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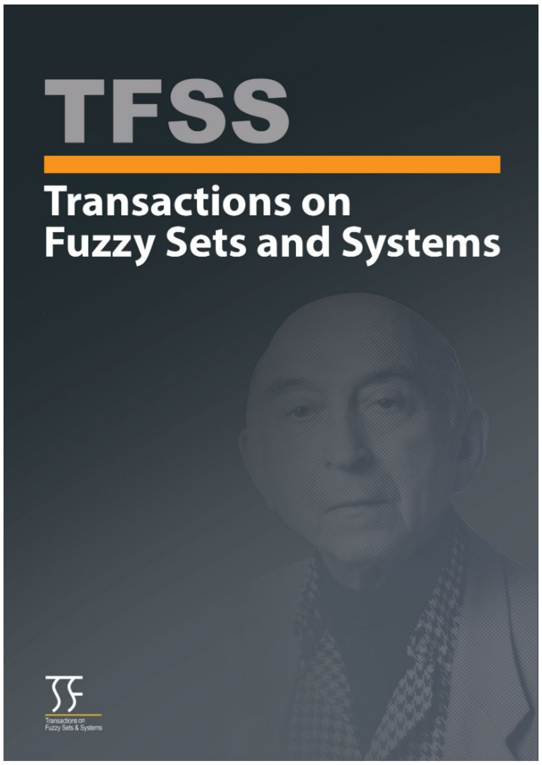

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Families of Fuzzy Sets and Lattice Isomorphisms Preparation

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Families of Fuzzy Sets and Lattice Isomorphisms Preparation

John N Mordeson , **Sunil Mathew***

Abstract. In this paper, we discuss how theoretical results from one family of fuzzy sets can be carried over immediately to another family of fuzzy sets by the use of lattice isomorphisms. We also show that these families can occur naturally and that applications may not necessarily be carried over using these isomorphisms. We illustrate this using techniques from the study of human trafficking and its analysis using mathematics of uncertainty. We also consider the new definition of fuzzy set provided by Trillas and de Soto.

AMS Subject Classification 2020: 03B52; 03E72
Keywords and Phrases: Modern slavery, Government response, Vulnerability, Lattice isomorphisms, Fuzzy sets.

1 Introduction

In this paper, we discuss how theoretical results from one family of fuzzy sets can be carried over immediately to another family of fuzzy sets by the use of lattice isomorphisms. We also show that these families can occur naturally and that applications may not necessarily be carried over using these isomorphisms. We illustrate this using techniques from the study of human trafficking and its analysis using mathematics of uncertainty. Mathematics of uncertainty is a very appropriate tool to use in the study of trafficking. This is because accurate data concerning trafficking in persons is impossible to obtain. The goal of the trafficker is to be undetected. The size of the problem also makes it very difficult to obtain accurate data. Victims are reluctant to report crimes or testify for fear of reprisals, disincentives, both structural and legal, for law enforcement to act against traffickers, a lack of harmony among existing data sources, and an unwillingness of some countries and agencies to share data. We also generalize some results concerning families of fuzzy sets involved with these lattice isomorphisms.

2 Fuzzy Sets and Lattice Isomorphisms

One of the most important papers concerning fuzzy set theory in recent years is one by Klement and Mesiar, [3]. In this paper, it is shown that differently defined families of fuzzy sets have lattice structures that are actually isomorphic and so theoretical results for one family can be carried over to another family.

We show by using a real world problem with real world data that even though theoretical results can be obtained for one family from another, the two families may arise naturally in an application.

We use the concepts of vulnerability and government response to modern slavery to illustrate our findings. In [12], it is stated that the departing point is the fact that not only fuzzy sets originate in Language, but

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that they are just 'linguistic entities' genetically different from the concept of 'crisp sets' whose origin is either in a physical collection of objects, or in a list of them. A new definition of a fuzzy set is presented by means of two magnitudes: A qualitative one, called a graph, the basic magnitude, and a quantitative one, a scalar magnitude. If the first reflects the language's relational ground of the fuzzy set, the second reflects the (numerical) extensional state in which it currently appears.

