes

$$\max\{\mathrm{w}, e\}^\alpha = \max\{\mathrm{w}, e\}^\alpha \max\left\{\frac{\max\{\mathrm{w}, \mathrm{k}\}^\alpha}{\max\{\mathrm{w}, \mathrm{k}\}^\alpha}, \frac{\max\{\mathrm{k}, e\}^\alpha}{\max\{\mathrm{k}, e\}^\alpha}\right\}.$$

Then

$$\max\{\mathrm{w}, e\}^\alpha \leq [\theta(\tau+\pi) + \max\{\mathrm{w}, e\}^\alpha] \max\left\{\frac{\max\{\mathrm{w}, \mathrm{k}\}^\alpha}{\max\{\mathrm{w}, \mathrm{k}\}^\alpha}, \frac{\max\{\mathrm{k}, e\}^\alpha}{\max\{\mathrm{k}, e\}^\alpha}\right\}.$$

That is,

$$\frac{\max\{\mathrm{w}, e\}^\alpha}{\theta(\tau+\pi) + \max\{\mathrm{w}, e\}^\alpha} \leq \max\left\{\frac{\max\{\mathrm{w}, \mathrm{k}\}^\alpha}{\tau + \max\{\mathrm{w}, \mathrm{k}\}^\alpha}, \frac{\max\{\mathrm{k}, e\}^\alpha}{\pi + \max\{\mathrm{k}, e\}^\alpha}\right\}.$$

Hence,

$$H(\mathrm{w}, e, \theta(\tau+\pi)) \leq H(\mathrm{w}, \mathrm{k}, \tau) \Delta H(\mathrm{k}, e, \pi).$$

Now, we show it's not an IFBM. Indeed, for $\pi = \tau = 1, \mathrm{w} = -1, \mathrm{k} = -\frac{1}{2}$ and $\alpha = 3$, (B4) and (B10) fail. □

Example 3.3. Let $\Xi = \mathbb{R}$ and define $\sigma * \theta = \sigma\theta$, $\sigma\Delta\theta = \min\{\sigma, \theta\}$ and $\perp$ by $\mathrm{w} \perp \mathrm{k}$ iff $\mathrm{w} + \mathrm{k} \geq 0$. Let

$$G(\mathrm{w}, \mathrm{k}, \tau) = \begin{cases} 1 \text{ if } \mathrm{w} = \mathrm{k}, \\ \left[e^{\frac{\max\{\mathrm{w}, \mathrm{k}\}^\alpha}{\tau}}\right]^{-1} \text{ otherwise.} \end{cases} \tag{3}$$

and

$$H(\mathrm{w},\mathrm{k},\tau)=\begin{cases}0 \text{ if } \mathrm{w}=\mathrm{k},\\ 1-\left[e^{\frac{\max\{\mathrm{w},\mathrm{k}\}^{\alpha}}{\tau}}\right]^{-1} \text{ otherwise.}\end{cases} \tag{4}$$

for all $\mathrm{w},\mathrm{k}\in\Xi,\tau>0$ with α belong to odd natural numbers.

Proof. $(B_{\perp}1)-(B_{\perp}3)$, $(B_{\perp}5)-(B_{\perp}9)$ and $(B_{\perp}11)$ are obvious. Here, we prove $(B_{\perp}4)$ and $(B_{\perp}10)$.
$(B_{\perp}4)$: for a random number $\theta\geq 1$, one writes

$$\max\{\mathrm{w},e\}^{\alpha}\leq\theta\left[\max\{\mathrm{w},\mathrm{k}\}^{\alpha}+\max\{\mathrm{k},e\}^{\alpha}\right].$$

Therefore,

$$\max\{\mathrm{w},e\}^{\alpha}\leq\theta\left[\frac{\tau+\pi}{\tau}\max\{\mathrm{w},\mathrm{k}\}^{\alpha}+\frac{\tau+\pi}{\pi}\max\{\mathrm{k},e\}^{\alpha}\right]$$

Then

$$\frac{\max\{\mathrm{w},e\}^{\alpha}}{\theta(\tau+\pi)}\leq\frac{\max\{\mathrm{w},\mathrm{k}\}^{\alpha}}{\tau}+\frac{\max\{\mathrm{k},e\}^{\alpha}}{\pi}$$

Since, e^{w} is an increasing function, one gets

$$e^{\frac{\max\{\mathrm{w},e\}^{\alpha}}{\theta(\tau+\pi)}}\leq e^{\frac{\max\{\mathrm{w},\mathrm{k}\}^{\alpha}}{\tau}}\cdot e^{\frac{\max\{\mathrm{k},e\}^{\alpha}}{\pi}}.$$

That is

$$\left[e^{\frac{\max\{\mathrm{w},e\}^{\alpha}}{\theta(\tau+\pi)}}\right]^{-1}\geq\left[e^{\frac{\max\{\mathrm{w},\mathrm{k}\}^{\alpha}}{\tau}}\right]^{-1}\cdot\left[e^{\frac{\max\{\mathrm{k},e\}^{\alpha}}{\pi}}\right]^{-1}.$$

Hence,

$$G(\mathrm{w},e,\theta(\tau+\pi))\geq G(\mathrm{w},\mathrm{k}\tau)*G(\mathrm{k},e,\pi).$$

$(B_{\perp}10)$: For a random $\theta\geq 1$, we write

$$\frac{\max\{\mathrm{w},e\}^{\alpha}}{\theta(\tau+\pi)}\leq\max\left\{\frac{\max\{\mathrm{w},\mathrm{k}\}^{\alpha}}{\tau},\frac{\max\{\mathrm{k},e\}^{\alpha}}{\pi}\right\}.$$

That is,

$$e^{\frac{\max\{\mathrm{w},e\}^{\alpha}}{\theta(\tau+\pi)}}\leq\max\left\{e^{\frac{\max\{\mathrm{w},\mathrm{k}\}^{\alpha}}{\tau}},e^{\frac{\max\{\mathrm{k},e\}^{\alpha}}{\pi}}\right\}.$$

Then,

$$\left[e^{\frac{\max\{\mathrm{w},e\}^{\alpha}}{\theta(\tau+\pi)}}\right]^{-1}\geq\max\left\{\left[e^{\frac{\max\{\mathrm{w},\mathrm{k}\}^{\alpha}}{\tau}}\right]^{-1},\left[e^{\frac{\max\{\mathrm{k},e\}^{\alpha}}{\pi}}\right]^{-1}\right\}.$$

That is,

$$1-\left[e^{\frac{\max\{\mathrm{w},e\}^{\alpha}}{\theta(\tau+\pi)}}\right]^{-1}\leq\max\left\{1-\left[e^{\frac{\max\{\mathrm{w},\mathrm{k}\}^{\alpha}}{\tau}}\right]^{-1},1-\left[e^{\frac{\max\{\mathrm{k},e\}^{\alpha}}{\pi}}\right]^{-1}\right\}.$$

Hence,

$$H(\mathrm{w},e,\theta(\tau+\pi))\leq H(\mathrm{w},\mathrm{k},\tau)\Delta H(\mathrm{k},e,\pi).\forall\mathrm{w},\mathrm{k},e\in\Xi,\forall\tau,\pi>0.$$

Now, we show it's not an IFBM. Indeed, for $\pi=\tau=1,\mathrm{w}=-1,\mathrm{k}=-\frac{1}{2},e=-2$ and $\alpha=3$, (B4) and (B10) is not satisfy. $\square$

Example 3.4. Let $\Xi = \mathbb{R}$ and define $\sigma * \theta = \sigma\theta, \sigma\Delta\theta = \max\{\sigma, \theta\}$ and $\perp$ by $\mathrm{w} \perp \mathrm{k}$ iff $\mathrm{w} + \mathrm{k} \geq 0$. Suppose

$$G(\mathrm{w}, \mathrm{k}, \tau) = \frac{\tau + \min\{\mathrm{w}, \mathrm{k}\}^{\alpha}}{\tau + \max\{\mathrm{w}, \mathrm{k}\}^{\alpha}} \tag{5}$$

and

$$H(\mathrm{w}, \mathrm{k}, \tau) = 1 - \frac{\tau + \min\{\mathrm{w}, \mathrm{k}\}^{\alpha}}{\tau + \max\{\mathrm{w}, \mathrm{k}\}^{\alpha}} \tag{6}$$

for all $\mathrm{w}, \mathrm{k} \in \Xi, \tau > 0$ with α belong to odd natural numbers. Here, $(\Xi, G, H, *, \Delta, \perp)$ is an OIFBMS. It is not an IFBMS. Indeed, if it is the case, from (B4),

$$\frac{\theta(\tau + \pi) + \min\{\mathrm{w}, \mathrm{k}\}^{\alpha}}{\theta(\tau + \pi) + \max\{\mathrm{w}, \mathrm{k}\}^{\alpha}} \geq \frac{\tau + \min\{\mathrm{w}, \mathrm{k}\}^{\alpha}}{\tau + \max\{\mathrm{w}, \mathrm{k}\}^{\alpha}} \cdot \frac{\pi + \min\{\mathrm{w}, \mathrm{k}\}^{\alpha}}{\pi + \max\{\mathrm{w}, \mathrm{k}\}^{\alpha}}$$

and from case (B10)

$$1 - \frac{\theta(\tau + \pi) + \min\{\mathrm{w}, \mathrm{k}\}^{\alpha}}{\theta(\tau + \pi) + \max\{\mathrm{w}, \mathrm{k}\}^{\alpha}} \leq \max\left[1 - \frac{\tau + \min\{\mathrm{w}, \mathrm{k}\}^{\alpha}}{\tau + \max\{\mathrm{w}, \mathrm{k}\}^{\alpha}} \cdot 1 - \frac{\pi + \min\{\mathrm{w}, \mathrm{k}\}^{\alpha}}{\pi + \max\{\mathrm{w}, \mathrm{k}\}^{\alpha}}\right].$$

Then by taking $\mathrm{w} = \mathrm{k}, e = -2$ and $\alpha = \frac{1}{2}$, the above inequalities are not satisfied.

Remark 3.5. Every IFBMS is an OIFBMS, but the converse is not true. The above examples confirm this reverse statement.

Definition 3.6. An O-sequence $\{\mathrm{w}_n\}$ is an OIFBMS $(\Xi, G, H, *, \Delta, \perp)$ is called an orthogonal convergent (O-convergent) to $\mathrm{w} \in \Xi$, if

$$\lim_{n\to\infty} G(\mathrm{w}_n, \mathrm{w}, \tau) = 1, \forall \tau > 0,$$

and

$$\lim_{n\to\infty} H(\mathrm{w}_n, \mathrm{w}, \tau) = 0, \forall \tau > 0,$$

Definition 3.7. An O-sequence $\{\mathrm{w}_n\}$ is an OIFBMS $(\Xi, G, H, *, \Delta, \perp)$ is known to be an orthogonal Cauchy (O-Cauchy) if

$$\lim_{n\to\infty} G(\mathrm{w}_n, \mathrm{w}, \tau) = 1,$$

and

$$\lim_{n\to\infty} H(\mathrm{w}_n, \mathrm{w}, \tau) = 0,$$

for all $\tau > 0, p \geq 1$.

Definition 3.8. Let $\xi : \Xi \to \Xi$ is $\perp$-continuous at $\mathrm{w} \in \Xi$ is an OIFBMS $(\Xi, G, H, *, \Delta, \perp)$, whenever for each O-sequence w_n for all $n \in \mathbb{N}$ in Ξ if $\lim_{n\to\infty} G(\mathrm{w}_n, \mathrm{w}, \tau) = 1$ and $\lim_{n\to\infty} H(\mathrm{w}_n, \mathrm{w}, \tau) = 0$ for all $\tau > 0$, then $\lim_{n\to\infty} G(\xi\mathrm{w}_n, \xi\mathrm{w}, \tau) = 1$ and $\lim_{n\to\infty} H(\xi\mathrm{w}_n, \xi\mathrm{w}, \tau) = 0$ for all $\tau > 0$. Furthermore, ξ is $\perp$-continuous on Ξ if ξ $\perp$-continuous at each $\mathrm{w} \in \Xi$. Also, ξ is $\perp$- preserving if $\xi\mathrm{w} \perp \xi\mathrm{k}$, whence $\mathrm{w} \perp \mathrm{k}$.

