Journal of Linear and Topological Algebra
Vol. 13*, No.* 02*,* 2024*,* 101- 112
DOR:
DOI: 10.71483/JLTA.2024.1115528

Star graphs for torsion elements in multiplication modules

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Received 29 April 2024; Revised 27 August 2024; Accepted 3 September 2024.
Communicated by Amin Mahmoodi

Abstract. Let R be a commutative ring with identity, M a multiplication R-module, and $T(M)^\star$ the set of non-zero torsion elements of M. We consider two graphs, the torsion graph and the annihilator graph of M that have $T(M)^\star$ as their set of vertices, and investigate the cases when these graphs are stars. The graph theoretic properties are reflected in the ring theoretic properties and vice versa. If a ring is considered as a module on itself, then the module is a multiplication module. Hence, our results directly generalize results about rings.

Keywords: Annihilator graphs, zero-divisor graphs, star graphs, torsion elements, annihilators, modules, multiplication modules, reduced modules.

2010 AMS Subject Classification: 13C70,05C25.

1. Introduction

For a commutative ring with identity R, and an R-module M, we define two related graphs. The torsion graph $\Gamma(M)$ was introduced by Ghalandarzadeh and Malakooti Rad [10], and the annihilator graph $\mathrm{AG}(M)$ was introduced by Abdollah et al. [1]. For both graphs, the set of vertices is the set $T(M)^\star$—consisting of non-zero torsion elements of M. (An element x of M is a torsion element if there exists a non-zero $r \in R$ with $rx = 0_M$.) Two vertices x and y are adjacent in $\Gamma(M)$ if and only if $[Rx : M][Ry : M]M = \{0_M\}$. In contrast, two vertices x and y are adjacent in $\mathrm{AG}(M)$ if and only if

$$\mathrm{Ann}_R([Rx : M]y) \neq \mathrm{Ann}_R(x) \cup \mathrm{Ann}_R(y), \text{ or } \mathrm{Ann}_R([Ry : M]x) \neq \mathrm{Ann}_R(x) \cup \mathrm{Ann}_R(y).$$

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Print ISSN: 2252-0201
Online ISSN: 2345-5934

(For N a submodule of M, $[N : M] = \{r \in R \mid rM \subseteq N\}$, and $\mathrm{Ann}_R(N) = [\{0_M\} : N]$.) It can be shown ([1, Proposition 3]) that $\Gamma(M)$ is a subgraph of $\mathrm{AG}(M)$ (but not necessarily vice versa). In the special case when $M = R$ is considered as an R-module, then the vertices of the graphs are the non-zero zero-divisors of R. Two vertices x and y are adjacent in $\Gamma(R)$ if $xy = 0$, while they are adjacent in $\mathrm{AG}(R)$ if $\mathrm{Ann}_R(xy) \neq \mathrm{Ann}_R(x) \cup \mathrm{Ann}_R(y)$. These special cases were first introduced by Beck [7] and Anderson and Livingston [4] for $\Gamma(R)$ and by Badawi [5] for $\mathrm{AG}(R)$. The zero-divisor graph of a ring and its generalizations and variants have been the object of intense recent study. For a survey, we refer the reader to Anderson et al. [3], or the recent monograph by Anderson, Asir, Badawi, and Chelvam [2]. For the more general torsion and annihilator graphs of a module over a commutative ring, see Abdollah et al.'s paper [1].

An R-module M is called a multiplication module, first introduced by Mehdi [14] but also see Barnard [6] and El Bast and Smith [9], if $N = [N : M]M$ for all submodules N of M. Every cyclic R-module is a multiplication module, and so, if R, a commutative ring with identity, is considered as a module over itself, then it is a multiplication module. Hence, results on multiplication modules directly generalize the corresponding results on rings.

We are interested in understanding multiplication R-modules M for which $\Gamma(M)$ or $\mathrm{AG}(M)$ are a star. Recall that a star is a graph where one vertex, called the central vertex, is adjacent to all the other vertices, and all the other vertices have degree 1.

Recall that $T(M)$ is the set of torsion elements of an R-module M, and $T(M)^\star = T(M) \backslash \{0_M\}$. A proper submodule P of M is called a prime submodule (see Lu [13]) if whenever $ax \in P$, for some $a \in R$, $x \in M$, then either $x \in P$ or $a \in [P : M]$. Establishing a connection between ring theoretic properties and graph theoretic properties, we prove

Theorem A. (Theorem 3.6) Let M be a multiplication R-module. Assume $\Gamma(M)$ is a star with x as its central vertex. Then $T(M) = Rx \cup \mathrm{Ann}_R(x)M$, $\mathrm{Ann}_R(x)M$ is a prime submodule of M, $\mathrm{Ann}_R(x)$ is a prime ideal of R, and exactly one of the following must be true:

a $T(M) = Rx = \mathrm{Ann}_R(x)M = \{0_M, x, 2x\}$, and $\Gamma(M)$ has two vertices and a single edge.

b $Rx = \{0_M, x\}$, $x \in \mathrm{Ann}_R(x)M$, and $T(M) = \mathrm{Ann}_R(x)M$.

c $Rx = \{0_M, x\}$, $\mathrm{Ann}_R(x)M = \{0_M\}$, $M = T(M) = Rx$, and $\Gamma(M)$ is a single vertex.

d $Rx = \{0_M, x\}$, $x \notin \mathrm{Ann}_R(x)M$, $M = Rx \oplus \mathrm{Ann}_R(x)M$, and $T(M)$ is not a submodule of M.

In Theorem 3.2, we extend parts of the above to the case when $\Gamma(M)$ is not necessarily a star, but continues to have a vertex x adjacent to all other vertices. Turning to sufficient conditions for $\Gamma(M)$ being a star, we show that it is uncommon for $\Gamma(M)$ to be a tree and yet not a star. In particular, we prove

Theorem B. (Theorem 4.2) Let M be a multiplication R-module. Assume $\Gamma(M)$ has no isolated vertices, and no cycles, and yet has a path of length 3. Then there exists $x \in T(M)^\star$ such that $Rx = \{0_M, x\}$, and $M = Rx \oplus \mathrm{Ann}_R(x)M$. Furthermore, $\mathrm{Ann}_R(x)M \backslash \{0_M\} \subseteq T(M)^\star$, and the subgraph of $\Gamma(M)$ induced by these vertices has no edges.

Finally, we turn to the graph $\mathrm{AG}(M)$, and show that, most often, if either $\Gamma(M)$ or $\mathrm{AG}(M)$ is a star, then the two graphs are identical. We prove

Theorem C. (Theorem 5.1) Let M be a multiplication R-module.

a If $\mathrm{AG}(M)$ is a star, then $\Gamma(M) = \mathrm{AG}(M)$ is a star as well. In particular, the conclusions of Theorem A remain valid.

b If $\Gamma(M)$ is a star, then, except for Case (b) of Theorem A, $\mathrm{AG}(M) = \Gamma(M)$ is a star as well.