We next illustrate these ideas using the concepts of vulnerability and government response with respect to modern slavery, [9].

Vulnerability Measures

(1) Government issues

(2) Nourishment and access

(3) Inequality

(4) Disenfranchised groups

(5) Effect of conflicts

Countries are scored with respect to these five measures. Then a weighted average of these scores is taken to provide a single score for each number. For example, the final score for Brazil is 36.4. The countries are placed into regions. Brazil is in the Americas. For this region, the highest score was 69.6 and the smallest was 10.2. The country scores were normalized using the formula (number - minimum)/(maximum - minimum) to obtain $(36.4 - 10.2)/(69.6 - 10.2) = 0.441$.

Government Response

(1) Support for survivors

(2) Criminal justice

(3) Coordination

(4) Response

(5) Supply chains

Similarly, as for the vulnerability measures, a final score is determined for each country with respect to government response. For example the final score for Brazil is 55.6. For the Americas, the maximum score was 71.7 and the minimum was 20.8. Hence the normalized value for Brazil was $(55.6-20.8)/(71.7-20.8) = 0.684$.

In [12], it is stated that shortening the statement x is less P, where P is a predicate, by $x \prec_P y$ facilitates the basic magnitude. That is, $x \prec_P y \subseteq X \times X$.

For our illustration, we let P denote the predicate vulnerable and X denote the set of countries under consideration. Now the final vulnerable score for Mexico was 57.3. Brazil's was 36.4. Hence Brazil $\prec_P$Mexico. The final value for government response for Mexico was 52.4 and for Brazil 55.6. In the case, we have Mexico $\prec_P$ Brazil if P denotes government response and $\prec_P$is the linguistic relation x has less government response than y.

In [12], a membership function $m_P : X \times X \to [0, 1]$ was introduced. It provides a numerical value for measuring the degree to which x is P. The membership function is required to satisfy the following three properties:

(i) $x \prec_P y$ implies $m_P(x) \leq m_P(y)$.

(ii) If z is minimal, then $m_P(z) = 0$.

(iii) If w is maximal, then $m_P(w) = 1$.

We see that our membership function $m_P(x) = \frac{\#(x)-\min}{\max-\min}$, where $\#(x)$ denotes the final score of x, satisfies these three properties. Thus $m_V(Brazil) = 0.441$, where V denotes vulnerable and $m_G(Brazil) = 0.684$, where G denotes government response. For Mexico, we have $m_V(Mexico) = 0.793$ and $m_G(Mexico) = 0.621$.

We present some isomorphisms and other methods in fuzzy set theory to obtain results from one family for another.

The following table is from [9]. See also [[4], p. 104].

Table 1: Global slavery index Americas

Country	Government Response	Vulnerability	Prevalence
Argentina	0.821	0.297	0.156
Barbados	0.365	0.533	0.431
Bolivia	0.402	0.570	0.313
Brazil	0.684	0.441	0.294
Canada	0.742	0.000	0.000
Chile	0.815	0.259	0.058
Columbia	0.398	0.696	0.431
Costa Rica	0.573	0.306	0.156
Cuba	0.000	0.710	0.647
Dominican Rep.	0.730	0.553	0.686
El Salvador	0.326	0.681	0.392
Ecuador	0.502	0.523	0.372
Guatemala	0.479	0.705	0.470
Guyana	0.210	0.592	0.509
Haiti	0.371	1.000	1.000
Honduras	0.318	0.762	0.568
Jamaica	0.742	0.572	0.411
Mexico	0.621	0.793	0.431
Nicaragua	0.500	0.567	0.470
Paraguay	0.394	0.516	0.215
Panama	0.453	0.441	0.313
Peru	0.622	0.574	0.411
Suriname	0.123	0.537	0.352
Trinidad and Tobago	0.571	0.486	0.490
United States	1.000	0.095	0.156
Uruguay	0.581	0.159	0.098
Venezuela	0.145	0.803	1.000