Definition 3.9. An OIFBMS $(\Xi, G, H, *, \Delta, \perp)$ is known to be orthogonally complete (O-complete) if every O-Cauchy O-sequence is O- convergent.

Remark 3.10. It is necessary that the limit of an O-convergent O-sequence is unique in an OIFBMS.

Remark 3.11. It is necessary that the limit of an O-convergent O-sequence is O-Cauchy in an OIFBMS.

Lemma 3.12. *If for some $v \in (0,1)$ and $\mathrm{w}, \mathrm{k} \in \Xi$,*

$$G(\mathrm{w}, \mathrm{k}, \tau) \geq G\left(\mathrm{w}, \mathrm{k}, \frac{\tau}{v}\right), \tau > 0,$$

and

$$H(\mathrm{w}, \mathrm{k}, \tau) \leq H\left(\mathrm{w}, \mathrm{k}, \frac{\tau}{v}\right), \tau > 0,$$

then $\mathrm{w} = \mathrm{k}$.
Proof. *The proof is follows from [8].* □

Definition 3.13. Suppose $(\Xi, G, H, *, \Delta, \perp)$ be an OIFBMS. A mapping $\xi : \Xi \to \Xi$ is an orthogonal contraction ($\perp$-contraction) if there exists $\varrho \in (0,1)$ such that for every $\tau > 0$ and $\mathrm{w}, \mathrm{k} \in \Xi$ with $\mathrm{w} \perp \mathrm{k}$, we have

$$G(\xi \mathrm{w}, \xi \mathrm{k}, \varrho\tau) \geq G(\mathrm{w}, \mathrm{k}, \tau), \tag{7}$$

$$H(\xi \mathrm{w}, \xi \mathrm{k}, \varrho\tau) \leq H(\mathrm{w}, \mathrm{k}, \tau). \tag{8}$$

Theorem 3.14. *Let $(\Xi, G, H, *, \Delta, \perp)$ be an O-complete IFBMS such that*

$$\lim_{\tau \to \infty} G(\mathrm{w}, \mathrm{k}, \tau) = 1,$$

and

$$\lim_{\tau \to \infty} H(\mathrm{w}, \mathrm{k}, \tau) = 0.$$

for all $\mathrm{w}, \mathrm{k} \in \Xi$. Suppose $\xi : \Xi \to \Xi$ be an $\perp$-continuous and $\perp$-preserving mapping. Thus, ξ has a unique FP, say $\mathrm{w}_* \in \Xi$. Furthermore,

$$\lim_{\tau \to \infty} G(\xi^n \mathrm{w}, \mathrm{k}, \tau) = 1,$$

and

$$\lim_{\tau \to \infty} H(\xi^n \mathrm{w}, \mathrm{k}, \tau) = 0.$$

for all $\mathrm{w}, \mathrm{k} \in \Xi$.
Proof. *Let $(\Xi, G, H, *, \Delta, \perp)$ be an O-complete IFBMS, there exists $\mathrm{w}_0 \in \Xi$ such that $\mathrm{w}_0 \perp \mathrm{k}$ for all $\mathrm{k} \in \Xi$, that is, $\mathrm{w}_0 \perp \xi \mathrm{w}_0$. Take $\mathrm{w}_n = \xi^n \mathrm{w}_0 = \xi \mathrm{w}_{n-1}$ for all $n \in \mathbb{N}$. Since ξ is $\perp$-preserving, $\{\mathrm{w}_n\}$ is an O-sequence. From assumption that ξ is an $\perp$-contraction, we have*

$$G(\mathrm{w}_{n+1}, \mathrm{w}_n, \varrho\tau) = G(\xi \mathrm{w}_n, \xi \mathrm{w}_{n-1}, \varrho\tau) \geq G(\mathrm{w}_n, \mathrm{w}_{n-1}, \tau)$$

for all $n \in \mathbb{N}$ and $\tau > 0$. Note that G is non-decreasing on $(0, \infty)$. By utilizing above inequality, we have

$$G(\mathrm{w}_{n+1}, \mathrm{w}_n, \tau) \geq G(\mathrm{w}_{n+1}, \mathrm{w}_n, \varrho\tau) = G(\xi \mathrm{w}_{n+1}, \xi \mathrm{w}_n, \varrho\tau) \geq G(\mathrm{w}_n, \mathrm{w}_{n-1}, \tau)$$
$$= G(\xi \mathrm{w}_{n-1}, \xi \mathrm{w}_{n-2}, \tau) \geq G\left(\mathrm{w}_{n-1}, \mathrm{w}_{n-2}, \frac{\tau}{\varrho}\right) \geq \cdots \geq G\left(\mathrm{w}_1, \mathrm{w}_0, \frac{\tau}{\varrho^n}\right) \tag{9}$$

for all $n \in \mathbb{N}$ and $\tau > 0$. Thus, from (9) and (B4), we deduce

$$G(\mathrm{w}_n, \mathrm{w}_{n+m}, \tau) \geq G\left(\mathrm{w}_n, \mathrm{w}_{n+1}, \frac{\tau}{\theta}\right) * G\left(\mathrm{w}_{n+1}, \mathrm{w}_{n+m}, \frac{\tau}{\theta}\right)$$

$$\geq G\left(\mathrm{w}_n, \mathrm{w}_{n+1}, \frac{\tau}{\theta}\right) * G\left(\mathrm{w}_{n+1}, \mathrm{w}_{n+2}, \frac{\tau}{\theta^2}\right) * G\left(\mathrm{w}_{n+2}, \mathrm{w}_{n+3}, \frac{\tau}{\theta^3}\right) * \cdots * G\left(\mathrm{w}_{n+m-1}, \mathrm{w}_{n+m}, \frac{\tau}{\theta^{n+m}}\right)$$

$$\geq G\left(\mathrm{w}_1, \mathrm{w}_0, \frac{\tau}{\theta\varrho^n}\right) * G\left(\mathrm{w}_{n+1}, \mathrm{w}_{n+2}, \frac{\tau}{\theta^2\varrho^n}\right) * G\left(\mathrm{w}_{n+2}, \mathrm{w}_{n+3}, \frac{\tau}{\theta^3\varrho^n}\right) * \cdots * G\left(\mathrm{w}_{n+m-1}, \mathrm{w}_{n+m}, \frac{\tau}{\theta^{n+m}\varrho^n}\right) \tag{10}$$

We know that $\lim_{\tau\to\infty} G(\mathrm{w}, \mathrm{k}, \tau) = 1$, *for all* $\mathrm{w}, \mathrm{k} \in \Xi$ *and* $\tau > 0$. *So, from (10), we have*

$$\lim_{\tau\to\infty} G(\mathrm{w}_n, \mathrm{w}_{n+m}, \tau) \geq 1 * 1 * \cdots * 1 = 1. \tag{11}$$

Similarly,

$$H(\mathrm{w}_{n+1}, \mathrm{w}_n, \varrho\tau) = H(\xi\mathrm{w}_n, \xi\mathrm{w}_{n-1}, \varrho\tau) \leq H(\mathrm{w}_n, \mathrm{w}_{n-1}, \tau)$$

for all $n \in \mathbb{N}$ *and* $\tau > 0$. *By utilizing above inequality, we have*

$$H(\mathrm{w}_{n+1}, \mathrm{w}_n, \tau) \leq H(\mathrm{w}_{n+1}, \mathrm{w}_n, \varrho\tau) = H(\xi\mathrm{w}_{n+1}, \xi\mathrm{w}_n, \varrho\tau) \leq H(\mathrm{w}_n, \mathrm{w}_{n-1}, \tau)$$

$$= H(\xi\mathrm{w}_{n-1}, \xi\mathrm{w}_{n-2}, \tau) \leq H\left(\mathrm{w}_{n-1}, \mathrm{w}_{n-2}, \frac{\tau}{\varrho}\right) \leq \cdots \leq H\left(\mathrm{w}_1, \mathrm{w}_0, \frac{\tau}{\varrho^n}\right) \tag{12}$$

for all $n \in \mathbb{N}$ *and* $\tau > 0$. *Thus, from (12) and (B10), we deduce*

$$H(\mathrm{w}_n, \mathrm{w}_{n+m}, \tau) \leq H\left(\mathrm{w}_n, \mathrm{w}_{n+1}, \frac{\tau}{\theta}\right) \Delta H\left(\mathrm{w}_{n+1}, \mathrm{w}_{n+m}, \frac{\tau}{\theta}\right)$$

$$\leq H\left(\mathrm{w}_n, \mathrm{w}_{n+1}, \frac{\tau}{\theta}\right) \Delta H\left(\mathrm{w}_{n+1}, \mathrm{w}_{n+2}, \frac{\tau}{\theta^2}\right) \Delta H\left(\mathrm{w}_{n+2}, \mathrm{w}_{n+3}, \frac{\tau}{\theta^3}\right) \Delta \cdots \Delta H\left(\mathrm{w}_{n+m-1}, \mathrm{w}_{n+m}, \frac{\tau}{\theta^{n+m}}\right)$$

$$\leq H\left(\mathrm{w}_1, \mathrm{w}_0, \frac{\tau}{\theta\varrho^n}\right) \Delta H\left(\mathrm{w}_{n+1}, \mathrm{w}_{n+2}, \frac{\tau}{\theta^2\varrho^n}\right) \Delta H\left(\mathrm{w}_{n+2}, \mathrm{w}_{n+3}, \frac{\tau}{\theta^3\varrho^n}\right) \Delta \cdots \Delta H\left(\mathrm{w}_{n+m-1}, \mathrm{w}_{n+m}, \frac{\tau}{\theta^{n+m}\varrho^n}\right) \tag{13}$$

We know that $\lim_{\tau\to\infty} H(\mathrm{w}, \mathrm{k}, \tau) = 0$, *for all* $\mathrm{w}, \mathrm{k} \in \Xi$ *and* $\tau > 0$. *So, from (13), we have*

$$\lim_{\tau\to\infty} H(\mathrm{w}_n, \mathrm{w}_{n+m}, \tau) \leq 0\Delta 0\Delta \cdots \Delta 0 = 0. \tag{14}$$

So, $\{\mathrm{w}_n\}$ *is an O-sequence. The O-sequence. The O-completeness of the IFBMS* $(\Xi, \mathrm{w}, \mathrm{k}, *, \Delta, \perp)$ *ensure that there exists* $\mathrm{w}_* \in \Xi$ *such that* $G(\mathrm{w}_n, \mathrm{w}_*, \tau) \to 1$, *and* $H(\mathrm{w}_n, \mathrm{w}_*, \tau) \to 0$, *as* $n \to +\infty$ *for all* $\tau > 0$. *Now, since* ξ *is an* $\perp$*-continuous mapping,* $G(\mathrm{w}_{n+1}, \xi\mathrm{w}_*, \tau) = G(\xi\mathrm{w}_{n+}, \xi\mathrm{w}_*, \tau) \to 1$ *and* $H(\mathrm{w}_{n+1}, \xi\mathrm{w}_*, \tau) = H(\xi\mathrm{w}_{n+}, \xi\mathrm{w}_*, \tau) \to 0$ *as* $n \to +\infty$. *Now, we have*

$$G(\mathrm{w}_*, \xi\mathrm{w}_*, \tau) \geq G\left(\mathrm{w}_*, \mathrm{w}_{n+1}, \frac{\tau}{2\theta}\right) * G\left(\mathrm{w}_{n+1}, \xi\mathrm{w}_*, \frac{\tau}{2\theta}\right),$$

$$H(\mathrm{w}_*, \xi\mathrm{w}_*, \tau) \leq H\left(\mathrm{w}_*, \mathrm{w}_{n+1}, \frac{\tau}{2\theta}\right) \Delta H\left(\mathrm{w}_{n+1}, \xi\mathrm{w}_*, \frac{\tau}{2\theta}\right).$$

Taking limit as $n \to \infty$, *we get* $G(\mathrm{w}_*, \xi\mathrm{w}_*, \tau) = 1 * 1 = 1$ *and* $H(\mathrm{w}_*, \xi\mathrm{w}_*, \tau) = 0\Delta 0 = 0$*and hence* $\xi\mathrm{w}_* = \mathrm{w}_*$.