For the particular case of rings (equivalent to considering a ring as a module over itself), star torsion and annihilator graphs were already studied, respectively, by Anderson and Livingston [4] and Badawi [5]. As explained in what follows, some of our results can be seen as partial generalizations of their results to the more general setting of modules.

2. Preliminaries

For convenience, we state a few oft used implications for multiplication modules.

Lemma 2.1 Let M be a multiplication R-module. Then

a if N and L are submodules of M, then $[N : M]L = [L : M]N$,

b if $x, y \in M$, then $[Rx : M]y = [Ry : M]x$,

c if $x \in M\backslash\{0_M\}$, then $[Rx : M]M = Rx$ and $[Rx : M] \neq \{0_R\}$,

d if $x \in M\backslash\{0_M\}$, then $\mathrm{Ann}_R(x)M$ is a proper submodule of M,

e if $x \in M$, then $\mathrm{Ann}_R(x)M = \{y \in M \mid [Rx : M]y = \{0_M\}\}$,

f if $x \in T(M)^\star$, then $\mathrm{Ann}_R(x)M\backslash\{0_M, x\}$ is exactly the set of neighbors of x in $\Gamma(M)$,

g in $\Gamma(M)$, if $x \in T(M)^\star$, and if y is a neighbor of x, then every element of $Ry\backslash\{0_M\}$ is adjacent to every element of $Rx\backslash\{0_M\}$,

h if $x, y \in T(M)^\star$, then x and y are adjacent vertices of $\mathrm{AG}(M)$ if and only if $\mathrm{Ann}_R(x) \cup \mathrm{Ann}_R(y)$ is a proper subset of $\mathrm{Ann}_R([Rx : M]y)$.

Proof. Some of this is adapted from Lemmas 5 & 6 of Abdollah et al. [1]. We include the proofs for completeness.

a By definition of a multiplication module, we have $[N : M]L = [N : M][L : M]M = [L : M]N$.

b A special case of Part (a), since $[Rx : M]y = [Rx : M]Ry$.

c By definition of a multiplication module, $Rx = [Rx : M]M$, and so since $0_M \neq x \in Rx$, $[Rx : M] \neq \{0_R\}$.

d $\mathrm{Ann}_R(x)$ is an ideal of R and so $\mathrm{Ann}_R(x)M$ is a submodule of M, and we have $\{0_M\} \neq Rx = [Rx : M]M$. However, if $\mathrm{Ann}_R(x)M = M$, then $[Rx : M]M = \mathrm{Ann}_R(x)[Rx : M]M \subseteq \mathrm{Ann}_R(x)Rx = \{0_M\}$. The contradiction implies that $\mathrm{Ann}_R(x)M$ is a proper submodule of M.

e If $y \in \mathrm{Ann}_R(x)M$, then $[Rx : M]y \subseteq [Rx : M]\,\mathrm{Ann}_R(x)M \subseteq Rx\,\mathrm{Ann}_R(x) = \{0_M\}$. For the reverse inclusion, if $y \in M$ and $[Rx : M]y = \{0_M\}$, then, by part (b), $[Ry : M]x = [Rx : M]y = \{0_M\}$ and $[Ry : M] \subseteq \mathrm{Ann}_R(x)$. Hence, by part (c), $Ry = [Ry : M]M \subseteq \mathrm{Ann}_R(x)M$.

f If y is adjacent to x in $\Gamma(M)$, then y is non-zero and not equal to x. Moreover, by part (c) and definition of adjacency in $\Gamma(M)$, we have $[Rx : M]y = [Rx : M][Ry : M]M = \{0_M\}$, and so $y \in \mathrm{Ann}_R(x)M$ by part (e). Conversely, if $y \in \mathrm{Ann}_R(x)M\backslash\{0_M, x\}$, then, by part (e), $[Rx : M]y = \{0_M\}$. So, using Part (c), $\{0_R\} \neq [Rx : M] \subseteq \mathrm{Ann}_R(y)$, and y is a non-zero torsion element of M. In addition, $[Rx : M][Ry : M]M = [Rx : M]y = \{0_M\}$, and so y and x are adjacent vertices in $\Gamma(M)$.

g This follows from parts (e) and (f) directly. If y is adjacent to x in $\Gamma(M)$, then $[Ry : M]Rx = \{0_M\}$, which in turn implies that every element of $Ry\backslash\{0_M\}$ is adjacent to every element of $Ry\backslash\{0_M\}$.

h Follows from the definition of AG(M), since, $\mathrm{Ann}_R(y) \subseteq \mathrm{Ann}_R([Rx : M]y])$, $\mathrm{Ann}_R(x) \subseteq \mathrm{Ann}_R([Ry : M]x])$, and by Part (a), $[Ry : M]Rx = [Rx : M]Ry$.

■

The following is immediate (also see El Bast and Smith [9, Corollary 2.11]):

Lemma 2.2 Let M be an R-module, and P a prime submodule of M. Then $[P : M]$ is a prime ideal of R.

Proof. Since $P < M$, $[P : M]$ is a proper ideal of R. By way of contradiction, assume that there exists $a, b \in R$, with $ab \in [P : M]$, and neither a nor b in $[P : M]$. Since $P < M$, there exists $m_0 \in M\backslash P$. Now, since $ab \in [P : M]$, $a(bm_0) = (ab)m_0 \in P$. Since P is a prime submodule and $a \notin [P : M]$, we must have $bm_0 \in P$. The latter implies that either $m_0 \in P$ or $b \in [P : M]$, and both possibilities contradict our assumptions. ■

In [1], Abdollah et al. investigated the relationship between the two graphs $\Gamma(M)$ and AG(M). We will need a few of those results, and restate them here for the record.

Proposition 2.3 Let M be an R-module. Then

a (Proposition 3 of [1]) $\Gamma(M)$ is a subgraph of AG(M).

b (Theorem 25 of [1]) If M is a multiplication module (or a reduced module or if $\mathrm{Nil}(M) = \{0_M\}$) Then
   i) A non-zero torsion element is an isolated vertex of $\Gamma(M)$ if and only if it is an isolated vertex of AG(M), and
   ii) AG(M) consists of a number (possibly zero) of isolated vertices and at most one connected component of diameter at most 2.

c (Corollary 22 of [1]) Assume that $\Gamma(M)$ has no isolated vertices. Then AG(M) is connected, and has diameter at most 2.

