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Research Article

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Fuzzy Predictor-Corrector Methods for Solving Fuzzy Impulsive Differential Equation

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Abstract

In this paper, we use fuzzy Adams-Bashforth method as well as fuzzy Adams-Moulton method both based on gH-differences to solve fuzzy impulsive differential equations with an initial value. We discuss the algorithm in details and finally, we solve a fuzzy impulsive differential equation with these methods.

The numerical results are shown in the table.

Keywords: Generalized Hukuhara difference; impulsive fuzzy differential equations.

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1. Introduction

Impulsive differential equations appear to represent a natural framework for mathematical modeling of several real-world phenomena. Impulsive differential equations are useful tools for modeling of evolution processes that are subjected to sudden changes in their state. But solving many impulsive differential equations analytically is very complicated or even impossible. Furthermore, to solve some practical prob- lems, we do not often need the analytic solution of impulsive differential equation, but just the numerical values of the exact solution. In addition, whenever a case of real-world phenomena is transformed into the deterministic differential equations with an initial value, we cannot usually be sure that this modeling is perfect. A wide number of works in this topic is done by Allahviranloo in the past two decades (see, e.g., [1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11])

Thus, we here consider an impulsive fuzzy differential equation, and then present a new numerical algorithm to solve this equation. So, consider the first order fuzzy impulsive differential equations

$$\begin{cases} u'(x) = f(x, u(x)), & x \in J = [0, T], x \neq x_k, k = 1, \dots, m, \quad (1.1) \\ u(x_k^+) = I_k(u(x_k^-)), & k = 1, \dots, m \quad (1.2) \\ u(x_0) = y_0 & \quad (1.3) \end{cases}$$

2. Basic Preliminaries

The definitions, lemmas, and theorems needed in this paper are given here.

The set of fuzzy numbers, denoted by $\mathrm{R_f}$, is the family of all normal, fuzzy convex, upper semi-continuous and compactly supported fuzzy sets.

Now we recall some properties of fuzzy numbers. For any $0 \le \alpha \le 1$, set

$$[\mathrm{p}]_\alpha = \{\mathrm{x} \in \mathrm{R} | \mathrm{p(x)} \geq \alpha\}, \qquad [\mathrm{p}]_{.0} = \mathrm{cl}\{\mathrm{x} \in \mathrm{R} | \mathrm{p(x)} > 0\}.$$

We represent $[p]_\alpha = [\underline{p}(\alpha), \bar{p}(\alpha)]$. Thus, for $p \in R_f$, the α-level set $[p]_\alpha$is a closed interval for all $\alpha \in [0,1]$.

Definition 2.1. For any $p, q \in R_f$ and $\lambda \in R$, we have $[p \oplus q]_\alpha = [p]_\alpha + [q]_\alpha = \{x + y | x \in [p]_\alpha, y \in [q]_\alpha\}$,

$$[\mathrm{p} \odot \mathrm{q}]_\alpha = [\min(\underline{\mathrm{pq}}, \underline{\mathrm{p}}\bar{\mathrm{q}}, \underline{\mathrm{p}}\bar{\mathrm{q}}, \overline{\mathrm{pq}}), \max(\underline{\mathrm{pq}}, \underline{\mathrm{p}}\bar{\mathrm{q}}, \underline{\mathrm{p}}\bar{\mathrm{q}}, \overline{\mathrm{pq}})].$$

Let T = [a,b] and S = [c,d] be two closed intervals of real numbers. Then

$$[\mathrm{a}, \mathrm{b}]/[\mathrm{c}, \mathrm{d}] = \left[\min(\frac{\mathrm{a}}{\mathrm{c}}, \frac{\mathrm{a}}{\mathrm{d}}, \frac{\mathrm{b}}{\mathrm{c}}, \frac{\mathrm{b}}{\mathrm{d}}), \max(\frac{\mathrm{a}}{\mathrm{c}}, \frac{\mathrm{a}}{\mathrm{d}}, \frac{\mathrm{b}}{\mathrm{c}}, \frac{\mathrm{b}}{\mathrm{d}})\right]$$

provided that $0 \notin [a, b]$.

A triangular fuzzy number is defined as a fuzzy set in Rf , that is specified by an ordered triple $u = (a, b, c) \in \mathbb{R}^3$with $a \le b \le c$ such that $u^-(\alpha) = a + (b - a)\alpha$ and $u^+(\alpha) = c - (c - b)\alpha$ are the endpoints of α-level sets for all $\alpha \in [0,1]$.