Uniqueness*:*

Let w_* *and* k_* *be two FPs of* ξ *such that* $\mathrm{w}_* \neq \mathrm{k}_*$. *We have* $\mathrm{w}_0 \perp \mathrm{w}_*$ *and* $\mathrm{w}_0 \perp \mathrm{k}_*$. *Since* T *is* $\perp$*-preserving, we have* $\xi\mathrm{w}_0 \perp \xi^n\mathrm{w}_*$ *and* $\xi^n\mathrm{w}_0 \perp \mathrm{k}_*$ *for all* $n \in \mathbb{N}$. *So from (7), we can drive*

$$G(\xi^n\mathrm{w}_0, \xi^n\mathrm{w}_*, \tau) \geq G(\xi^n\mathrm{w}_0, \xi^n\mathrm{w}_*, \varrho\tau) \geq G\left(\mathrm{w}_0, \mathrm{w}_*, \frac{\tau}{\varrho^n}\right)$$

and

$$G(\xi^n\mathrm{w}_0, \xi^n\mathrm{k}_*, \tau) \geq G(\xi^n\mathrm{w}_0, \xi^n\mathrm{k}_*, \varrho\tau) \geq G\left(\mathrm{w}_0, \mathrm{k}_*, \frac{\tau}{\varrho^n}\right)$$

Therefore,

$$G(\mathrm{w}_*, \mathrm{k}_*, \tau) = G(\xi^n \mathrm{w}_*, \xi^n \mathrm{k}_*, \tau) \geq G\left(\xi^n \mathrm{w}_0, \xi^n \mathrm{w}_*, \frac{\tau}{2\theta}\right) * G\left(\xi^n \mathrm{w}_0, \xi^n \mathrm{k}_*, \frac{\tau}{2\theta}\right)$$

$$\geq G\left(\mathrm{w}_0, \mathrm{w}_*, \frac{\tau}{2\theta\varrho^n}\right) * G\left(\mathrm{w}_0, \mathrm{k}_*, \frac{\tau}{2\theta\varrho^n}\right) \to 1$$

as $n \to \infty$ *So from (8), we can derive*

$$H(\xi^n \mathrm{w}_0, \xi^n \mathrm{w}_*, \tau) \leq H(\xi^n \mathrm{w}_0, \xi^n \mathrm{w}_*, \varrho\tau) \leq H\left(\mathrm{w}_0, \mathrm{w}_*, \frac{\tau}{\varrho^n}\right)$$

and

$$H(\xi^n \mathrm{w}_0, \xi^n \mathrm{k}_*, \tau) \leq H(\xi^n \mathrm{w}_0, \xi^n \mathrm{k}_*, \varrho\tau) \leq H\left(\mathrm{w}_0, \mathrm{k}_*, \frac{\tau}{\varrho^n}\right)$$

Therefore,

$$H(\mathrm{w}_*, \mathrm{k}_*, \tau) = H(\xi^n \mathrm{w}_*, \xi^n \mathrm{k}_*, \tau) \leq H\left(\xi^n \mathrm{w}_0, \xi^n \mathrm{w}_*, \frac{\tau}{2\theta}\right) * H\left(\xi^n \mathrm{w}_0, \xi^n \mathrm{k}_*, \frac{\tau}{2\theta}\right)$$

$$\leq H\left(\mathrm{w}_0, \mathrm{w}_*, \frac{\tau}{2\theta\varrho^n}\right) \Delta H\left(\mathrm{w}_0, \mathrm{k}_*, \frac{\tau}{2\theta\varrho^n}\right) \to 0$$

as $n \to \infty$ *So,* $\mathrm{w}_* = \mathrm{k}_*$*, hence* w_* *is the unique FP.* □

Corollary 3.15. *Suppose* $(\Xi, G, H, *, \Delta, \perp)$ *be an O-complete IFBMS. Assume* $\xi : \Xi \to \Xi$ *be* $\perp$*-contraction and* $\perp$*-preserving. Assume that if* $\{\mathrm{w}\}$ *is an O-sequence with* $\mathrm{w}_n \to \mathrm{w} \in \Xi$*, Then* $\mathrm{w} \perp \mathrm{w}_n$ *for all* $n \in \mathbb{N}$*. Then* ξ *has a unique FP, say* $\mathrm{w}_* \in \Xi$*, Moreover,* $\lim_{n\to\infty} G(\xi^n \mathrm{w}, \mathrm{w}_*, \tau) = 1$ *and* $\lim_{n\to\infty} H(\xi^n, \mathrm{w}, \mathrm{w}_*, \tau) = 0$*, for all* $\mathrm{w} \in \Xi$ *and* $\tau > 0$*.*

Proof. *Follows from Theorem 2.1 that* w_n *is a O-Cauchy O-sequence and so it O-converges to* $\mathrm{w}_* \in \Xi$*. Hence* $\mathrm{w}_* \perp \mathrm{w}_n$ *for all* $n \in \mathbb{N}$ *from (7), we have*

$$G(\xi \mathrm{w}_*, \mathrm{w}_{n+1}, \tau) = G(\xi \mathrm{w}_*, \xi \mathrm{w}_n, \tau) \geq G(\xi \mathrm{w}_*, \xi \mathrm{w}_n, \tau\varrho) \geq G(\mathrm{w}_*, \mathrm{w}_n, \tau)$$

and

$$\lim_{n\to\infty} G(\xi \mathrm{w}_*, \mathrm{w}_{n+1}, \tau) = 1.$$

Then, we can write

$$G(\mathrm{w}_*, \xi \mathrm{w}_*, \tau) \geq G\left(\mathrm{w}_*, \xi \mathrm{w}_{n+1}, \frac{\tau}{2\theta}\right) * G\left(\mathrm{w}_{n+1}, \xi \mathrm{w}_*, \frac{\tau}{2\theta}\right)$$

Taking limit as $n \to +\infty$*, We get* $G(\mathrm{w}_*, \xi \mathrm{w}_*, \tau) = 1 * 1 = 1$
and from (8)

$$H(\xi \mathrm{w}_*, \mathrm{w}_{n+1}, \tau) = H(\xi \mathrm{w}_*, \xi \mathrm{w}_n, \tau) \leq H(\xi \mathrm{w}_*, \xi \mathrm{w}_n, \tau\varrho) \leq H(\mathrm{w}_*, \mathrm{w}_n, \tau)$$

and

$$\lim_{n\to\infty} H(\xi \mathrm{w}_*, \mathrm{w}_{n+1}, \tau) = 0.$$

Then, we can write

$$H(\mathrm{w}_*, \xi \mathrm{w}_*, \tau) \leq H\left(\mathrm{w}_*, \xi \mathrm{w}_{n+1}, \frac{\tau}{2\theta}\right) \Delta H\left(\mathrm{w}_{n+1}, \xi \mathrm{w}_*, \frac{\tau}{2\theta}\right)$$

Taking limit as $n \to +\infty$*, We get* $H(\mathrm{w}_*, \xi \mathrm{w}_*, \tau) = 0\Delta 0 = 0$*, So* $\xi \mathrm{w}_* = \mathrm{w}_*$*. Next follows from Theorem 3.13.*
□

Example 3.16. Let $\Xi = [-2, 2]$. We define $\perp$ by

$$\mathrm{w} \perp \mathrm{k} \Leftrightarrow \mathrm{w} + \mathrm{k} \in \{| \mathrm{w} |, | \mathrm{k} | \tag{15}$$

$$G(\mathrm{w}, \mathrm{k}, \tau) = \begin{cases} 1 \text{ if } \mathrm{w} = \mathrm{k}, \\ \left[e^{\frac{\max\{\mathrm{w},\mathrm{k}\}^{\alpha}}{\tau}}\right]^{-1} \text{ otherwise.} \end{cases} \tag{16}$$

and

$$H(\mathrm{w}, \mathrm{k}, \tau) = \begin{cases} 0 \text{ if } \mathrm{w} = \mathrm{k}, \\ 1 - \left[e^{\frac{\max\{\mathrm{w},\mathrm{k}\}^{\alpha}}{\tau}}\right]^{-1} \text{ otherwise.} \end{cases} \tag{17}$$

for all $\mathrm{w}, \mathrm{k} \in \Xi, \tau > 0$ with $\sigma \times \theta = \sigma \cdot \theta$ and $\sigma \Delta \theta = \max\{\sigma, \theta\}$. Then $(\Xi, G, *, \Delta, \perp)$ is an O-complete IFBMS. Define $\xi : \Xi \to \Xi$ by

$$\xi(\mathrm{w}) = \begin{cases} \frac{\mathrm{w}}{4}, \text{ if } \mathrm{w} \in [-2, 0] \\ 0, \text{ if } \mathrm{w} \in (0, 2]. \end{cases} \tag{18}$$

Then the below cases fulfilled:

1. if $\mathrm{w} \in [-2, 0]$ and $\mathrm{k} \in (0, 2]$, then $\xi(\mathrm{w}) = \frac{\mathrm{w}}{4}$ and $\xi(\mathrm{k}) = 0$,
2. if $\mathrm{w}, \mathrm{k} \in [-2, 0]$, then $\xi(\mathrm{w}) = \frac{\mathrm{w}}{4}$ and $\xi(\mathrm{k}) = \frac{\mathrm{k}}{4}$,
3. if $\mathrm{w}, \mathrm{k} \in (0, 2]$, then $\xi(\mathrm{w}) = 0$ and $\xi(\mathrm{k}) = 0$,
4. if $\mathrm{w} \in (0, 2]$ and $\mathrm{k} \in [-2, 0]$, then $\xi(\mathrm{w}) = 0$ and $\xi(\mathrm{k}) = \frac{\mathrm{k}}{4}$,

This is easy to see that $\xi((w)) + \xi(\mathrm{k}) \in \{| \xi(\mathrm{w}) |, | \xi(\mathrm{k}) |\}$. Hence, ξ is $\perp$-preserving. Let $\{\mathrm{w}_n\}$ be an arbitrary O-sequence in Ξ that $\{\mathrm{w}_n\}$ O-converges to $\mathrm{w} \in \Xi$. That is

$$\lim_{n\to\infty} G(\mathrm{w}_n, \mathrm{w}, \tau) = \lim_{n\to\infty} \left[e^{\frac{\max\{\mathrm{w}_n,\mathrm{w}\}^{\alpha}}{\tau}}\right]^{-1} = 1,$$

$$\lim_{n\to\infty} H(\mathrm{w}_n, \mathrm{w}, \tau) = 1 - \lim_{n\to\infty} \left[e^{\frac{\max\{\mathrm{w}_n,\mathrm{w}\}^{\alpha}}{\tau}}\right]^{-1} = 0.$$