3. Necessary condition for $\Gamma(M)$ to be a star

Our first result already connects graph theoretic properties with ring theoretic properties. In Theorem 3.2, we show that, for a multiplication R-module M, if $\Gamma(M)$ has a vertex x adjacent to all other vertices (something that happens in a star), then $\mathrm{Ann}_R(x)M$ is a prime submodule of M, and $\mathrm{Ann}_R(x)$ is a prime ideal of R. This result, and the more detailed description of Theorem 3.6, for the more special case when $\Gamma(M)$ is a star, give partial generalizations, to multiplication modules, of the result of Anderson and Livingston [4, Theorem 2.5] that states that for a commutative ring R, $\Gamma(R)$ is a star if and only if either $R = \mathbb{Z}/2\mathbb{Z} \oplus D$ where D is an integral domain or the set of zero divisors of R is an annihilator ideal (and hence a prime ideal) of R. Our Theorems 3.2 and 3.6 also refine a result of Ghalandarzadeh and Malakooti Rad [11, Theorem 2.9]. They prove, for a multiplication R-module M, that $\Gamma(M)$ has a vertex x adjacent to all other vertices if and only if one of two possibilities occurs. Either $M = Rx \oplus \mathrm{Ann}_R(x)M$ is a faithful module, $|Rx| = 2$, $\mathrm{Ann}_R(x)M$ is finitely generated, and $T(M) = Rx \cup \mathrm{Ann}_R(x)M$, or $T(M) = \mathrm{Ann}_R(x)M$.

Lemma 3.1 Assume that M is a multiplication R-module, and that $\Gamma(M)$ has a vertex x adjacent to every other vertex. Further assume that $[Rx : M]x = \{0_M\}$, and $\alpha \in R$ with $\alpha x \neq 0_M$. Then $\mathrm{Ann}_R(\alpha x)M = \mathrm{Ann}_R(x)M$.

Proof. Clearly $\mathrm{Ann}_R(x)M \subseteq \mathrm{Ann}_R(\alpha x)M$. To show the reverse inclusion, let $y \in$

$\mathrm{Ann}_R(\alpha x)M$. To show that $y \in \mathrm{Ann}_R(x)M$, we can assume $y \neq 0_M$. Since $x \in T(M)^\star$, and $\alpha x \neq 0_M$, we have $\alpha x \in T(M)^\star$. By Lemma 2.1(f), $y = \alpha x$ or y is a neighbor of αx. In either case, $y \in T(M)^\star$. Since x is adjacent to every vertex, either $y = x$ or y is adjacent to x. In the former case, we are done by Lemma 2.1(e). In the latter case, $y \in \mathrm{Ann}_R(x)\backslash\{0_M, x\}$ by Lemma 2.1(f). ■

Theorem 3.2 Assume that M is a multiplication R-module, and that $\Gamma(M)$ has a vertex x adjacent to every other vertex. Then $T(M) = Rx \cup \mathrm{Ann}_R(x)M$, $\mathrm{Ann}_R(x)M$ is a prime submodule of M, and $\mathrm{Ann}_R(x)$ is a prime ideal of R.

Proof. By assumption, all elements of $T(M)\backslash\{0_M, x\}$ are adjacent to x in $\Gamma(M)$. Hence, by Lemma 2.1(f), $T(M) = Rx \cup \mathrm{Ann}_R(x)$. To prove that $\mathrm{Ann}_R(x)M$ is a prime submodule, first note that by Lemma 2.1(d), $\mathrm{Ann}_R(x)M$ is proper submodule of M. Now, let $\alpha \in R$ and $y \in M$ be arbitrary, and assume that $\alpha y \in \mathrm{Ann}_R(x)M$. By definition, $\mathrm{Ann}_R(x)M$ is a prime submodule of M, if we show that either $\alpha \in [\mathrm{Ann}_R(x)M : M]$ or $y \in \mathrm{Ann}_R(x)M$. If $\alpha x = 0_M$, then $\alpha \in \mathrm{Ann}_R(x) \subseteq [\mathrm{Ann}_R(x)M : M]$, and we would be done. Assuming $\alpha x \neq 0_M$, if $\alpha y = 0_M$, then either $y = 0_M \in \mathrm{Ann}_R(x)M$ or $y \in T(M)^\star\backslash\{x\}$ is adjacent to x. The latter would mean, by Lemma 2.1(f), that $y \in \mathrm{Ann}_R(x)M$ as desired. So wlog assume $\alpha x \neq 0$ and $\alpha y \neq 0_M$.

We claim that $[R\alpha y : M][Rx : M]M = \{0_M\}$. Since $\alpha y \in \mathrm{Ann}_R(x)M\backslash\{0_M\}$, either $\alpha y = x$ or, by Lemma 2.1(f), αy is adjacent to x in $\Gamma(M)$. In the latter case, the claim follows from the definition of adjacency in $\Gamma(M)$. In the former case, $x = \alpha y \in \mathrm{Ann}_R(x)M$ and so, by Lemma 2.1(e), $[Rx : M]x = \{0_M\}$. As a result, $[R\alpha y : M][Rx : M]M = [Rx : M]Rx = \{0_M\}$, and the claim is proved.

In a multiplication module, since $[R\alpha y : M]M = R\alpha y$, we have $\{0_M\} = [R\alpha y : M][Rx : M]M = \alpha[Rx : M]y = \alpha[Ry : M]x$. Hence, $[Ry : M] \subseteq \mathrm{Ann}_R(\alpha x)$. Now, applying Lemma 3.1, we have $Ry = [Ry : M]M \subseteq \mathrm{Ann}_R(\alpha x)M = \mathrm{Ann}_R(x)M$ completing the proof that $\mathrm{Ann}_R(x)M$ is a prime submodule of M.

To show that $\mathrm{Ann}_R(x)$ is a prime ideal of R, by Lemma 2.2, it is enough to show that $\mathrm{Ann}_R(x) = [\mathrm{Ann}_R(x)M : M]$. It is clear that $\mathrm{Ann}_R(x) \subseteq [\mathrm{Ann}_R(x)M : M]$. To show the converse, note that, by Lemma 2.1(e), $[Rx : M]\,\mathrm{Ann}_R(x)M = \{0_M\}$, and so, using Lemma 2.1(c),

$$[\mathrm{Ann}_R(x)M : M]Rx = [\mathrm{Ann}_R(x)M : M][Rx : M]M \subseteq \mathrm{Ann}_R(x)M[Rx : M] = \{0_M\}.$$

Hence, $[\mathrm{Ann}_R(x)M : M] \subseteq \mathrm{Ann}_R(x)$ as desired. ■

If the ring $\mathbb{Z}/16\mathbb{Z}$ is considered as a module over itself, then, in $\Gamma(M)$, the vertex 8 is adjacent to all other vertices, and the vertices 4, 8, and 12 form a triangle. (See Figure 1). If we require that $\Gamma(M)$ be a star (and so have no cycles), then we get more restrictions on the module M.