Neutrosophic fuzzy sets and Pythagorean fuzzy sets: Recall that a neutrosophic fuzzy set is a triple (σ, τ, μ) of fuzzy subsets of a set. It is based on the lattice of elements $(x_1, x_2, x_3) \in [0, 1]^3$,where $(x_1, x_2, x_3) \leq (y_1, y_2, y_3)$ if and only if $x_1 \leq y_1, x_2 \leq y_2$, and $x_3 \geq y_3$. Also, a Pythagorean fuzzy set is a pair of fuzzy subsets (σ, τ) of a set X such that for all $x \in X, \sigma(x)^2 + \tau(x)^2 \leq 1$, [5].We can see that vulnerability and government response corresponding to modern slavery are opposites, [13]. That is, an increase in government response by a country would lower the country's vulnerability. However, $m_V(Brazil) + m_G(Brazil) = 0.441 + 0.684 > 1$. This gives meaning to neutrosophic fuzzy sets, [8], even though certain theoretical results can following immediately from other types of fuzzy sets. Also, $(0.441)^2 + (0.684)^2 = 0.194 + 0.468 < 1$. Consequently, similar comments might be able to be made here even though Pythagorean fuzzy sets and intuitionistic fuzzy sets, [1], have corresponding isomorphic lattices. However, this isomorphism may make the situation different to the neutrosophic case since it is so straight forward. The lattice isomorphism f involved here is $f : P^* \to L^*$ defined by $f((x_1, x_2)) = (x_1^2, x_2^2)$, where $L^* = \{(x_1, x_2)|x_1, x_2 \epsilon[0, 1], |x_1 + x_2 \leq 1\}$ and $P^* = \{(x_1, x_2)|x_1, x_2\epsilon[0, 1], x_1^2 + x_2^2 \leq 1\}$. The paper by Klement and Mesiar contains many other cases, where various families of fuzzy sets have isomorphic lattices.

Let X be a set with n elements, say $X = \{x_1, ..., x_n\}$. Let μ, ν be fuzzy subsets of X. Consider the fuzzy similarity measures, $M(\mu, \nu) = \frac{\sum_{x\in X} \mu(x)\wedge\nu(x)}{\sum_{x\in X} \mu(x)\vee\nu(x)}$ and $S(\mu, .\nu) = 1 - \frac{\sum_{x\in X} |\mu(x)-\nu(x)|}{\sum_{x\in X}(\mu(x)+\nu(x))}$. Let m be a positive real number. Then $\frac{\sum_{x\in X} \mu(x)\wedge\nu(x)}{\sum_{x\in X} \mu(x)\vee\nu(x)} = \frac{\sum_{x\in X} \mu(x)/m\wedge\nu(x)/m}{\sum_{x\in X} \mu(x)/m\vee\nu(x)/m}$ and $\frac{\sum_{x\in X} |\mu(x)-\nu(x)|}{\sum_{x\in X}(\mu(x)+\nu(x))} = \frac{\sum_{x\in X} |\mu(x)/m-\nu(x)/m|}{\sum_{x\in X}(\mu(x)/m+\nu(x)/m)}$. Suppose there exists $x \in X$ such that $\mu(x) + \nu(x) > 1$. Let m denote the maximal such $\mu(x) + \nu(x)$. Then we see that we get the same M and S values if we divide all the $\mu(x)$ and $\nu(x)$ by m.

Example 2.1. Let $X = \{x_1, x_2\}$. Define the fuzzy subsets μ, ν of X as follows:

	μ	ν
x_1	0.1	0.1
x_2	0.2	0.91

Then $\mu(x_2) + \nu\,(x_2) = 0.2 + 0.91 = 1.11 > 1$. Now $M(\mu, \nu) = \frac{0.1\wedge0.1+0.2\wedge0.91}{0.1\vee0.1+0.2\vee0.91} = \frac{0.3}{1.01}$.