We can easily see that if $\lim_{n\to\infty} G(\mathrm{w}_n, \mathrm{w}, \tau) = 1$, then $\lim_{n\to\infty} G(\xi\mathrm{w}_n, \xi\mathrm{w}, \tau) = 1$, and if $\lim_{n\to\infty} H(\mathrm{w}_n, \mathrm{w}, \tau) = 0$, then $\lim_{n\to\infty} H(\xi\mathrm{w}_n, \xi\mathrm{w}, \tau) = 0$, for all $\mathrm{w} \in \Xi$ and $\tau > 0$. That is, ξ is $\perp$-continuous. if $\mathrm{w} = \mathrm{k}$, then it is obvious. Suppose $\mathrm{w} \neq \mathrm{k}$, then there are following four cases for $\varrho \in [\frac{1}{2}, 1)$:
Case 1) if $\mathrm{w} \in [-2, 0]$ and $\mathrm{k} \in (0, 2]$, then $\xi\mathrm{w} = \frac{\mathrm{w}}{4}$ and $\xi\mathrm{k} = 0$. Here,

$$G(\xi\mathrm{w}_n, \xi\mathrm{w}, \varrho\tau) = G(\frac{\mathrm{w}}{4}, 0, \varrho\tau) = \left[e^{\frac{[\frac{\mathrm{w}}{4}]^{\alpha}}{\varrho\tau}}\right]^{-1} \geq \left[e^{\frac{\max\{\mathrm{w},\mathrm{k}\}^{\alpha}}{\tau}}\right]^{-1} = G(\mathrm{w}, \mathrm{k}, \tau),$$

$$H(\xi\mathrm{w}_n, \xi\mathrm{w}, \varrho\tau) = H(\frac{\mathrm{w}}{4}, 0, \varrho\tau) = 1 - \left[e^{\frac{[\frac{\mathrm{w}}{4}]^{\alpha}}{\varrho\tau}}\right]^{-1} \leq 1 - \left[e^{\frac{\max\{\mathrm{w},\mathrm{k}\}^{\alpha}}{\tau}}\right]^{-1} = H(\mathrm{w}, \mathrm{k}, \tau),$$

Case 2) If $\mathrm{w}, \mathrm{k} \in [-2, 0)$, then $\xi\mathrm{w} = \frac{\mathrm{w}}{4}$ and $\xi\mathrm{k} = \frac{\mathrm{k}}{4}$. We have

$$G(\xi\mathrm{w}_n, \xi\mathrm{w}, \varrho\tau) = G(\frac{\mathrm{w}}{4}, \frac{\mathrm{k}}{4}, \varrho\tau) = \left[e^{\frac{\max\left\{\frac{\mathrm{w}}{4}, \frac{\mathrm{k}}{4}\right\}^{\alpha}}{\varrho\tau}}\right]^{-1} \geq \left[e^{\frac{\max\{\mathrm{w},\mathrm{k}\}^{\alpha}}{\tau}}\right]^{-1} = G(\mathrm{w}, \mathrm{k}, \tau),$$

$$H(\xi \mathrm{w}_n, \xi \mathrm{w}, \varrho\tau) = H(\frac{\mathrm{w}}{4}, \frac{\mathrm{k}}{4}, \varrho\tau) = 1 - \left[e^{\frac{\max\left\{\frac{\mathrm{w}}{4}, \frac{\mathrm{k}}{4}\right\}^{\alpha}}{\varrho\tau}} \right]^{-1} \leq 1 - \left[e^{\frac{\max\{\mathrm{w},\mathrm{k}\}^{\alpha}}{\tau}} \right]^{-1} = H(\mathrm{w}, \mathrm{k}, \tau),$$

Case 3) If $\mathrm{w}, \mathrm{k} \in (0, 2]$, then $\xi\mathrm{w} = 0$ and $\xi\mathrm{k} = 0$. Here,

$$G(\xi \mathrm{w}, \xi \mathrm{k}, \varrho\tau) = G(0, 0, \varrho\tau) = e^0 \geq \left[e^{\frac{\max\{\mathrm{w},\mathrm{k}\}^{\alpha}}{\tau}} \right]^{-1} = G(\mathrm{w}, \mathrm{k}, \tau),$$

$$H(\xi \mathrm{w}, \xi \mathrm{k}, \varrho\tau) = H(0, 0, \varrho\tau) = 1 - e^0 \leq 1 - \left[e^{\frac{\max\{\mathrm{w},\mathrm{k}\}^{\alpha}}{\tau}} \right]^{-1} = H(\mathrm{w}, \mathrm{k}, \tau),$$

Case 4) If $\mathrm{w} \in (0, 2]$ and $\mathrm{k} \in [-2, 0]$, then $\xi\mathrm{w} = 0$ and $\xi\mathrm{k} = \frac{\mathrm{k}}{4}$. We have

$$G(\xi \mathrm{w}, \xi \mathrm{k}, \varrho\tau) = G(0, \frac{\mathrm{k}}{4}, \varrho\tau) = \left[e^{\frac{\max\left\{0, \frac{\mathrm{k}}{4}\right\}^{\alpha}}{\varrho\tau}} \right]^{-1} \geq \left[e^{\frac{\max\{\mathrm{w},\mathrm{k}\}^{\alpha}}{\tau}} \right]^{-1} = G(\mathrm{w}, \mathrm{k}, \tau),$$

$$H(\xi \mathrm{w}, \xi \mathrm{k}, \varrho\tau) = H(0, \frac{\mathrm{k}}{4}, \varrho\tau) = 1 - \left[e^{\frac{\max\left\{0, \frac{\mathrm{k}}{4}\right\}^{\alpha}}{\varrho\tau}} \right]^{-1} \leq 1 - \left[e^{\frac{\max\{\mathrm{w},\mathrm{k}\}^{\alpha}}{\tau}} \right]^{-1} = H(\mathrm{w}, \mathrm{k}, \tau),$$

From all the above cases, We obtain that

$$G(\xi \mathrm{w}, \xi \mathrm{k}, \varrho\tau) \geq G(\mathrm{w}, \mathrm{k}, \tau), \tag{19}$$

$$G(\xi \mathrm{w}, \xi \mathrm{k}, \varrho\tau) \geq G(\mathrm{w}, \mathrm{k}, \tau), \tag{20}$$

Hence, ξ is an orthogonal contraction. But, ξ is not a contraction. In fact, let $\mathrm{w} = -1$ and $\mathrm{k} = -2$ and $\alpha = 3$, then

$$G(\xi \mathrm{w}, \xi \mathrm{k}, \varrho\tau) = \left[e^{\frac{\max\left\{\frac{\mathrm{w}}{4}, \frac{\mathrm{k}}{4}\right\}^{\alpha}}{\varrho\tau}} \right]^{-1} \geq 1,$$

$$H(\xi \mathrm{w}, \xi \mathrm{k}, \varrho\tau) = 1 - \left[e^{\frac{\max\left\{\frac{\mathrm{w}}{4}, \frac{\mathrm{k}}{4}\right\}^{\alpha}}{\varrho\tau}} \right]^{-1} \leq 0.$$

Which is not true. Hence, all assumptions of Theorem 3.13 are fulfilled and 0 is the unique FP of ξ. Also,

$$G(\mathrm{w}, \mathrm{w}, \tau) = G(0, 0, \tau) = e^0 = 1, \forall \tau > 0 \tag{21}$$

and

$$H(\mathrm{w}, \mathrm{w}, \tau) = H(0, 0, \tau) = 1 - e^0 = 0. \forall \tau > 0 \tag{22}$$

Theorem 3.17. *Suppose* $(\Xi, G, H, *, \Delta, \perp)$ *be an O-complete IFBMS such that* $\lim_{t\to\infty} G(\mathrm{w}, \mathrm{k}, \tau) = 1$, *and* $\lim_{t\to\infty} H(\mathrm{w}, \mathrm{k}, \tau) = 0, \forall \mathrm{w}, \mathrm{k} \in \Xi$ *and* $\tau > 0$. *Suppose* $\xi : \Xi \to \Xi$ *be* $\perp$*-continuous,* $\perp$*-contraction, and* $\perp$*-preserving. Suppose* $\varrho \in \left(0, \frac{1}{\theta}\right)$ *and* $\tau > 0$, *such that*

$$G(\xi \mathrm{w}, \xi \mathrm{k}, \varrho\tau) \geq \min\{G(\xi \mathrm{w}, \mathrm{w}, \tau), G(\xi \mathrm{k}, \mathrm{k}, \tau)\} \tag{23}$$

$$H(\xi \mathrm{w}, \xi \mathrm{k}, \varrho\tau) \leq \min\{H(\xi \mathrm{w}, \mathrm{w}, \tau), H(\xi \mathrm{k}, \mathrm{k}, \tau)\} \tag{24}$$

for all $\mathrm{w}, \mathrm{k} \in \Xi, \tau > 0$. *Then* ξ *has a unique FP, say* $\mathrm{w}_* \in \Xi$. *Moreover,* $\lim_{n\to\infty} G(\xi^n \mathrm{w}, \mathrm{w}_*, \tau) = 1$ *and* $\lim_{n\to\infty} H(\xi^n \mathrm{w}, \mathrm{w}_*, \tau) = 0$ *for all* $\mathrm{w} \in \Xi$ *and* $\tau > 0$.

Proof. *Let* $(\Xi, G, H, *, \Delta, \perp)$ *be an O-complete IFBMS, There exists* $\mathrm{w}_0 \in \Xi$ *such that*

$$\mathrm{w}_0 \perp \mathrm{k} \forall \mathrm{k} \in \Xi \tag{25}$$

Therefore, ξ is $\perp$-preserving, and $\{\mathrm{w}_n\}$ is an O-sequence. We have

$$G(\mathrm{w}_{n+1},\mathrm{n},\tau)\geq G(\mathrm{w}_{n+1},\mathrm{n},\varrho\tau)=G(\xi\mathrm{w}_n,\xi\mathrm{w}_{n-1},\varrho\tau)\geq\min\{G(\xi\mathrm{w}_n,\mathrm{n},\tau),G(\xi\mathrm{w}_{n-1},\mathrm{w}_{n-1},\tau)$$

$$H(\mathrm{w}_{n+1},\mathrm{n},\tau)\leq H(\mathrm{w}_{n+1},\mathrm{n},\varrho\tau)=H(\xi\mathrm{w}_n,\xi\mathrm{w}_{n-1},\varrho\tau)\leq\min\{H(\xi\mathrm{w}_n,\mathrm{n},\tau),H(\xi\mathrm{w}_{n-1},\mathrm{w}_{n-1},\tau)$$

Two cases arise.
Case 1: If $G(\mathrm{w}_{n+1},\mathrm{n},\tau)\geq G(\xi\mathrm{w}_n,\mathrm{w}_n,\tau)$, then

$$G(\mathrm{w}_{n+1},\mathrm{w}_n,\tau)\geq G(\mathrm{w}_{n+1},\mathrm{w}_n,\varrho\tau)\geq G(\xi\mathrm{w}_n,\mathrm{w}_n,\tau)=G(\mathrm{w}_{n+1},\mathrm{w}_n,\tau)$$

and

$$H(\mathrm{w}_{n+1},\mathrm{w}_n,\tau)\leq H(\mathrm{w}_{n+1},\mathrm{w}_n,\varrho\tau)\leq H(\xi\mathrm{w}_n,\mathrm{w}_n,\tau)=H(\mathrm{w}_{n+1},\mathrm{w}_n,\tau)$$

Then, by Lemma 3.12, $\mathrm{w}_n=\mathrm{w}_{n+1}$ for all $n\in\mathbb{N}$
Case 2): If $G(\mathrm{w}_{n+1},\mathrm{n},\tau)\geq G(\xi\mathrm{w}_{n-1},\mathrm{w}_{n-1},\tau)$, then

$$G(\mathrm{w}_{n+1},\mathrm{w}_n,\tau)\geq G(\mathrm{w}_{n+1},\mathrm{w}_n,\varrho\tau)\geq G(\xi\mathrm{w}_{n-1},\mathrm{w}_{n-1},\tau)\geq G(\mathrm{w}_n,\mathrm{w}_{n-1},\tau)$$

and $H(\mathrm{w}_{n+1},\mathrm{n},\tau)\leq H(\xi\mathrm{w}_{n-1},\mathrm{w}_{n-1},\tau)$, then

$$H(\mathrm{w}_{n+1},\mathrm{w}_n,\tau)\leq H(\mathrm{w}_{n+1},\mathrm{w}_n,\varrho\tau)\leq H(\xi\mathrm{w}_{n-1},\mathrm{w}_{n-1},\tau)\leq H(\mathrm{w}_n,\mathrm{w}_{n-1},\tau)$$

for all $n\in\mathbb{N}$ and $\tau>0$. By utilizing Theorem 3.13, we have an O-Cauchy sequence. Since $(\Xi,G,H,*,\Delta,\perp)$ is complete, there exists $\mathrm{w}_*\in\Xi$, such that

$$\lim_{n\to\infty}G(\mathrm{w}_n,\mathrm{w}_*,\tau)=1,\tag{26}$$

and

$$\lim_{n\to\infty}H(\mathrm{w}_n,\mathrm{w}_*,\tau)=0,\tag{27}$$

for all $\tau>0$. Science,ξ is an $\perp$-continuous, We have

$$\lim_{n\to\infty}G(\mathrm{w}_{n+1},\mathrm{w}_*,\tau)=G(\xi\mathrm{w}_n,\xi\mathrm{w}_*,\tau)=1,$$

and

$$\lim_{n\to\infty}H(\mathrm{w}_{n+1},\mathrm{w}_*,\tau)=H(\xi\mathrm{w}_n,\xi\mathrm{w}_*,\tau)=0,$$