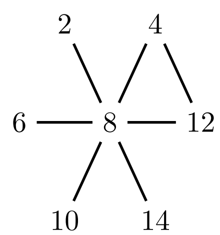

Figure 1. $\Gamma(M)$ for $M = R = \mathbb{Z}/16\mathbb{Z}$

Lemma 3.3 Assume M is a multiplication R-module, and $\Gamma(M)$ is a star with x as its central vertex. Then $|Rx| = 2$ or 3.

Proof. Since $x \neq 0_M$, $|Rx| > 1$. Now, if $|Rx| > 3$, then $Rx = \{0_M, x, \alpha x, \beta x\}$ for some $\alpha, \beta \in R$. Since x is the central vertex of a star, x is adjacent to αx and βx. But by Lemma 2.1(g), αx is also adjacent to βx, and we have a cycle contrary to assumption. ∎

If $\Gamma(M)$ is a star with x as its central vertex, Theorem 3.2 applies and we know that, for multiplication modules, $\mathrm{Ann}_R(x)$ is a prime ideal of R and $\mathrm{Ann}_R(x)M$ is a prime submodule of M. However, in the particular case of a star, because of Lemmas 3.3 and 2.1(d), we can give a more direct proof.

Lemma 3.4 Let M be an R-module, $x \in M$, and $p \in \mathbb{Z}$ an ordinary prime integer. Assume $|Rx| = p$. Then

a $Rx = \{0_M, x, 2x, \ldots, (p-1)x\}$ with $px = 0_M$.
b $\mathrm{Ann}_R(x)$ is a prime ideal.
c $\mathrm{Ann}_R(x)M$ is a prime submodule of M as long as it is a proper submodule.

Proof.

a Rx is a submodule of M and $(Rx, +)$ is an abelian group of order p. As a result, since p is a prime, the additive order of all non-zero elements of $(Rx, +)$ is p. So $px = 0_M$. Now, if m and n are non-negative integers, $m > n$, and $mx = nx$, then $(m-n)x = 0_M$. This implies that $p \mid m-n$. So, the set $\{0_M, x, 2x, \ldots, (p-1)x\}$ consists of p distinct elements of Rx and so we must have $Rx = \{0_M, x, 2x, \ldots, (p-1)x\}$ with $px = 0_M$.
b $\mathrm{Ann}_R(x)$ is a proper ideal since otherwise $Rx = \{0_M\}$. If $a, b \in R \backslash \mathrm{Ann}_R(x)$, then $ax \in Rx \backslash \{0_M\}$ and so $ax = mx$ for some integer m with $1 \leqslant m \leqslant p-1$. Likewise, $bx = nx$ for some integer n with $1 \leqslant n \leqslant p-1$. But then $(ab)x = a(bx) = a(nx) = mnx$. Since p does not divide mn, $(ab)x \neq 0_M$. We conclude that $\mathrm{Ann}_R(x)$ is a prime ideal of R.
c El Bast and Smith [9, Corollary 2.11] proves that a proper submodule N of an R-mdoule M is a prime submodule, if $N = PM$ for some prime ideal P of R with $\mathrm{Ann}_R(M) \subseteq P$. Our assertion follows by replacing P with $\mathrm{Ann}_R(x)$ and using the previous part.

∎

Lemma 3.5 Let M be a multiplication R-module. Assume $\Gamma(M)$ is a star with x as its central vertex, and with $|Rx| = 3$. Then $T(M) = Rx = \mathrm{Ann}_R(x)M = \{0_M, x, 2x\}$ is a submodule, and $\Gamma(M)$ has two vertices and a single edge.

Proof. If $|Rx| = 3$, then $Rx = \{0_M, x, 2x\}$ (by Lemma 3.4(a)). Since x is the central vertex and $2x \in T(M)^\star$, x is adjacent to $2x$. If $y \in T(M) \backslash Rx$, then x, as the central vertex, would be adjacent to y. By Lemma 2.1(g), $2x$ would also be adjacent to y, creating a triangle. The contradiction proves that $T(M) = Rx$, and that $\Gamma(M)$ is a single edge (with vertices x and $2x$). Now, by Lemma 2.1(f), $\mathrm{Ann}_R(x)M \backslash \{0_M, x\} = 2x$. Since $\mathrm{Ann}_R(x)M$ is a submodule, it must include $2(2x) = x$, and so $\mathrm{Ann}_R(x)M = \{0_M, x, 2x\}$. ∎

For N a submodule of an R-module M, we define $D(N)$, a submodule of N, by $D(N) = \{n \in N \mid \exists\, 0_M \neq n' \in N \text{ with } [Rn : M][Rn' : M]M = \{0_M\}\}$. Putting together what we have, we now state our main result on modules M for which $\Gamma(M)$ is star.

Theorem 3.6 Let M be a multiplication R-module. Assume $\Gamma(M)$ is a star with x as its central vertex. Then $T(M) = Rx \cup \mathrm{Ann}_R(x)M$, $\mathrm{Ann}_R(x)M$ is a prime submodule of M, $\mathrm{Ann}_R(x)$ is a prime ideal of R, and exactly one of the following must be true:

a $T(M) = Rx = \mathrm{Ann}_R(x)M = \{0_M, x, 2x\}$, and $\Gamma(M)$ has two vertices and a single edge.
b $Rx = \{0_M, x\}$, $x \in \mathrm{Ann}_R(x)M$, and $T(M) = \mathrm{Ann}_R(x)M$.
c $Rx = \{0_M, x\}$, $\mathrm{Ann}_R(x)M = \{0_M\}$, $M = T(M) = Rx$, and $\Gamma(M)$ is a single vertex.
d $Rx = \{0_M, x\}$, $x \notin \mathrm{Ann}_R(x)M$, $M = Rx \oplus \mathrm{Ann}_R(x)M$, $T(M)$ is not a submodule of M, and $D(\mathrm{Ann}_R(x)M) = \{0_M\}$.

Proof. We already proved in Theorem 3.2 that $T(M) = Rx \cup \mathrm{Ann}_R(x)M$, and $\mathrm{Ann}_R(x)M$ is a prime submodule. By Lemma 3.3, $|Rx| = 2$ or 3. In the latter case, by Lemma 3.5, we are exactly in the case described by option (a). So assume $Rx = \{0_M, x\}$. If $x \in \mathrm{Ann}_R(x)M$, then $T(M) = \mathrm{Ann}_R(x)M$, and we are in the case described by (b).