Define the fuzzy subsets μ', ν' of X as follows:

	μ'	ν'
x_1	0.1	0.2
x_2	0.2	0.82

Then $\mu'(x_2) + \nu'\,(x_2) = 0.2 + 0.82 = 1.02 > 1$. Now $M(\mu', \nu') = \frac{0.1\wedge0.2+0.2\wedge0.82}{0.1\vee0.2+0.2\vee0.82} = \frac{0.3}{1.02}$.

Thus $M(\mu, \nu) > M(\mu', \nu')$. We have a Pythagorean situation since $(0.2)^2 + (0.91)^2 < 1$ and $(0.2)^2 + (0.82)^2 < 1$.

Squaring the values of μ and ν, we obtain $\mu_2(x_1) = 0.01, \mu_2(x_2) = 0.04$ and $\nu_2(x_1) = 0.01, \nu_2(x_2) = 0.8281$. Hence

$$M(\mu_2, \nu_2) = \frac{0.01 \wedge 0.01 + 0.4 \wedge 0.8281}{0.01 \vee 0.01 + 0.4 \vee 0.8281} = \frac{0.01 + 0.04}{0.01 + 0.8281}.$$

Also, $\mu_2'(x_1) = 0.01, \mu_2'(x_2) = 0.04, \nu_2'(x_1) = 0.04, \nu_2'(x_2) = 0.6724$. Thus

$$M(\mu_2', \nu_2') = \frac{0.01 \wedge 0.04 + 0.04 \wedge 0.6724}{0.01 \vee 0.04 + 0.04 \vee 0.6724} = \frac{0.01 + 0.04}{0.04 + 0.6724}.$$

Hence $M(\mu_2, \nu_2) < M\mu_2', \nu_2')$. That is, the inequalities have switched. They were not preserved.

We next consider $S(\mu, \nu) = 1 - \frac{\sum_{x\in X} |\mu(x)-\nu(x)|}{\sum_{x\in X}(\mu(x)+\nu(x))}$. For the previous situation, we have $S(\mu, \nu) = 1 - \frac{0+0.71}{0.2+1.11} = \frac{0.71}{1.31} = 1 - 0.542$ and $S(\mu', \nu') = 1 - \frac{0.1+0.62}{0.3+1.02} = 1 - \frac{0.72}{1.32} = 1 - 0.545$. Thus $S(\mu, \nu) > S(\mu', \nu')$.

We also have $S(\mu_2, \nu_2) = 1 - \frac{0+0.7881}{0.04+0.8281} = 1 - \frac{0.7881}{0.8681} = 1 - 0.9078$ and $S(\mu_2', \nu_2') = 1 - \frac{0.03+.6324}{0.05+0.7164} = 1 - \frac{0.6624}{0.7664} = 1 - 0.8643$. Hence $S(\mu_2, \nu_2) < S(\mu_2', \nu_2')$. Once again the inequalities were not preserved.

We have shown with this example that the isomorphism $f : P^* \to L^*$ defined by $f((x_1, x_2)) = (x_1^2, x_2^2)$, where $L^* = \{(x_1, x_2)|x_1, x_2\epsilon[0, 1], x_1 + x_2 \leq 1\}$ and $P^* = \{(x_1, x_2)|x_1, x_2 \in [0, 1]x_1^2 + x_2^2 \leq 1\}$, [1], shows that although theoretical results can determined between Pythagorean fuzzy sets and intuitionistic fuzzy sets, the isomorphism may not be suitable in changing a data set from one to another in applications. Also, isomorphisms in general preserve certain structural properties, but not all outside functions defined on the sets.