Next, we examine that w_* is a FP of ξ. Let $\tau_1\in(\varrho\theta,1)$ and $\tau_2=1-\tau_1$. then

$$G(\xi\mathrm{w}_*,\mathrm{w}_*,\tau)\geq G\left(\xi\mathrm{w}_*,\mathrm{w}_{n+1},\frac{\tau\tau_1}{\theta}\right)*G\left(\xi\mathrm{w}_{n+1},\mathrm{w}_*,\frac{\tau\tau_2}{\theta}\right),$$

$$=G\left(\xi\mathrm{w}_*,\xi\mathrm{w}_n,\frac{\tau\tau_1}{\theta}\right)*G\left(\xi\mathrm{w}_{n+1},\mathrm{w}_*,\frac{\tau\tau_2}{\theta}\right),$$

$$\geq\min\left\{G\left(\xi\mathrm{w}_*,\mathrm{w}_*,\frac{\tau\tau_1}{\varrho\theta}\right)*G\left(\xi\mathrm{w}_n,\mathrm{w}_n,\frac{\tau\tau_2}{\varrho\theta}\right)\right\}*G\left(\xi\mathrm{w}_{n+1},\mathrm{w}_*,\frac{\tau\tau_2}{\theta}\right)$$

$$=\min\left\{G\left(\xi\mathrm{w}_*,\mathrm{w}_*,\frac{\tau\tau_1}{\varrho\theta}\right)*G\left(\xi\mathrm{w}_{n+1},\mathrm{w}_n,\frac{\tau\tau_2}{\varrho\theta}\right)\right\}*G\left(\xi\mathrm{w}_{n+1},\mathrm{w}_*,\frac{\tau\tau_2}{\theta}\right)$$

Taking $n\to\infty$, We get

$$G(\xi\mathrm{w}_*,\mathrm{w}_*,\tau)\geq\min\left\{G\left(\xi\mathrm{w}_*,\mathrm{w}_*,\frac{\tau\tau_1}{\varrho\theta}\right),1\right\}*1,$$

$$G(\xi \mathrm{w}_*, \mathrm{w}_*, \tau) \geq G\left(\xi \mathrm{w}_*, \mathrm{w}_*, \frac{\tau}{\nu}\right) \tau > 0, ,$$

and

$$H(\xi \mathrm{w}_*, \mathrm{w}_*, \tau) \leq H\left(\xi \mathrm{w}_*, \mathrm{w}_{n+1}, \frac{\tau\tau_1}{\theta}\right) \Delta H\left(\xi \mathrm{w}_{n+1}, \mathrm{w}_*, \frac{\tau\tau_2}{\theta}\right),$$
$$= H\left(\xi \mathrm{w}_*, \xi \mathrm{w}_n, \frac{\tau\tau_1}{\theta}\right) \Delta H\left(\xi \mathrm{w}_{n+1}, \mathrm{w}_*, \frac{\tau\tau_2}{\theta}\right),$$
$$\leq \min\left\{H\left(\xi \mathrm{w}_*, \mathrm{w}_*, \frac{\tau\tau_1}{\varrho\theta}\right) \Delta H\left(\xi \mathrm{w}_n, \mathrm{w}_n, \frac{\tau\tau_2}{\varrho\theta}\right)\right\} \Delta H\left(\xi \mathrm{w}_{n+1}, \mathrm{w}_*, \frac{\tau\tau_2}{\theta}\right)$$
$$= \min\left\{H\left(\xi \mathrm{w}_*, \mathrm{w}_*, \frac{\tau\tau_1}{\varrho\theta}\right) \Delta H\left(\xi \mathrm{w}_{n+1}, \mathrm{w}_n, \frac{\tau\tau_2}{\varrho\theta}\right)\right\} \Delta H\left(\xi \mathrm{w}_{n+1}, \mathrm{w}_*, \frac{\tau\tau_2}{\theta}\right)$$

Taking $n \to \infty$, *We get*

$$H(\xi \mathrm{w}_*, \mathrm{w}_*, \tau) \leq \min\left\{H\left(\xi \mathrm{w}_*, \mathrm{w}_*, \frac{\tau\tau_1}{\varrho\theta}\right), 0\right\} * 0,$$
$$H(\xi \mathrm{w}_*, \mathrm{w}_*, \tau) \leq H\left(\xi \mathrm{w}_*, \mathrm{w}_*, \frac{\tau}{\nu}\right) \tau > 0, ,$$

There is $\nu = \frac{\theta\varrho}{\tau_1} \in (0,1)$, *and by utilizing Lemma 3.12, we get* $\xi \mathrm{w}_* = \mathrm{w}_*$.
Uniqueness : *Suppose* $\mathrm{w}_* \neq \mathrm{k}_*$ *are two FPs of* ξ. *We get* $\mathrm{w}_0 \perp \mathrm{w}_*$ *and* $\mathrm{w}_0 \perp \mathrm{k}_*$. *Therefore, since* ξ *is an* $\perp$*-preserving, we have* $\xi^n \mathrm{w}_0 \perp \xi^n \mathrm{w}_*$ *and* $\xi^n \mathrm{w}_0 \xi^n \perp \mathrm{k}_*$. *for all* $n \in \mathbb{N}$. *we can write*

$$G(\xi^n \mathrm{w}_0, \xi^n \mathrm{w}_*, \tau) \geq G(\xi^n \mathrm{w}_0, \xi^n \mathrm{w}_*, \varrho\tau) \geq \min\{G(\xi^n \mathrm{w}_0, \mathrm{w}_0, \tau), G(\xi^n \mathrm{w}_*, \mathrm{w}_*, \tau)\},$$

and

$$G(\xi^n \mathrm{w}_0, \xi^n \mathrm{k}_*, \tau) \geq G(\xi^n \mathrm{w}_0, \xi^n \mathrm{k}_*, \varrho\tau) \geq \min\{G(\xi^n \mathrm{w}_0, \mathrm{w}_0, \tau), G(\xi^n \mathrm{k}_*, \mathrm{k}_*, \tau)\},$$

Hence, we write that

$$G(\mathrm{w}_0, \mathrm{k}_*, \tau) = G(\xi^n \mathrm{w}_*, \xi^n \mathrm{k}_*, \tau) \geq \min\left\{G\left(\xi^n \mathrm{w}_*, \mathrm{w}_*, \frac{\tau}{\varrho}\right), G\left(\xi^n \mathrm{k}_*, \mathrm{k}_*, \frac{\tau}{\varrho}\right)\right\},$$

and

$$H(\xi^n \mathrm{w}_0, \xi^n \mathrm{w}_*, \tau) \leq G(\xi^n \mathrm{w}_0, \xi^n \mathrm{w}_*, \varrho\tau) \leq \min\{H(\xi^n \mathrm{w}_0, \mathrm{w}_0, \tau), H(\xi^n \mathrm{w}_*, \mathrm{w}_*, \tau)\},$$

and

$$H(\xi^n \mathrm{w}_0, \xi^n \mathrm{k}_*, \tau) \leq H(\xi^n \mathrm{w}_0, \xi^n \mathrm{k}_*, \varrho\tau) \leq \min\{H(\xi^n \mathrm{w}_0, \mathrm{w}_0, \tau), H(\xi^n \mathrm{k}_*, \mathrm{k}_*, \tau)\},$$

Hence, we write that

$$H(\mathrm{w}_0, \mathrm{k}_*, \tau) = H(\xi^n \mathrm{w}_*, \xi^n \mathrm{k}_*, \tau) \leq \min\left\{H\left(\xi^n \mathrm{w}_*, \mathrm{w}_*, \frac{\tau}{\varrho}\right), H\left(\xi^n \mathrm{k}_*, \mathrm{k}_*, \frac{\tau}{\varrho}\right)\right\},$$

for all $\tau > 0$. *Thus,* $\mathrm{w}_* = \mathrm{k}_*$. □

Corollary 3.18. *Suppose* $(\Xi, G, H, *, \Delta, \perp)$ *be a complete OIFBMS and* $\xi : \Xi \to \Xi$ *be an* $\perp$*-continuous and* $\perp$*-preserving. Let* $\varrho \in \left(0, \frac{1}{\theta}\right)$ *for all* $\tau > 0$,*with*

$$G(\xi \mathrm{w}, \xi \mathrm{k}, \varrho\tau) \geq \min\{G(\xi \mathrm{w}, \mathrm{w}, \tau), G(\xi \mathrm{k}, \mathrm{k}, \tau),$$

$$H(\xi \mathrm{w}, \xi \mathrm{k}, \varrho\tau) \leq \min\{H(\xi \mathrm{w}, \mathrm{w}, \tau), H(\xi \mathrm{k}, \mathrm{k}, \tau).$$

Then, ξ *has a unique FP. Furthermore,* $\lim_{n\to\infty} G(\xi^n \mathrm{w}, \mathrm{w}_*, \tau) = 1$ *and* $\lim_{n\to\infty} H(\xi^n \mathrm{w}, \mathrm{w}_*, \tau) = 0$, *for all* $\mathrm{w} \in \Xi$ *and* $\tau > 0$.
Proof. *It is obvious from Theorem 3.14 and 3.17* □

Example 3.19. Suppose $\Xi = [-2, 2]$ and by $\perp$ by $\mathrm{w} \perp \mathrm{k} \Leftrightarrow \mathrm{w} + \mathrm{k} \geq 0$. Define G and H by

$$G(\mathrm{w}, \mathrm{k}, \tau) = \begin{cases} 1 \text{ ifw} = \mathrm{k}, \\ \frac{\tau}{\tau + \max\{\mathrm{w}, \mathrm{k}\}^{\alpha}} \text{ otherwise} \end{cases} \tag{28}$$

$$H(\mathrm{w}, \mathrm{k}, \tau) = \begin{cases} 0 \text{ ifw} = \mathrm{k}, \\ \frac{\max\{\mathrm{w}, \mathrm{k}\}^{\alpha}}{\tau + \max\{\mathrm{w}, \mathrm{k}\}^{\alpha}} \text{ otherwise} \end{cases} \tag{29}$$

for all $\mathrm{w}, \mathrm{k} \in \Xi$ and $\tau > 0$, with $\sigma * \theta = \sigma \cdot \theta$ and $\sigma \Delta \theta = \max\{\sigma, \theta\}$, Then $(\Xi, G, H, *, \Delta, \perp)$ is an O-complete IFBMS. Note that $\lim_{n \to \infty} G\mathrm{w}, \mathrm{k}, \tau = 1$ and $\lim_{n \to \infty} H\mathrm{w}, \mathrm{k}, \tau = 0$. Define $\xi : \Xi \to \Xi$ by

$$\xi(\mathrm{w}) = \begin{cases} \frac{\mathrm{w}}{4}, \ \mathrm{w} \in \left[-2, \frac{2}{3}\right], \\ 1 - \mathrm{w}, \ \mathrm{w} \in \left(\frac{2}{3}, 1\right], \\ \mathrm{w} - \frac{1}{2}, \ \mathrm{w} \in (1, 2]. \end{cases} \tag{30}$$