Hence, we can assume $Rx = \{0, x\}$, $x \notin \mathrm{Ann}_R(x)M$, and, by Lemma 2.1(c), $[Rx : M]x \neq \{0_M\}$. Let $\alpha \in [Rx : M]$ with $\alpha x \neq 0_M$. Since $Rx = \{0_M, x\}$, we have $\alpha x = x$ and so $1 - \alpha \in \mathrm{Ann}_R(x)$. Thus $1 \in \mathrm{Ann}_R(x) + [Rx : M]$, and $M \subseteq \mathrm{Ann}_R(x)M + \underbrace{[Rx : M]M}_{Rx}$.

Now since $x \notin \mathrm{Ann}_R(x)M$, $\mathrm{Ann}_R(x)M \cap Rx = \{0_M\}$, and $M = Rx \oplus \mathrm{Ann}_R(x)M$.

If $\mathrm{Ann}_R(x)M = \{0_M\}$, then $M = Rx = T(M)$ and we are in case (c). So it only remains to show that if $|Rx| = 2$, $\mathrm{Ann}_R(x)M \neq \{0_M\}$ and $x \notin \mathrm{Ann}_R(x)M$, then $T(M)$ is not a submodule of M and $D(\mathrm{Ann}_R(x)M) = \{0_M\}$, and hence we are in case (d).

Now $T(M) = Rx \cup \mathrm{Ann}_R(x)M$, and Rx and $\mathrm{Ann}_R(x)M$ are both additive subgroups of M. The union of two subgroups is a subgroup if and only if one is contained in the other. But this cannot happen if $x \notin \mathrm{Ann}_R(x)M$, and $\mathrm{Ann}_R(x)M \neq \{0_M\}$.

Finally, by way of contradiction, assume $0_M \neq n \in D(\mathrm{Ann}_R(x)M)$. Then, by definition, there exists a non-zero element $n' \in \mathrm{Ann}_R(x)M$ with $[Rn : M][Rn' : M]M = \{0_M\}$. Since $x \notin \mathrm{Ann}_R(x)M$, by Lemma 2.1(f), non-zero elements of $\mathrm{Ann}_R(x)M$ are vertices of $\Gamma(M)$ adjacent to x. Therefore both n and n' are adjacent to x in $\Gamma(M)$. But since, by Lemma 2.1(c), $[Rn' : M]M = Rn'$, we have $[Rn : M]n' = [Rn : M][Rn' : M]M = \{0_M\}$. We conclude that $n = n'$, since otherwise, by Lemma 2.1(e) and 2.1(f), n and n' would be adjacent in $\Gamma(M)$, and $x - n - n' - x$ would be a triangle. Thus $n \in \mathrm{Ann}_R(x)M \backslash \{0_M, x\} \subseteq T(M)^{\star}$, $[Rn : M]n = \{0_M\}$, and, since x and n are adjacent, $[Rn : M]x = \{0_M\}$. But this means that $[Rn : M](x + n) = [Rn : M]x + [Rn : M]n = \{0_M\}$, and so, by Lemma 2.1(e), $x + n \in \mathrm{Ann}_R(x)M$. But $\mathrm{Ann}_R(x)M$ is a submodule, and if both n and $x + n$ are in this submodule, then so is x, which is a contradiction. The proof is now complete. ■

***Example* 3.7** Four examples show that each of the cases of Theorem 3.6 are possible. Also, see Figure 2.

Let $M = R = \mathbb{Z}/9\mathbb{Z}$, and $x = 3$. Then $T(M) = Rx = \{0, 3, 6\}$, and $\Gamma(M)$ is a single edge.

Let $M = R = \mathbb{Z}/8\mathbb{Z}$, and $x = 4$. Then $Rx = \{0, 4\}$, $T(M) = \mathrm{Ann}_R(x)M = \{0, 2, 4, 6\}$, and $\Gamma(M)$ is a path of length 2.

Let $R = \mathbb{Z}$, $M = \mathbb{Z}/2\mathbb{Z}$, and $x = 1$. Then $Rx = \{0, 1\} = M = T(M)$, $\mathrm{Ann}_R(x)M = \{0_M\}$, and $\Gamma(M)$ is a single vertex.

If $M = R = \mathbb{Z}/2\mathbb{Z} \oplus D$ where D is a non-trivial integral domain (finite or infinite), and $x = (1, 0)$, then $M = Rx \oplus \mathrm{Ann}_R(x)M$, and $\Gamma(M)$ is a star with x as its central vertex and all elements of the form $(0, y)$ with $0 \neq y \in D$ as vertices of degree 1.

Remark 1 *In Theorem 3.6, note that $T(M)$ is not a submodule of M only for case* (d). *Also, by Lemma 2.1*(e), $[Rx : M]x = \{0_M\}$ *only for cases* (a) *and* (b).

6 4

\ /

3 — 6 2 — 4 — 6 8 — 5 — 2

Figure 2. $\Gamma(M)$ for $M = R = \mathbb{Z}/9\mathbb{Z}$ (left), $M = R = \mathbb{Z}/8\mathbb{Z}$ (middle), and $M = R = \mathbb{Z}/10\mathbb{Z} \cong \mathbb{Z}/2\mathbb{Z} \oplus \mathbb{Z}/5\mathbb{Z}$ (right).

An R-module M is called reduced (Lee and Zhou [12]) if, for all $\alpha \in R$ and $x \in M$, we have $Rx \cap \alpha M = \{0_M\}$ whenever $\alpha x = 0$.

Proposition 3.8 Let M be a reduced multiplication R-module, and assume $\Gamma(M)$ is a star with central vertex x. Then $[Rx : M]x \neq \{0_M\}$, and only cases (c) and (d) of Theorem 3.6 are possible.

Proof. Assume $[Rx : M]x = \{0_M\}$. By the definition of a reduced module, $Rx \cap [Rx : M]M = \{0_M\}$. But since M is a multiplication module $[Rx : M]M = Rx$ and $Rx \cap Rx$ is not $\{0_M\}$. The contradiction proves that $[Rx : M]x \neq \{0_M\}$, and the rest follows from Remark *1*. ■

4. Sufficient conditions for $\Gamma(M)$ to be a star

If R is a commutative ring with identity, and M is a faithful R-module, then Ghalandarzadeh and Malekooti Rad [11, Theorem 2.6] showed that the torsion graph $\Gamma(M)$ is connected and its diameter is at most 3. Let $R = \mathbb{Z}/2\mathbb{Z} \oplus \mathbb{Z}/4\mathbb{Z}$, and let $M = R$ considered as an R-module. Then M is a faithful multiplication module, and $\Gamma(M)$ has no cycles with diameter equal to 3 (See Figure 3). So, in this case, $\Gamma(M)$ is a tree and yet not a star. In this section, we characterize faithful multiplication modules M for which $\Gamma(M)$ has no cycles, and yet is not a star. As an aside, we note that Abdollah et al. [1, Theorem 28(a)] showed that if a torsion graph (for any module—not necessarily a multiplication module or faithful—over a commutative ring with identity) has a cycle, then its girth is either 3 or 4.