3 Lack of Accurate Data

Linguistic variables: The size of flow of trafficked people from country to country is given in [10]. It is reported in linguistic terms since accurate data concerning the size of the flow is impossible to obtain. Information is provided with respect to the reported human trafficking in terms of origin, transit, and/or destination according to the citation index. The data is provided in two columns. Information in the left column as to whether a country ranks (very) low, medium (very) high depends upon the total number of sources which made reference to this country as one of origin, transit, or destination. Information provided in then the right column provides further detail to the information provided in the left column. If a country in the right column was mentioned by one or two sources, the related country was ranked low. If linkage between the countries in the two columns was reported by 3-5 sources, the related country was ranked medium. If 5 or more sources linked the two countries, the country in the right was ranked high. This method of combining linguistic data provides an ideal reason for the use of mathematics of uncertainty to study the problem of trafficking by persons. For example, by assigning numbers in the interval $[0,1]$ to the linguistic data, the data can be combined in a mathematical way. In [6], the notions of t-norms and t-conorms were used. The number 0.1 can be assigned very low, 0.3 to low. 0.5 to medium, 0.7 to high and 0.9 to very high. Using the notation and ideas from [7], we have $x \prec_P y$ if and only if country x's linguistic rank is less than country y's linguistic rank. We have $m_P(x) = 0.1, 0.3, 0.5, 0.7$, or 0.9 if x is assigned very low, low, medium, high, or very high, respectively. We note that here m_P does not satisfy (ii) and (iii).

Colors: In [7], colors are used to determine how well a country is achieving the Sustainable Development Goals (SDGs). A green rating on the SDG dashboard is assigned to a country if all the indicators under that goal are labeled green. Yellow, orange and red indicate increasing distance from the SDG achievement. The worst two colors of a target were averaged to determine the color for its SDG. In [5], the numbers $0.2, 0.4, 0.6.0.8$ are assigned to the colors red, orange, yellow, and green, respectively. Consequently, the results in [7] are placed into the context of mathematics of uncertainty.

4 Theoretical Results

Let m and n be positive real numbers. Let $P_{m,n} = \{(x,y)|x,y \in [0,1] \text{ and } x^m + y^n \leq 1\}$. Let $L^* = \{(x,y)|x,y \in [0,1] \text{ and } x+y \leq 1\}$. Define $\leq_{P\,m,n}$ on $P_{m,n}$ by for all $(x,y),(u,v)\epsilon P_{m,n}(u,v)$ if and only if $x \leq u$ and $y \geq v$. This includes L^* since $L^* = P_{1,1}$. The following result extends the theory for Pythagorean fuzzy sets to m,n-rung fuzzy sets, [2], since $P_{2,2}$ is a Pythagorean fuzzy set.

The following result follows from Theorem 4.2, but we place it here since it motivates Theorem 4.2.

Theorem 4.1. *Define $f : P_{m,n} \to L^*$ by for all $(x,y) \in P_{m,n}, f((x,y)) = (x^m, y^n)$. Then f is a lattice isomorphism of $P_{m,n}$ onto L^*.*

In the above table, we see that for Mexico, $(0.621, 0.793) \notin P_{2,2} \cup P_{3,1} \cup P_{1,3}$. We have $(0.62, 1.793) \in P_{2,3} \cap P_{3,2}$.

For the United States, $\nexists m,n$ such that $(1^m, 0.095^n) \leq 1$.

Consider Mexico again. Now $P_{2,2}$ is the set of all Pythagorean fuzzy sets. Define $g : P_{2,3} \to P_{2,2}$ by for all $(x,y) \in P_{2,3}, g((x,y)) = (x^{\frac{3}{2}}, y)$. Note $(x^{\frac{3}{2}}, y) \in P_{2,2}$ since $(x^{\frac{3}{2}})^2 + y^2 = x^3 + y^2 \leq 1$.

Let $f_1, f_2 : [0,1] \to [0,1]$ be one-to-one functions of $[0,1]$ onto $[0,1]$ such that for all $x, y\epsilon[0,1], x \leq y$ implies $f_i(x) \leq f_i(y), i = 1,2$.Then $f_i(0) = 0$ and $f_i(1) = 1, i = 1,2$. Assume $f_i(x \wedge y) = f_i(x_i) \wedge f_i(y)$ and $f_i(x \vee y) = f_i(x) \vee f_i(y), i = 1,2$.