There are following four cases:

1. If $\mathrm{w}, \mathrm{k} \in \left[-2, \frac{2}{3}\right]$ then $\xi(\mathrm{w}) = \frac{\mathrm{w}}{4}$ and $\xi(\mathrm{k}) = \frac{\mathrm{k}}{4}$.
2. If $\mathrm{w}, \mathrm{k} \in \left(\frac{2}{3}, 1\right]$ then $\xi(\mathrm{w}) = 1 - \mathrm{w}$ and $\xi(\mathrm{k}) = 1 - \mathrm{k}$.
3. If $\mathrm{w}, \mathrm{k} \in (1, 2]$ then $\xi(\mathrm{w}) = \mathrm{w} - \frac{1}{2}$ and $\xi(\mathrm{k}) = \mathrm{k} - \frac{1}{2}$.
4. If $\mathrm{w}, \in \left[-2, \frac{2}{3}\right]$ and $\mathrm{k} \in \left(\frac{2}{3}, 1\right]$ then $\xi(\mathrm{w}) = \frac{\mathrm{w}}{4}$ and $\xi(\mathrm{k}) = \mathrm{k} - \frac{1}{2}$.
5. If $\mathrm{w}, \in \left[-2, \frac{2}{3}\right]$ and $\mathrm{k} \in (1, 2]$ then $\xi(\mathrm{w}) = \frac{\mathrm{w}}{4}$ and $\xi(\mathrm{k}) = \mathrm{k} - \frac{1}{2}$.
6. If $\mathrm{w}, \in \left(\frac{2}{3}, 1\right]$ and $\mathrm{k} \in \left(\frac{2}{3}, 1\right]$ then $\xi(\mathrm{w}) = 1 - \mathrm{w}$ and $\xi(\mathrm{k}) = \mathrm{k} - \frac{1}{2}$.
7. If $\mathrm{w}, \in (1, 2]$ and $\mathrm{k} \in \left(\frac{2}{3}, 1\right]$ then $\xi(\mathrm{w}) = \mathrm{w} - \frac{1}{2}$ and $\xi(\mathrm{k}) = 1 - \mathrm{k}$.
8. If $\mathrm{w} \in (1, 2]$ and $\mathrm{k} \in \left[-2, \frac{2}{3}\right]$ then $\xi(\mathrm{w}) = \mathrm{w} - \frac{1}{2}$ and $\xi(\mathrm{k}) = \frac{\mathrm{k}}{4}$.
9. If $\mathrm{w} \in \left(\frac{2}{3}, 1\right]$ and $\mathrm{k} \in \left[-2, \frac{2}{3}\right]$ then $\xi(\mathrm{w}) = 1 - \mathrm{w}$ and $\xi(\mathrm{k}) = \frac{\mathrm{k}}{4}$.

Because $\mathrm{w} \perp \mathrm{k} \Leftrightarrow \mathrm{w} + \mathrm{k} \geq 0$, it is clearly implies that $\xi\mathrm{w} + \mathrm{k} \geq 0$. that is, ξ is $\perp$-preserving. Suppose $\{\mathrm{w}_n\}$ be any O-sequence in Ξ that O-converges to $\mathrm{w} \in \Xi$. We get

$$\lim_{n \to \infty} G(\mathrm{w}_n, \mathrm{w}, \tau) = \lim_{n \to \infty} \frac{\tau}{\tau + \max\{\mathrm{w}_n, \mathrm{w}\}^3} = 1, \tag{31}$$

$$\lim_{n \to \infty} H(\mathrm{w}_n, \mathrm{w}, \tau) = \lim_{n \to \infty} \frac{\max\{\mathrm{w}_n, \mathrm{w}\}^3}{\tau + \max\{\mathrm{w}_n, \mathrm{w}\}^3} = 0, \tag{32}$$

Note that if $G(\mathrm{w}_n, \mathrm{w}, \tau) = 1$ and $H(\mathrm{w}_n, \mathrm{w}, \tau) = 0$, then $G(\xi\mathrm{w}_n, \xi\mathrm{w}, \tau) = 1$ and $H(\xi\mathrm{w}_n, \xi\mathrm{w}, \tau) = 0$ for all $\tau > 0$. that is, ξ is orthogonal continuous. For $\mathrm{w} = \mathrm{k}$, it is obvious. Assume $\mathrm{w} \neq \mathrm{k}$. We get

$$G(\xi\mathrm{w}, \xi\mathrm{k}, \varrho\tau) \geq \min\{G(\xi\mathrm{w}, \mathrm{w}, \tau), G(\xi\mathrm{k}, \mathrm{k}, \tau)\}$$

$$H(\xi\mathrm{w}, \xi\mathrm{k}, \varrho\tau) \leq \min\{H(\xi\mathrm{w}, \mathrm{w}, \tau), H(\xi\mathrm{k}, \mathrm{k}, \tau)\}.$$

It fulfilled above all cases. Now, we show that ξ is not a contraction. Suppose

$$\min\{G(\xi\mathrm{w}, \mathrm{w}, \tau), G(\xi\mathrm{k}, \mathrm{k}, \tau)\} = G(\xi\mathrm{w}, \mathrm{w}, \tau)$$

$$\min\{H(\xi\mathrm{w},\mathrm{w},\tau),H(\xi\mathrm{k},\mathrm{k},\tau)\}=H(\xi\mathrm{w},\mathrm{k},\tau).$$

then for $\mathrm{w}=-1$ and $\mathrm{k}=-2$, we have

$$G(\xi\mathrm{w},\xi\mathrm{k},\varrho\tau)=\frac{\varrho\tau}{\varrho\tau+\max\left\{\frac{\mathrm{w}}{4},\frac{\mathrm{k}}{4}\right\}^3}=\frac{64\varrho\tau}{64\varrho\tau-1}\geq 1,$$

$$H(\xi\mathrm{w},\xi\mathrm{k},\varrho\tau)=\frac{\max\left\{\frac{\mathrm{w}}{4},\frac{\mathrm{k}}{4}\right\}^3}{\varrho\tau+\max\left\{\frac{\mathrm{w}}{4},\frac{\mathrm{k}}{4}\right\}^3}=\frac{-1}{64\varrho\tau-1}\leq 0.$$

Which is not true. That is, all assumptions of Theorem 2.2 are fulfilled, and 0 is a unique FP of ξ.

Definition 3.20. Suppose $(\Xi,G,H,*,\Delta,\perp)$ be an OIFBMS. A mapping $\xi:\Xi\to\Xi$ is called a fuzzy $\theta-\perp$-contraction if their exists $\varrho\in(0,1)$ such that

$$\frac{1}{G(\xi\mathrm{w},\xi\mathrm{k},\tau)}-1\leq\varrho\left[\frac{1}{G(\mathrm{w},\mathrm{k},\tau)}-1\right]\tag{33}$$

$$H(\xi\mathrm{w},\xi\mathrm{k},\tau)\leq\varrho H(\mathrm{w},\mathrm{k},\tau)\tag{34}$$

for all $\mathrm{w},\mathrm{k}\in\Xi$ and $\tau>0$. Where ϱ is said to be an IFB-$\perp$-contractive constant of ξ.

Theorem 3.21. *Suppose $(\Xi,G,H,*,\Delta,\perp)$ be an OIFBMS. Such that*

$$\lim_{\tau\to\infty}G(\mathrm{w},\mathrm{k},\tau)=1,\tag{35}$$

$$\lim_{\tau\to\infty}H(\mathrm{w},\mathrm{k},\tau)=0,\forall\mathrm{w},\mathrm{k}\in\Xi.\tag{36}$$

Assume a mapping $\xi:\Xi\to\Xi$ be a $\perp$-continuous, IFB-$\perp$-contraction and $\perp$-preserving mapping. Thus, ξ has a FP, call $\nu\in\Xi$. Moreover, $G(\nu,\nu,\alpha)=1$ and $H(\nu,\nu,\alpha)=0$ for all $\alpha>0$.

Proof. *Suppose $(\Xi,G,H,*,\Delta,\perp)$ be an O-complete IFBMS. For any point $\mathrm{w}_0\in\Xi$, $\mathrm{w}_0\perp\mathrm{k}$, for all $\mathrm{k}\in\Xi$. That is, $\mathrm{w}_0\perp\xi\mathrm{w}_0$. Consider $\mathrm{w}_n=\xi^n\mathrm{w}_0=\xi\mathrm{w}_{n-1}$ for all $n\in\mathbb{N}$. Therefore, ξ is $\perp$-preserving and $\{\mathrm{w}_n\}$ is an O-sequence. If $\mathrm{w}_n=\mathrm{w}_{n-1}$ for some $n\in\mathbb{N}$ then w_n is a FP of ξ. We suppose that $\mathrm{w}_n\neq\mathrm{w}_{n-1}$ for all $n\in\mathbb{N}$. For all $\tau>0$, $n\in\mathbb{N}$ and utilizing (9), we have*

$$\frac{1}{G(\mathrm{w}_n,\mathrm{w}_{n+1},\tau)}-1=\frac{1}{G(\xi\mathrm{w}_{n-1},\xi\mathrm{w}_n,\tau)}-1\leq\varrho\left[\frac{1}{G(\mathrm{w}_{n-1},\mathrm{w},\tau)}-1\right]$$

$$H(\mathrm{w}_n,\mathrm{w}_{n+1},\tau)=H(\xi\mathrm{w}_{n-1},\xi\mathrm{w}_n,\tau)\leq\varrho H(\mathrm{w}_{n-1},\mathrm{w}_n,\tau).$$

We have

$$\frac{1}{G(\mathrm{w}_n,\mathrm{w}_{n+1},\tau)}-1=\frac{\varrho}{G(\mathrm{w}_{n-1},\mathrm{w}_n,\tau)}+(1-\varrho),\forall\tau>0$$

$$\frac{\varrho}{G(\xi\mathrm{w}_{n-2},\xi\mathrm{w}_{n-1},\tau)}+(1-\varrho)\leq\frac{\varrho^2}{G(\mathrm{w}_{n-2},\mathrm{w}_{n-1},\tau)}+\varrho(1-\varrho)+(1-\varrho).$$

Continuing in this way, we get

$$\frac{\varrho}{G(\mathrm{w}_n,\mathrm{w}_{n+1},\tau)}\leq\frac{\varrho^n}{G(\mathrm{w}_0,\mathrm{w}_1,\tau)}+\varrho^{n-1}(1-\varrho)+\varrho^{n-2}(1-\varrho)+\cdots+\varrho(1-\varrho)+(1-\varrho).$$

$$\leq\frac{\varrho^n}{G(\mathrm{w}_0,\mathrm{w}_1,\tau)}+(\varrho^{n-1}+\varrho^{n-2}+\cdots 1)(1-\varrho)$$

$$\leq \frac{\varrho^n}{G(\mathrm{w}_0, \mathrm{w}_1, \tau)} + (1 - \varrho^n)$$

We have

$$\frac{1}{\frac{\varrho^n}{G(\mathrm{w}_0,\mathrm{w}_1,\tau)} + (1 - \varrho^n)} \leq G(\mathrm{w}_n, \mathrm{w}_{n+1}, \tau), \forall \tau > 0, n \in \mathbb{N} \tag{37}$$

and

$$H(\mathrm{w}_n, \mathrm{w}_{n+1}, \tau) = H(\xi\mathrm{w}_{n-1}, \xi\mathrm{w}_n, \tau) \leq \varrho H(\mathrm{w}_{n-1}, \mathrm{w}_n, \tau) = \varrho H(\xi\mathrm{w}_{n-2}, \xi\mathrm{w}_{n-1}, \tau)$$

$$\leq \varrho^2 H(\mathrm{w}_{n-2}, \mathrm{w}_{n-1}, \tau) \leq \cdots \leq \varrho^n H(\mathrm{w}_0, \mathrm{w}_1, \tau) \forall \tau > 0, n \in \mathbb{N} \tag{38}$$