(0, 3)

|

(0, 1) — (1, 0) — (0, 2) — (1, 2)

Figure 3. Let $R = \mathbb{Z}/2\mathbb{Z} \oplus \mathbb{Z}/4\mathbb{Z}$, and consider $M = R$ as an R-module. The torsion graph $\Gamma(M)$ is a tree but not a star.

Our main theorem of this section shows that the example of the multiplication module $R = \mathbb{Z}/2\mathbb{Z} \oplus \mathbb{Z}/4\mathbb{Z}$ (as an R-module) of a torsion graph that is a tree but not a star (see Figure 3) is quite unusual.

Lemma 4.1 Let M be a multiplication R-module, and assume that $\Gamma(M)$ contains a path $a - x - b$ of length 2 and no cycles. Then $\{0_M, x\} = \mathrm{Ann}_R(b)M \cap \mathrm{Ann}_R(a)M$ is a submodule of M.

Proof. Since x is assumed to be distinct from and adjacent to both a and b in $\Gamma(M)$, by Lemma 2.1f, we have $x \in \mathrm{Ann}_R(a)M \cap \mathrm{Ann}_R(b)M$. Conversely, let $z \in \mathrm{Ann}_R(a)M \cap \mathrm{Ann}_R(b)M$, and, by way of contradiction assume $z \notin \{0_M, x\}$. Again by Lemma 2.1f, either $z = a$ or z is a vertex of $\Gamma(M)$ adjacent to a. Likewise, either $z = b$ or $z \in T(M)^\star$ is adjacent to b. Hence, the vertex z is either the same as one of a or b (and adjacent to the other one), or distinct from both. In the former case, $a - x - b$ is a triangle, and in the latter case, $a - x - b - z - a$ is a four cycle. Both cases contradict the assumption that $\Gamma(M)$ has no cycles, completing the proof. ■

In the case of a commutative ring R, DeMeyer and Schneider [8, Theorem 1.6] showed that if $\Gamma(R)$ is not the empty graph, has no cycles, and yet is not a star, then $R \cong \mathbb{Z}/2\mathbb{Z} \oplus \mathbb{Z}/4\mathbb{Z}$ or $R \cong \mathbb{Z}/2\mathbb{Z} \oplus \mathbb{Z}/2\mathbb{Z}[t]/\langle t^2 \rangle$. This section's main theorem, Theorem 4.2, is a partial generalization to the more general case of multiplication modules over commutative rings. Recall that for N a submodule of an R-module M, we have defined a submodule of N, denoted $D(N)$, by $D(N) = \{n \in N \mid \exists\ 0_M \neq n' \in N \text{ with } [Rn : M][Rn' : M]M = \{0_M\}\}$.

Theorem 4.2 Let M be a multiplication R-module. Assume $\Gamma(M)$ has no isolated vertices, and no cycles, and yet has a path of length 3. Then there exists $x \in T(M)^\star$ such that $Rx = \{0_M, x\}$, and $M = Rx \oplus \operatorname{Ann}_R(x)M$. Furthermore, $\operatorname{Ann}_R(x)M \backslash \{0_M\} \subseteq T(M)^\star$, the subgraph of $\Gamma(M)$ induced by these vertices has no edges, and yet $|D(\operatorname{Ann}_R(x)M)| = 2$.

Proof. By hypothesis, we have a path $a - x - z - b$ in $\Gamma(M)$ of length 3. By Lemma 2.1(e) and 2.1(f), $[Ra : M]x = [Rx : M]a = [Rz : M]b = [Rb : M]z = \{0_M\}$, and, by Lemma 4.1, both $\{0_M, x\}$ and $\{0_M, z\}$ are submodules of M.
Claim: It is not possible for both $[Rx : M]x$ and $[Rz : M]z$ to be equal to $\{0_M\}$.
Proof of Claim: By way of contradiction, assume $[Rx : M]x = [Rz : M]z = \{0_M\}$. Consider the element $x + z$. Since x and z are non-zero, $x + z$ is distinct from x and z. If $x + z = 0_M$, then $z = -x \in Rx$, and, in $\Gamma(M)$, by Lemma 2.1f, z is adjacent to all vertices that x is adjacent to. As a result, $a - x - z - a$ would be a cycle of length 3 contradicting one of the assumptions. Hence, $x + z \neq 0_M$. Since x and z are adjacent in $\Gamma(M)$, by Lemma 2.1(e) and 2.1(f), $[Rx : M]z = \{0_M\}$, and we are assuming $[Rx : M]x = \{0_M\}$. So $0 \neq [Rx : M] \subseteq \operatorname{Ann}_R(x + z)$ since $[Rx : M](x + z) = [Rx : M]x + [Rx : M]z = \{0_M\}$. Hence, $x + z$ is a vertex in $\Gamma(M)$ adjacent to x. Likewise, $x + z$ is adjacent to z. This means that $x - (x + z) - z - x$ is a cycle of length 3 which contradicts our hypothesis. The contradiction completes the proof of the claim.

Because of the claim, and without loss of generality, assume that $[Rx : M]x \neq \{0_M\}$—in fact, we will prove below that, given this assumption, $[Rz : M]z$ will have to be equal to $\{0_M\}$. Now, let $\alpha \in [Rx : M]$ with $\alpha x \neq 0_M$. By Lemma 4.1, $Rx = \{0_M, x\}$, and so $\alpha x = x$. In addition, $\alpha \neq 1$, since otherwise $M = Rx = \{0_M, x\}$ will not have enough elements for a path of length 3 in $\Gamma(M)$. From $\alpha x = x$, we get that $1 - \alpha \in \operatorname{Ann}_R(x)$. Thus $1 \in \operatorname{Ann}_R(x) + [Rx : M]$, and as a result, $M \subseteq \operatorname{Ann}_R(x)M + \underbrace{[Rx : M]M}_{Rx} \subseteq M$.

Hence, $M = Rx + \operatorname{Ann}_R(x)M$.