Let $\widehat{L}(f_1, f_2) = \{(x, y)|x, y\epsilon[0, 1], f_1(x) + f_2(y) \leq 1\}$. For ease of notation, let $\widehat{L} = \widehat{L}(f_1, f_2)$. Define $\leq_{\widehat{L}}$ on $\widehat{L}$ by for all $(x, y), (w, z)\epsilon\widehat{L}, (x, y) \leq_{\widehat{L}} (w, z)$ if and only if $x \leq w$ and $y \geq z$. Then $\leq_{\widehat{L}}$ is a partial order on $\leq_{\widehat{L}}$ such that any two elements $\leq_{\widehat{L}}$ have a greatest lower bound and a least upper bound. Thus $\leq_{\widehat{L}}$ is a lattice. The greatest lower bound of $(x, y), (w, z)\epsilon \leq_{\widehat{L}}$ is $(x \wedge w, y \vee z)$ and the least upper bound is $(x \vee w, y \wedge z)$.

Theorem 4.2. *Define* $f : \widehat{L} \to L^*$ *by for all* $(x_1, x_2)\epsilon\widehat{L}, f((x_1, x_2)) = (f(x_1), f_2(x_2))$. *Then* f *is a lattice isomorphism of* $\widehat{L}$ *onto* L^*.

Proof. Clearly, f maps $\widehat{L}$ into L^*. Now $(x_1, x_2) = (y_1, y_2) \Leftrightarrow x_1 = y_1$ and $x_2 = y_2 \Leftrightarrow f_1x_1) = f_1(y_1)$ and $f_2(x_2) = f_2(y_2) \Leftrightarrow f((x_1, x_2)) = (f_1(x_1), f_2(x_2)) = (f_1(y_1), f_2(y_2)) = f((y_1, y_2))$. Hence f is single-valued and one-to-one. Let $(x_1, x_2)\epsilon L^*$. Then $(f_1^{-1}(x_1), f_2^{-1}(x_2))\epsilon\widehat{L}$. Thus f maps $\widehat{L}$ onto L^*. Let $(x_1, x_2), (y_1, y_2)\epsilon\widehat{L}$. Then

$$\begin{aligned} f((x_1, x_2) \wedge_{\widehat{L}} (y_1, y_2)) &= f((x_1 \wedge y_1, x_2 \vee y_2)) \\ &= (f_1(x_1 \wedge y_1), f_2(x_2 \vee y_2)) \\ &= (f_1(x_1) \wedge f_1(y_1)), (f_2(x_2) \vee f_2(y_2)) \\ &= (f_1(x_1), f_2(x_2)) \wedge_{L^*} (f_1(y_1), x_{1x} f_2(y_2)) \\ &= f((x_1, x_2) \wedge_{L^*} f(y_1, y_2)). \end{aligned}$$

Similarly, $f((x_1, x_2) \vee_{\widehat{L}} (y_1, y_2)) = f((x_1, x_2) \vee_{L^*} f(y_1, y_2))$.

Suppose $(x_1, x_2) \leq_{\widehat{L}} (y_1, y_2)$. Then $x_1 \leq y_1$ and $x_2 \geq y_2$ and so $f_1(x_1) \leq f_1(y_1)$ and $f_2(x_2) \geq f_2(y_2)$. Thus $(f_1(x_1), f_2(x_2)) \leq_{\widehat{L}} (f_1(y_1), f_2(y_2))$. That is, $f((x_1, x_2) \leq_{\widehat{L}} f((y_1, y_2))$. □

If we let m, n be positive real numbers. Define $f_1, f_2 : [0, 1] \to [0, 1]$ by for all $x\epsilon[0, 1], f_1(x) = x^m$ and $f_2(x) = x^n$, then Theorem 4.1 follows from Theorem 4.2.

Example 4.3. Let i, j be positive real numbers. Define $f_1, f_2 : [0, 1] \to [0, 1]$ by for all $x\epsilon[0, 1], f_1(x) = x^i$ and $f_2(x) = x^j$. Then f_1 and f_2 satisfy the above properties. Thus the above Theorem holds for (m, n)-rung fuzzy sets, where m, n are positive integers.