Now, for $m \geq 1$ and $n \in \mathbb{N}$, we have

$$G(\mathrm{w}_n, \mathrm{w}_{n+m}, \tau) \geq G\left(\mathrm{w}_n, \mathrm{w}_{n+1}, \frac{\tau}{\theta}\right) * G\left(\mathrm{w}_{n+1}, \mathrm{w}_{n+m}, \frac{\tau}{\theta}\right)$$

$$\geq G\left(\mathrm{w}_n, \mathrm{w}_{n+1}, \frac{\tau}{\theta}\right) * G\left(\mathrm{w}_{n+1}, \mathrm{w}_{n+2}, \frac{\tau}{\theta^2}\right) * G\left(\mathrm{w}_{n+2}, \mathrm{w}_{n+m}, \frac{\tau}{\theta^2}\right)$$

Again, continuing in this way, we get

$$G(\mathrm{w}_n, \mathrm{w}_{n+m}, \tau) \geq G\left(\mathrm{w}_n, \mathrm{w}_{n+1}, \frac{\tau}{\theta}\right) * G\left(\mathrm{w}_{n+1}, \mathrm{w}_{n+2}, \frac{\tau}{\theta^2}\right) * \cdots * G\left(\mathrm{w}_{n+m-1}, \mathrm{w}_{n+m}, \frac{\tau}{\theta^{m-1}}\right)$$

and

$$H(\mathrm{w}_n, \mathrm{w}_{n+m}, \tau) \leq H\left(\mathrm{w}_n, \mathrm{w}_{n+1}, \frac{\tau}{\theta}\right) \Delta H\left(\mathrm{w}_{n+1}, \mathrm{w}_{n+m}, \frac{\tau}{\theta}\right)$$

$$\leq H\left(\mathrm{w}_n, \mathrm{w}_{n+1}, \frac{\tau}{\theta}\right) \Delta H\left(\mathrm{w}_{n+1}, \mathrm{w}_{n+2}, \frac{\tau}{\theta^2}\right) \Delta H\left(\mathrm{w}_{n+2}, \mathrm{w}_{n+m}, \frac{\tau}{\theta^2}\right)$$

Continuing in this way, we get

$$H(\mathrm{w}_n, \mathrm{w}_{n+m}, \tau) \leq H\left(\mathrm{w}_n, \mathrm{w}_{n+1}, \frac{\tau}{\theta}\right) \Delta H\left(\mathrm{w}_{n+1}, \mathrm{w}_{n+2}, \frac{\tau}{\theta^2}\right) \Delta \cdots \Delta H\left(\mathrm{w}_{n+m-1}, \mathrm{w}_{n+m}, \frac{\tau}{\theta^{m-1}}\right)$$

By utilizing (37) in the above inequality, we get

$$G(\mathrm{w}_n, \mathrm{w}_{n+m}, \tau) \geq \frac{1}{\frac{\varrho^n}{G\left(\mathrm{w}_0,\mathrm{w}_1,\frac{\tau}{\theta}\right)} + (1 - \varrho^n)} * \frac{1}{\frac{\varrho^{n+1}}{G\left(\mathrm{w}_0,\mathrm{w}_1,\frac{\tau}{\theta^2}\right)} + (1 - \varrho^n)} * \cdots$$

$$* \frac{1}{\frac{\varrho^{n+m-1}}{G\left(\mathrm{w}_0,\mathrm{w}_1,\frac{\tau}{\theta^{m-1}}\right)} + (1 - \varrho^{n+m-1})}$$

$$\geq \frac{1}{\frac{\varrho^n}{G\left(\mathrm{w}_0,\mathrm{w}_1,\frac{\tau}{\theta}\right)}+1} * \frac{1}{\frac{\varrho^{n+1}}{G\left(\mathrm{w}_0,\mathrm{w}_1,\frac{\tau}{\theta^2}\right)}+1} * \cdots * \frac{1}{\frac{\varrho^{n+m-1}}{G\left(\mathrm{w}_0,\mathrm{w}_1,\frac{\tau}{\theta^{m-1}}\right)}+1}$$

Also, using (38), we have

$$H(\mathrm{w}_n, \mathrm{w}_{n+p}, \tau) \leq H\left(\mathrm{w}_n, \mathrm{w}_{n+1}, \frac{\tau}{\theta}\right) \Delta H\left(\mathrm{w}_{n+1}, \mathrm{w}_{n+2}, \frac{\tau}{\theta^2}\right) \Delta \cdots \Delta H\left(\mathrm{w}_{n+m-1}, \mathrm{w}_{n+m}, \frac{\tau}{\theta^{m-1}}\right)$$

As $\varrho \in (0,1)$, we have $\lim_{n\to\infty} G(\mathrm{w}_n, \mathrm{w}_{n+m}, \tau) = 1$ and $\lim_{n\to\infty} H(\mathrm{w}_n, \mathrm{w}_{n+m}, \tau) = 0$ for all $\tau > 0$, $m \geq 1$. Therefore, a sequence $\{\mathrm{w}\}$ is an O-Cauchy in $(\Xi, G, H, *, \Delta, \perp)$ is complete, and we have ξ is an $\perp$-continuous, there exist $\nu \in \Xi$ such that

$$\lim_{n\to\infty} G(\mathrm{w}_{n+1}, \nu, \tau) = \lim_{n\to\infty} G(\xi\mathrm{w}_n, \xi\nu, \tau) = 1, \forall \tau > 0, \tag{39}$$

$$\lim_{n\to\infty} H(\mathrm{w}_{n+1},\nu,\tau)=\lim_{n\to\infty} H(\xi\mathrm{w}_n,\xi\nu,\tau)=0,\forall\tau>0, \tag{40}$$

Now, we show that ν is a FP of ξ. By utilizing (33), we have

$$\frac{1}{G(\xi\mathrm{w},\xi\nu,\tau)}-1\leq\varrho\left[\frac{1}{G(\mathrm{w}_n,\xi\nu,\tau)}-1\right]=\frac{\varrho}{G(\mathrm{w},\xi\nu,\tau)}-\varrho.$$

That is,

$$\frac{1}{G(\xi\mathrm{w},\xi\nu,\tau)+1-\varrho}\leq G(\xi\mathrm{w}_n,\xi\nu,\tau).$$

Using the above inequality, we obtain

$$G(\nu,\xi\nu,\tau)\geq G\left(\nu,\mathrm{w}_{n+1},\frac{\tau}{2\theta}\right)*G\left(\mathrm{w}_{n+1},\xi\nu,\frac{\tau}{2\theta}\right)$$

$$=G\left(\nu,\mathrm{w}_{n+1},\frac{\tau}{2\theta}\right)*G\left(\xi\mathrm{w}_n,\xi\nu,\frac{\tau}{2\theta}\right)$$
$$\geq G\left(\nu,\mathrm{w}_{n+1},\frac{\tau}{2\theta}\right)*\frac{\varrho}{G\left(\mathrm{w}_n,\nu,\frac{\tau}{2\theta}\right)+1-\varrho}$$

and

$$H(\mathrm{w},\nu,\tau)=H(\xi\mathrm{w},\xi\nu,\tau)\leq\varrho H(\mathrm{w},\nu,\tau)<H(\mathrm{w},\nu,\tau)$$
$$=H\left(\mathrm{w},\mathrm{w}_{n+1},\frac{\tau}{2\theta}\right)\Delta H\left(\xi\mathrm{w}_n,\xi\mathrm{w},\frac{\tau}{2\theta}\right)$$
$$\leq H\left(\mathrm{w},\mathrm{w}_{n+1},\frac{\tau}{2\theta}\right)\Delta\varrho H\left(\mathrm{w}_n,\mathrm{w},\frac{\tau}{2\theta}\right)$$

Taking limit as $n\to\infty$ and using (39) and (40) in the above expression, we get $G(\nu,\xi\nu,\tau)=1$, that is, $\xi\nu=\nu$. Therefore, ν is a FP of ξ,and $G(\nu,\nu,\tau)=1$ and $H(\nu,\nu,\tau)=0$ for all $\tau>0$. □

Corollary 3.22. *Suppose $(\Xi,G,H,*,\Delta,\perp)$ be an O-complete IFBMS such that $\lim_{nto\infty} G(\mathrm{w},\mathrm{k},\tau)=1$ and $\lim_{nto\infty} H(\mathrm{w},\mathrm{k},\tau)=0$, for all $\mathrm{w},,\mathrm{k}\in\Xi$ and $\xi:\Xi to\Xi$ satisfy*

$$\frac{1}{G(\xi^n\mathrm{w},\xi^n\mathrm{k},\tau)}-1\leq\varrho\left[\frac{1}{G(\mathrm{w},\mathrm{k},\tau)}-1\right] \tag{41}$$
$$H(\xi^n\mathrm{w},\xi^n\mathrm{k},\tau)\leq\varrho H(\mathrm{w},\mathrm{k},\tau) \tag{42}$$

for all $n\in\mathbb{N},\mathrm{w},\mathrm{k}\in\Xi,\tau>0$,where $0<\varrho<1$. Then ξ has a FP, say $\nu\in\Xi$ and $G(\nu,\nu,\tau)=1$, for all $\tau>0$.
Proof. *$\nu\in\Xi$ is a unique FP of ξ^n by utilizing Theorem 3.22, and $G(\nu,\nu,\tau)=1$, for all $\tau>0$. $\xi\nu$ is also a FP of $\xi^n(\xi\nu)=\xi\nu$ from Theorem 3.22, $\xi\nu=\nu$. Hence, the FP of ξ is also a FP of ξ^n.* □

Example 3.23. Suppose $\Xi=[-1,2]$ and define $\perp$ by $\mathrm{w}\perp\mathrm{k}\Leftrightarrow\mathrm{w}+\mathrm{k}\geq 0$. Define G,H as in Example 3.4 with $\alpha=3$,

$$G(\mathrm{w},\mathrm{k},\tau)=\frac{\tau+\min\{\mathrm{w},\mathrm{k}\}^3}{\tau+\min\{\mathrm{w},\mathrm{k}\}^3}\forall\mathrm{w},\mathrm{k}\in\Xi,\tau>0, \tag{43}$$

and

$$H(\mathrm{w},\mathrm{k},\tau)=1-\frac{\tau+\min\{\mathrm{w},\mathrm{k}\}^3}{\tau+\min\{\mathrm{w},\mathrm{k}\}^3}\forall\mathrm{w},\mathrm{k}\in\Xi,\tau>0, \tag{44}$$

with $\sigma*\theta=\sigma\cdot\theta$ and $\sigma\Delta\theta=\max\{\sigma,\theta\}$, then $(\Xi,G,H,*,\Delta,\perp)$ is an O-complete IFBMS. see that $\lim_{\tau\to\infty} G(\mathrm{w},\mathrm{k},\tau)=1$ and $\lim_{\tau\to\infty} H(\mathrm{w},\mathrm{k},\tau)=0$ for all $\mathrm{w},\mathrm{k}\in\Xi$. Define $\xi:\Xi\to\Xi$ by

$$G(\mathrm{w},\mathrm{k},\tau)=\begin{cases}2-\mathrm{w}\ \mathrm{w}\in[-1,1),\\ 1\ \mathrm{w}\in[1,2),\end{cases} \tag{45}$$

We have the following four cases:

1. if $\mathrm{w}, \mathrm{k} \in [-1,1)$ then $\xi\mathrm{w} = 2 - \mathrm{w}$ and $\xi\mathrm{k} = 2 - \mathrm{k}$,
2. if $\mathrm{w}, \mathrm{k} \in [1,2]$ then $\xi\mathrm{w} = \xi\mathrm{k} = 1$,
3. if $\mathrm{w}, \in [-1,1)$ and $\mathrm{k} \in [1,2]$ then $\xi\mathrm{w} = 2 - \mathrm{w}$ and $\xi\mathrm{k} = 1$,
3. if $\mathrm{w}, \in [1,2]$ and $\mathrm{k} \in [-1,1)$ then $\xi\mathrm{w} = 1$ and $\xi\mathrm{k} = 2 - \mathrm{k}$,