Since $Rx = \{0_M, x\}$, to show that $Rx \cap \operatorname{Ann}_R(x)M = \{0_M\}$, we need to show that x is not an element of $\operatorname{Ann}_R(x)M$. If it were, and recalling that $x = \alpha x$ with $\alpha \in [Rx : M]$, we would have $x = \alpha x \in [Rx : M] \operatorname{Ann}_R(x)M \subseteq \operatorname{Ann}_R(x)Rx = \{0_M\}$, a contradiction. Thus, $M = Rx \oplus \operatorname{Ann}_R(x)M$.

By Lemma 2.1f, every non-zero element of $\operatorname{Ann}_R(x)M$ is a vertex of $\Gamma(M)$ and adjacent to x. There cannot be two distinct elements in $\operatorname{Ann}_R(x)M$ that are adjacent in $\Gamma(M)$ since otherwise those two elements and x would make a cycle of length 3 contrary to assumption. We conclude that the subgraph of $\Gamma(M)$ induced by the vertices $T(M)^\star \cap \operatorname{Ann}_R(x)M$ has no edges.

It remains to show that, even though the graph induced by the vertices $T(M)^\star \cap \operatorname{Ann}_R(x)M$ has no edges, $|D(\operatorname{Ann}_R(x)M)| = 2$. By assumption, $a - x - z - b$ is a path of length 3 in $\Gamma(M)$. By Lemma 2.1f, a and z are both elements of $T(M)^\star \cap \operatorname{Ann}_R(x)M$.

One consequence is that $0_M \in D(\mathrm{Ann}_R(x)M)$ since $[\{0_M\} : M][Rz : M]M = \{0_M\}$.

Let $0_M \neq y \in \mathrm{Ann}_R(x)M$. Then, since no two non-zero torsion elements of $\mathrm{Ann}_R(x)M$ are adjacent in $\Gamma(M)$, $y \in D(\mathrm{Ann}_R(x)M)$ if and only if $[Ry : M]y = [Ry : M][Ry : M]M = \{0_M\}$.

We claim that z is the unique non-zero element of $D(\mathrm{Ann}_R(x)M)$. Vertex b (from the path $a-x-z-b$) is adjacent to z, and is not equal to x. As a result, $b \notin Rx \cup \mathrm{Ann}_R(x)M$, and, since $M = Rx \oplus \mathrm{Ann}_R(x)M$, we have $b = x + y$ for some $y \in \mathrm{Ann}_R(x)M$. Invoking Lemma 2.1f, $\{0_M\} = [Rz : M]b = \underbrace{[Rz : M]x}_{\{0_M\}} + [Rz : M]y = [Rz : M]y$. This implies that either $y = z$ or y and z are adjacent in $\Gamma(M)$. However, both y and z are elements of $\mathrm{Ann}_R(x)M$ and no two elements of $\mathrm{Ann}_R(x)M$ can be adjacent. We conclude that $y = z$, $b = x + z$, and $[Rz : M]z = \{0_M\}$. The latter means that $z \in D(\mathrm{Ann}_R(x)M)$. To complete the proof that $D(\mathrm{Ann}_R(x)M) = \{0_M, z\}$, assume y is yet another element of $D(\mathrm{Ann}_R(x)M)$. This means that $[Ry : M]y = \{0_M\}$. Since y and z are not adjacent vertices, we have $[Ry : M]z \neq \{0_M\}$, and so there exists $\beta \in [Ry : M]$ with $\beta z \neq 0_M$. By definition of β, we have $\beta z \in Ry$, and so $\beta[Ry : M]z \subseteq R[Ry : M]y = \{0_M\}$. Since βz and y are not adjacent, this means that $y = \beta z$, but that would imply that $[Rz : M]y = [Rz : M]\beta z = \{0_M\}$ contradicting the fact that y and z are not adjacent. ■

***Example* 4.3** Let $R = \mathbb{Z}/2\mathbb{Z} \oplus \mathbb{Z}/4\mathbb{Z}$, and let $M = R$ considered as an R-module. Then M is a faithful multiplication module, and $\Gamma(M)$ has no isolated vertices, no cycles, and yet has a path of length 3 (See Figure 3). As a result,Theorem 4.2 and its proof apply. Since $(0,1) - (1,0) - (0,2) - (1,2)$ is the only path of length 3, the candidates for x and z (from the proof of Theorem 4.2) are $(1,0)$ and $(0,2)$. Indeed, $x = (1,0)$ and $[Rx : M]x = \{(0,0),(1,0)\}$, while $z = (0,2)$ and $[Rz : M]z = \{(0,0)\}$. In this example, $\mathrm{Ann}_R(x)M = \{(0,0),(0,1),(0,2),(0,3)\}$, $M = Rx \oplus \mathrm{Ann}_R(x)M$, there are no edges among the nonzero elements of $\mathrm{Ann}_R(x)M$, and $D(\mathrm{Ann}_R(x)M) = \{(0,0), z\}$, as predicted by the Theorem.

As pointed out earlier in the case of faithful multiplication modules, Ghalandarzadeh and Malakooti Rad [11, Theorem 2.6] showed that the torsion graph $\Gamma(M)$ is connected. Therefore in this case, Theorem 4.2 can be restated to say that if $\Gamma(M)$ has no cycles, then it is either a star or $M \cong M_1 \oplus M_2$ with $|M_1| = |D(M_2)| = 2$.

5. Stars and the Annihilator graph AG(M)

We now turn to the annihilator graph AG(M). Recall that $T(M)^\star$—the set of non-zero torsion elements of the R-module M—continues to be the set of vertices, and, by Lemma 2.1(h), in the case of multiplication modules, two vertices x and y are adjacent in AG(M) if and only if

$$\mathrm{Ann}_R([Rx : M]y) \neq \mathrm{Ann}_R(x) \cup \mathrm{Ann}_R(y).$$

Consider $\mathbb{Z}/8\mathbb{Z}$, the integers modulo 8, as a modulo over itself. Then this is a multiplication module, where $\Gamma(M)$ is a star (see Example 3.7 and Figure 4) while AG(M) is a triangle and not equal to $\Gamma(M)$. As the next Proposition shows, for multiplication modules—and this includes the case of any ring considered as a module over itself—this is an anomaly, and, most often, if one of the graphs is a star, then the two graphs are the same.

Figure 4. If $M = \mathbb{Z}/8\mathbb{Z}$ is considered as a module over itself, then $\Gamma(M)$, on the left, is a star, while $\mathrm{AG}(M)$, on the right, is a triangle.

Theorem 5.1 Let M be a multiplication R-module.

a If $\mathrm{AG}(M)$ is a star, then $\Gamma(M) = \mathrm{AG}(M)$ is a star as well. In particular, the conclusions of Theorem 3.6 remain valid.

b If $\Gamma(M)$ is a star, then, except for Case (b) of Theorem 3.6, $\mathrm{AG}(M) = \Gamma(M)$ is a star as well.