Let $f_1, f_2 : [0, 1] \to [0, 1]$ be defined by for all $x\epsilon[0, 1], f_1(x) = x$ and $f_2(x) = 1 - x$. Let $\mathcal{I} = \{[x, y]|0 \leq x \leq y \leq 1\}$. Define $f : L^* \to \mathcal{I}$ by for all $(x, y)\epsilon L^*, f((x, y)) = [f_1(x), f_2(y)]$. Then $f((x, y)) = [x, 1 - y]$. Clearly f is single-valued. Note that since $x + y \leq 1, x \leq 1 - y$. Let $[x, y]\epsilon\mathcal{I}$. Then $f((x, 1 - y)) = [x, y]$ and $(x, 1 - y)\epsilon L^*$ since $x + 1 - y \leq 1$, i.e., $(x, 1 - y)\epsilon L^*$. Thus f maps L^* onto $\mathcal{T}$. (Note $x \leq y$ so $x + 1 - y \leq 1$.). Now f is one-to-one since f_1 and f_2 are.

Define $\wedge_{\mathcal{I}}, \vee_{\mathcal{I}}$ on $\mathcal{I}$by for all $[x, 1 - y], [w, 1 - z]\epsilon\mathcal{I}$, $[x, 1 - y] \wedge_{\mathcal{I}} [w, 1 - z] = [x \wedge w, (1 - y) \wedge (1 - z)]$, and $[x, 1 - y] \vee_{\mathcal{I}} [w, 1 - z] = [\ x \vee w, (1 - y) \vee (1 - z)]$. Define $\leq_{\mathcal{I}}$by on $\mathcal{I}$ by for all $[x, 1 - y], [w, 1 - z]\epsilon I$, $[x, 1 - y] \leq_{\mathcal{I}} [w, 1 - z]$ if and only if $x \leq w$ and $y \geq z$.

Theorem 4.4. f *is a lattice isomorphism of* L^* *onto* $\mathcal{I}$.

Proof. By the discussion above f is a one-to-one function of L^* onto $\mathcal{I}$. Let $(x_1, x_2), (y_1, y_2)\epsilon L^*$. Then

$$\begin{aligned} f((x_1, x_2) \wedge_{L^*} (y_1, y_2)) &= f((x_1 \wedge y_1, x_2 \vee y_2)) = (f_1(x_1 \wedge y_1), , f_2(x_2 \vee y_2)) \\ &= (x_1 \wedge y_1, 1 - (x_2 \vee y_2)) = (x_1 \wedge y_1, (1 - x_2) \wedge (1 - y_2)) \\ &= (x_1, (1 - x_2)) \wedge_{\mathcal{I}} (y_1, (1 - y_2)) \\ &= (f_1(x_1), f_2(x_2)) \wedge_{\mathcal{I}} (f_1(y_1), f_2(y_2) \\ &= f((x_1, x_2) \wedge_{\mathcal{I}} f((y_1, y_2)) \end{aligned}$$

Similarly, $f((x_1, x_2) \vee_{L^*} (y_1, y_2)) = f((x_1, x_2) \vee_{\mathcal{I}} f((y_1, y_2))$.

Now $(x_1, x_2) \leq_{L^*} (y_1, y_2) \Leftrightarrow x_1 \leq y_1$ and $x_2 \geq y_2 \Leftrightarrow x_1 \leq y_1$ and $1 - x_2 \leq 1 - y_2 \Leftrightarrow [x_1, 1 - x_2] \leq_{\mathcal{I}^*} [y_1, 1 - y_2] \Leftrightarrow f((x_1, x_2)) \leq_{\mathcal{I}^*} f((y_1, y_2))$. □

5 Conclusion

In this paper, we discussed the important paper by Klement and Mesiar that shows, using lattice isomorphisms, how theoretical results can be carried over immediately from one family of fuzzy sets to another. We show that these families of fuzzy sets can arise naturally in applications. We also show that newly developed families of fuzzy sets may also have these isomorphic lattices.

Conflict of Interest: The authors declare no conflict of interest.

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