Because $\mathrm{w} \perp \mathrm{k} \Leftrightarrow \mathrm{w} + \mathrm{k} \geq 0$, it is clearly implies that $\xi(\mathrm{w}) + \xi(\mathrm{k}) \geq 0$. That is, ξ is $\perp$-preserving. Suppose $\{\mathrm{w}_n\}$ be any O-sequence in Ξ that O-converges to $\mathrm{w} \in \Xi$. we get

$$\lim_{n\to\infty} G(\mathrm{w}, \mathrm{k}, \tau) = \lim_{n\to\infty} \frac{\tau + \min\{\mathrm{w}, \mathrm{k}\}^3}{\tau + \min\{\mathrm{w}, \mathrm{k}\}^3} = 1 \forall \mathrm{w}, \mathrm{k} \in \Xi, \tau > 0,$$

and

$$\lim_{n\to\infty} H(\mathrm{w}, \mathrm{k}, \tau) = 1 - \lim_{n\to\infty} \frac{\tau + \min\{\mathrm{w}, \mathrm{k}\}^3}{\tau + \min\{\mathrm{w}, \mathrm{k}\}^3} = 0 \forall \mathrm{w}, \mathrm{k} \in \Xi, \tau > 0,$$

we can easily see that if $\lim_{n\to\infty} G(\mathrm{w}_n, \mathrm{w}, \tau) = 1$, and $\lim_{n\to\infty} H(\mathrm{w}_n, \mathrm{w}, \tau) = 0$, then $\lim_{n\to\infty} G(\xi\mathrm{w}_n, \xi\mathrm{w}, \tau) = 1$ and $\lim_{n\to\infty} H(\xi\mathrm{w}_n, \xi\mathrm{w}, \tau) = 0$ for all $\tau > 0$. That is, ξ is orthogonal continuous. For $\mathrm{w} = \mathrm{k}$, it is obvious.

$$\frac{1}{G(\xi\mathrm{w}, \xi\mathrm{k}, \tau)} - 1 \leq \varrho \left[\frac{1}{G(\mathrm{w}, \mathrm{k}, \tau)} - 1\right]$$

$$H(\xi\mathrm{w}, \xi\mathrm{k}, \tau) leq \varrho H(\mathrm{w}, \mathrm{k}, \tau).$$

All conditions of Theorem 3.21 are satisfied and 1 is a FP of ξ

4 An Application to an Integeal Equation

Let $\Xi = C([\sigma, \theta], \mathbb{R})$ be the set of all continuous real valued functions defined on $[\sigma, \theta]$. Now, we consider the Fredholm type integral equation of fiest kind:

$$\mathrm{w}(\eta) = \int_\sigma^\theta F(\eta, j)\mathrm{w}(\eta)\mathrm{k}j, for\ \eta, j \in [\sigma, \theta] \tag{46}$$

Where, $F \in \Xi$. Define G as in Example 3.2, That is

$$G(\mathrm{w}(\eta), \mathrm{k}(\eta), \tau) = \sup_{\eta\in[\sigma,\theta]} \begin{cases} 1 \text{ if } \mathrm{w} = \mathrm{k}, \\ \left[e^{\frac{\max\{\mathrm{w}(\eta), \mathrm{k}(\eta)\}^\alpha}{\tau}}\right]^{-1} \text{ otherwise,} \end{cases} \tag{47}$$

and

$$H(\mathrm{w}(\eta), \mathrm{k}(\eta), \tau) = \sup_{\eta\in[\sigma,\theta]} \begin{cases} 0 \text{ if } \mathrm{w} = \mathrm{k}, \\ 1 - \left[e^{\frac{\max\{\mathrm{w}(\eta), \mathrm{k}(\eta)\}^\alpha}{\tau}}\right]^{-1} \text{ otherwise,} \end{cases} \tag{48}$$

for all $\mathrm{w}, \mathrm{k} \in \Xi$ and $\tau > 0$. Then $(\Xi, G, H, *, \Delta, \perp)$ is an O-complete IFBMS.

Theorem 4.1. *Assume that* $\max\{F(\eta, j)\mathrm{w}(\eta), F(\eta, j)\mathrm{k}(\eta)\} \leq \varrho \max\{\mathrm{w}(\eta), \mathrm{k}(\eta)\} for\ \mathrm{w}, \mathrm{k} \in \Xi, \varrho \in (0,1)$ *and* $\eta, j \in [\sigma, \theta]$. *Also, consider* $\int_\sigma^\theta \mathrm{k}j = 1$. *Then the Fredholm type integral equation of first kind in equation (46) has a unique solution.*

Proof. *Define* $\xi : \Xi \to \Xi$ *by* $\mathrm{w}(\eta) = \int_\sigma^\theta F(\eta, j)\mathrm{w}(\eta)\mathrm{k}j, for\ \eta, j \in [\sigma, \theta]$. *Define Orthogonality as:* $\mathrm{w}(\eta) \perp \mathrm{k}(\eta) \Leftrightarrow \mathrm{w}(\eta)\mathrm{k}(\eta) \in \{|\mathrm{w}(\eta)|, |\mathrm{k}(\eta)|\}$. *We see that* $\mathrm{w}(\eta)$ *and* $\xi\mathrm{w}(\eta)$ *belong to* Ξ. *So, observe that if* $\mathrm{w}(\eta) \perp \mathrm{k}(\eta)$, *then must be* $\xi \mathrm{w}(\eta) \perp \xi \mathrm{k}(\eta)$. *Observe that the existence of a FP of the operator* ξ *is equivalent to the existance of a solution of the Fredholm type integral equation* (46). *Now, for* $\mathrm{w}(\eta) = \mathrm{k}(\eta)$, *the contraction condition holds. While for* $\mathrm{w} \neq \mathrm{k}$, *We have*

$$G(\xi \mathrm{w}(\eta), \xi \mathrm{k}(\eta), \varrho\tau) = \left[e^{\frac{\max\{\mathrm{w}(\eta),\mathrm{k}(\eta)\}^{\alpha}}{\varrho\tau}}\right]^{-1}$$

$$= \left[e^{\frac{\max\left\{\int_\sigma^\theta F(\eta,j)\mathrm{w}(\eta)\mathrm{k}j, \int_\sigma^\theta F(\eta,j)\mathrm{k}(\eta)\mathrm{k}j\right\}^{\alpha}}{\varrho\tau}}\right]^{-1}$$

$$= \left[e^{\frac{\left(\int_\sigma^\theta \max\{F(\eta,j)\mathrm{w}(\eta)\mathrm{k}j, F(\eta,j)\mathrm{k}(\eta)\mathrm{k}j\}\right)^{\alpha}}{\varrho\tau}}\right]^{-1}$$

$$\geq \left[e^{\frac{\left(\int_\sigma^\theta \max\{\mathrm{w}(\eta)\mathrm{k}j, \mathrm{k}(\eta)\mathrm{k}j\}\right)^{\alpha}}{\varrho\tau}}\right]^{-1}$$

$$\geq \sup_{\eta\in[\sigma,\theta]} \left[e^{\frac{(\varrho \max\{\mathrm{w}(\eta)\mathrm{k}j, \mathrm{k}(\eta)\})^{\alpha}\left(\int_\sigma^\theta \mathrm{k}j\right)^{\alpha}}{\varrho\tau}}\right]^{-1}$$

$$= \sup_{\eta\in[\sigma,\theta]} \left[e^{\frac{(\max\{\mathrm{w}(\eta)\mathrm{k}j, \mathrm{k}(\eta)\})^{\alpha}}{\tau}}\right]^{-1}$$

$$= G(\mathrm{w}(\eta), \mathrm{k}(\eta), \tau),$$

and

$$H(\xi \mathrm{w}(\eta), \xi \mathrm{k}(\eta), \varrho\tau) = 1 - \left[e^{\frac{\max\{\mathrm{w}(\eta),\mathrm{k}(\eta)\}^{\alpha}}{\varrho\tau}}\right]^{-1}$$

$$= 1 - \left[e^{\frac{\max\left\{\int_\sigma^\theta F(\eta,j)\mathrm{w}(\eta)\mathrm{k}j, \int_\sigma^\theta F(\eta,j)\mathrm{k}(\eta)\mathrm{k}j\right\}^{\alpha}}{\varrho\tau}}\right]^{-1}$$

$$1 - \left[e^{\frac{\left(\int_\sigma^\theta \max\{F(\eta,j)\mathrm{w}(\eta)\mathrm{k}j, F(\eta,j)\mathrm{k}(\eta)\mathrm{k}j\}\right)^{\alpha}}{\varrho\tau}}\right]^{-1}$$

$$\leq 1 - \left[e^{\frac{\left(\int_\sigma^\theta \max\{\mathrm{w}(\eta)\mathrm{k}j, \mathrm{k}(\eta)\mathrm{k}j\}\right)^{\alpha}}{\varrho\tau}}\right]^{-1}$$

$$\leq 1 - \sup_{\eta\in[\sigma,\theta]} \left[e^{\frac{(\varrho \max\{\mathrm{w}(\eta)\mathrm{k}j, \mathrm{k}(\eta)\})^{\alpha}\left(\int_\sigma^\theta \mathrm{k}j\right)^{\alpha}}{\varrho\tau}}\right]^{-1}$$

$$= 1 - \sup_{\eta\in[\sigma,\theta]} \left[e^{\frac{(\max\{\mathrm{w}(\eta)\mathrm{k}j, \mathrm{k}(\eta)\})^{\alpha}}{\tau}}\right]^{-1}$$

$$= H(\mathrm{w}(\eta), \mathrm{k}(\eta), \tau),$$

Hence, ξ *is an* $\perp$*-contraction. Let* $\{\mathrm{w}_n\}$ *be an O-sequence in* Ξ *O-converging to* $\mathrm{w} \in \Xi$. *Because* ξ *is an* $\perp$*-preserving, then* $\{\xi \mathrm{w}_n\}$ *is an O-sequence for each* $n \in \mathbb{N}$. *We have*

$$G(\xi \mathrm{w}_n(\eta), \xi \mathrm{w}, \varrho\tau) \geq G(\mathrm{w}_n(\eta), \mathrm{w}(\eta), \tau) \tag{49}$$

and

$$H(\xi \mathrm{w}_n(\eta), \xi \mathrm{w}, \varrho\tau) \leq H(\mathrm{w}_n(\eta), \mathrm{w}(\eta), \tau) \tag{50}$$

As $\lim_{n\to\infty} G(\xi \mathrm{w}_n(\eta), \xi \mathrm{w}, \varrho\tau) = 1$ *and* $\lim_{n\to\infty} H(\xi \mathrm{w}_n(\eta), \xi \mathrm{w}, \varrho\tau) = 0$ *for all* $\tau > 0$*, it is clear that*

$$\lim_{n\to\infty} G(\xi \mathrm{w}_n(\eta), \xi \mathrm{w}, \varrho\tau) = 1, \tag{51}$$

$$\lim_{n\to\infty} H(\xi \mathrm{w}_n(\eta), \xi \mathrm{w}, \varrho\tau) = 0, \tag{52}$$

Hence, ξ *is* $\perp$*-continuous. Therefore, all conditions of Theorem 3.13 are satisfied. Hence, the operator* ξ *has a unique FP. That is, the Fredholm type integral equation (46) has a unique solution.* □

5 Conclusion

In this study, we established the concept of an OIFBMS as a generalization of an IFBMS. We established some fixed point theorems and solved some non-trivial examples with an application to Fredholm integral equations. This work is extendable in the structure of orthogonal neutrosophic b-metric spaces,and orthogonal inutionistic fuzzy controlled metric spaces and we can increase self mappings to get new results.

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