Proof.

a By Proposition 2.3(a), $\Gamma(M)$ is a subgraph of $\mathrm{AG}(M)$, and, for multiplication modules, by Proposition 2.3(b) a vertex is an isolated vertex of one if and only if it is an isolated vertex of the other.

b If $\Gamma(M)$ is a star, then Theorem 3.6 applies, and M is in one of the four cases of that theorem. Moreover, by Proposition 2.3(a), $\Gamma(M)$ is a subgraph of $\mathrm{AG}(M)$, and so we just have to show that $\mathrm{AG}(M)$ does not have any extra edges. In Cases (a) and (c), $\Gamma(M)$ is the complete graph on respectively 2 and 1 vertices, and hence $\mathrm{AG}(M) = \Gamma(M)$ is a star as well. It remains to show that in Case (d), other than the edges from the central vertex x to all other vertices, there are no other adjacencies in $\mathrm{AG}(M)$.

Hence, we can assume that M is a multiplication module, $\Gamma(M)$ is a star with $x \in M\setminus \mathrm{Ann}_R(x)M$ as its central vertex, $Rx = \{0, x\}$, $T(M) = Rx \cup \mathrm{Ann}_R(x)M$, and $M = Rx \oplus \mathrm{Ann}_R(x)M$. Let y and z be non-zero elements of $\mathrm{Ann}_R(x)M$. The proof will be complete when we show that y and z, which are not adjacent in $\Gamma(M)$, are also not adjacent in $\mathrm{AG}(M)$. By way of contradiction, assume they are. By Lemma 2.1(h), $\mathrm{Ann}_R(y) \cup \mathrm{Ann}_R(z)$ is a proper subset of $\mathrm{Ann}_R([Ry : M]z)$. Let $\alpha \in \mathrm{Ann}_R([Ry : M]z)\setminus \mathrm{Ann}_R(y)\cup\mathrm{Ann}_R(z)$. Hence, $\alpha[Ry : M]z = \{0_M\}$, and, by Lemma 2.1(e), $[Ry : M]x = \{0_M\}$. Note that since y and z are not adjacent in $\Gamma(M)$, $[Ry : M] \neq \{0_R\}$ (Lemma 2.1(e) and 2.1(f)), and $[Ry : M](x + \alpha z) = [Ry : M]x + \alpha[Ry : M]z = \{0_M\}$. Hence, $x + \alpha z \in T(M)$, and, if $x + \alpha z \neq 0_M$, then, in $\Gamma(M)$, y is adjacent to $x + \alpha z$. But in $\Gamma(M)$, y is adjacent only to x. However, since $\alpha \notin \mathrm{Ann}_R(z)$, $x + \alpha z \neq x$. We conclude that $x + \alpha z = 0_M$. But this means that $x = -\alpha z \in \mathrm{Ann}_R(x)M$ contradicting one of the assumptions.

■

Corollary 5.2 Let M be a multiplication R-module, and assume $\Gamma(M)$ is a star. If M is a reduced R-module, or alternatively, $T(M)$ is not a submodule of M, then $\mathrm{AG}(M) = \Gamma(M)$ is a star as well.

Proof. Follows immediately from Remark *1*, Proposition 3.8, and Theorem 5.1. ■

We note that in the special case when a commutative ring R is considered as a module over itself, then Badawi [5, Theorem 3.17] has characterized the rings where $\mathrm{AG}(R) \neq \Gamma(R)$ and yet $\Gamma(R)$ is a star. In such a case, $\Gamma(R)$ must be a path of length 2, and $\mathrm{AG}(R)$ a triangle. In addition, Badawi [5, Theorem 3.18] gives various characterizations of non-reduced rings R with at least two non-zero zero divisors where $\mathrm{AG}(R)$ is a star.

In Section 4, we saw that, while rare, it is possible for $\Gamma(M)$ to be a tree without being a star. A straightforward consequence of our results in Abdollah et al. [1] for AG(M) shows that, even without assuming that M is a multiplication module, this does not happen for AG(M).

Proposition 5.3 Let M be an R-module. If AG(M) has no isolated vertices and no cycles, then AG(M) is a star graph.

Proof. By Proposition 2.3(c), if AG(M) has no isolated vertices, then AG(M) is connected and has diameter at most 2. If the diameter is 1, then the graph must be complete, but since we are assuming no cycles, then AG(M) has two vertices and a single edge and is a star graph. If the diameter is 2, then the graph has a path $y - x - z$ of length 2. Since the graph has no cycles, all the other vertices must be adjacent to x. Hence, AG(M) is a star with x as its central vertex. ■

References

[1] Z. Abdollah, P. Malakooti, Sh. Ghalandarzadeh, Sh. Shahriari, On torsion elements and their annihilators, J. Algebra. 610 (2022), 199-222.
[2] D. F. Anderson, T. Asir, A. Badawi, T. Tamizh Chelvam, Graphs from Rings, Springer, Cham, 2021.
[3] D. F. Anderson, M. C. Axtell, J. A. Stickles, Zero-divisor graphs in commutative rings, Commutative Algebra-Noetherian and Non-Noetherian Perspectives, Springer, New York, 2011.
[4] D. F. Anderson, Ph. S. Livingston, The zero-divisor graph of a commutative ring, J. Algebra. 217 (2) (1999), 434-447.
[5] A. Badawi, On the annihilator graph of a commutative ring, Comm. Algebra. 42 (1) (2014), 108-121.
[6] A. Barnard, Multiplication modules, J. Algebra. 71 (1) (1981), 174-178.
[7] I. Beck, Coloring of commutative rings, J. Algebra. 116 (1) (1988), 208-226.
[8] F. DeMeyer, K, Schneider, Automorphisms and Zero Divisor Graphs of Commutative Rings, Commutative Rings, Nova Sci. Publ. NY, 2002.
[9] Z. Abd El-Bast, P. F. Smith, Multiplication modules, Comm. Algebra. 16 (4) (1988), 755-779.
[10] Sh. Ghalandarzadeh, P. Malakooti Rad, Torsion graph over multiplication modules, Extracta Math. 24 (3) (2009), 281-299.
[11] Sh. Ghalandarzadeh, P. Malakooti Rad, Torsion graph of modules, Extracta Math. 26 (1) (2011), 153-163.
[12] T. K. Lee, Y. Zhou, Reduced modules, rings, modules, algebras, and abelian groups, Lecture Notes in Pure and Appl. Math. 236 (2004), 365-377.
[13] Ch. P. Lu, Prime submodules of modules, Rikkyo Daigaku Sugaku Zasshi. 33 (1) (1984), 61-69.
[14] F. Mehdi, On multiplication modules, Math. Student. 42 (1974), 149-153.