For any $p, q, r \in R_f$ Hukuhara di_erence is showed by $\ominus_H$ and it means that r$\ominus_H$ q = p if and only if $p \oplus q = r$. The Hausdor_ distance of the set of fuzzy numbers is defined by $D: R_f \times R_f \to R^+ \cup \{0\}$ as in [12],

$$D(p,q) = sup_{0\le\alpha\le1}\, max\{\,|\underline{p}(\alpha) - \underline{q}(\alpha)|, |\bar{p}(\alpha) - \bar{q}(\alpha)|\}[6,12,13].$$

Consider p,q,r,s $\in R_f$ and $\lambda \in R$, then the following properties are well-known for metric D. See [14]

1) $D(p \oplus r, q \oplus r) = D(p,q)$.
2) $D(\lambda_p, \lambda_q) = |\lambda| D(p,q)$,
3) $D(p \oplus q, r \oplus s) \le D(p,r) + D(q,s)$,

Definition 2.2 ([15]). The gH-difference of two fuzzy numbers p and q is defined as follows

$$p\ominus_{gH}\ q = r \Leftrightarrow \begin{cases}(1)p = q \oplus r,\\ or(2)q = p \oplus (-1)r\end{cases} \quad (2.4)$$

and $[p\Theta_g H\ q]_\alpha = [min\{\underline{p}(\alpha) - \underline{q}(\alpha) - \bar{q}(\alpha)\}, max\{\underline{p}(\alpha) - \underline{q}(\alpha), \bar{p}(\alpha) - \bar{q}(\alpha)\}]$.

The conditions for the existence of $p\Theta_{gH}\ q \in R_f$ are given in [15].

Definition 2.3 ([15]). Let $p, q \in R_f$, then:

1. If the gH-difference of p and q exists, it is unique.
2. $p\Theta_{gH}\ q = p\Theta_H\ q\ or\ p\Theta_{gH}\ q = -(q\Theta_H\ p)$ if all the expressions on the right side exist. In particular, $p\Theta_{gH}p = p\Theta_H\ p = 0$,
3. if $p\Theta_{gH}q$exists in the sense of (i), then $q\Theta_{gH}p$ exists in the sense if (ii) and vice versa.
4. $(p \oplus q)\Theta_{gH}q = p$,
5. $0\Theta_{gH}(p\Theta_{gH}q) = q\Theta_{gH}p$,

$p\Theta_{gH}q = q\Theta_{gH}p = r$if and only if $r = -r$. Furthermore, r = 0 if and only if p = q.

Remark 2.1. In this study, we assume that p $\ominus_{gH}\ q \in R_f$.

In [16], Park and Han formulated the Lipschitz condition as an inequality and as an informal definition and proved the existence and uniqueness of a solution for an FIVP in the n-dimensional space $\boldsymbol{R_f^n}$. Here, we rewrite the Lipschitz condition in a formal definition and present a special case of the existence and uniqueness theorem for n = 1 which are needed in this paper.

Definition 2.4 ([16]). Assume that $f: I \times R_f \to R_f$ is a fuzzy number-valued function. It is said that f satisfies the Lipschitz condition if there exists a constant k > 0 such that

$$D(f(t,y), f(t,z)) \le kD(y,z)$$

for all $t \in I$ and all $y, z \in R_f$.

Definition 2.5 ([17]). Let $f: [t,s] \to R_f$ be a fuzzy number-valued function. The gH-derivative of at $a \in (t,s)$ is de_ned by

$$f'_{gH}(a) = \lim_{h\to 0} \frac{f(a+h)\Theta_{gH} f(a)}{h} \tag{2.5}$$

If $f'_{gH}(a) \in R_f$ defined by (2.5) exists, it is called that f is gH-differentiable at a.

Definition 2.6 ([18]). Let $f: [t, s] \to R_f$ and $a \in (t, s)$ be such that $\underline{f}(x; \alpha)$ and $\bar{f}(x; a)$ are both differentiable at a for all $\alpha \in [0,1]$. Then it is said that f is differentiable of first type and denoted by

(1)-gH-differentiable at a if

$$f'_{1-gH}(a, \alpha) = [\underline{f'}(a; \alpha), \overline{f'}(a; \alpha)], \tag{2.6}$$

and f is differentiable of second type and denoted by (2)-gH-differentiable at a if

$$f'_{2-gH}(a, \alpha) = [\overline{f'}(a; \alpha), \underline{f'}(a; \alpha)], \tag{2.7}$$

Definition 2.7 ([19]). A point $a \in (t, s)$, is a switching point for differentiability of the fuzzy number- valued function f, if for any neighborhood V of a there exist points $t_1; t_2 \in V$ with $t_1 < a < t_2$ such that for i = 1; 2, (2.6) holds at t_i if and only if (2.7) does not hold at t_i.

Theorem 2.1 ([6]). Let $T = [a, b, \beta] \subset \mathbb{R}$, with $\beta > 0$ and $f \in C^n_{gH}([a, b], R_f)$. For $s \in T$

1. if $f^{(i)}, i = 0,1, \dots, n-1$ are (1)-gH-differentiable, provided that type of gH-differentiability has no change, then

$$f(s) = f(a) \oplus f'_{1-gH}(a) \odot (s-a) \oplus f''_{1-gH}(a) \odot \frac{(s-a)^2}{2!} \oplus \dots$$

$$\oplus f^{(n-1)}_{1-gH}(a) \odot \frac{(s-a)^{n-1}}{(n-1)!} \oplus R_n(a, s),$$

Where

$$R_n(a, s) = \int_a^s \left(\int_a^{s_1} \dots \left(\int_a^{s_{n-1}} f^{(n)}_{1-gH}(s_n) ds_n \right) d_{s_{n-1}} \dots \right) ds_1$$

2. if $f^{(i)}, i = 0,1, \dots, n-1$ are (2)-gH-differentiable, provided that type of gH-differentiability has no change, then

$$f(s) = f(a) \Theta (-1) f'_{2-gH}(a) \odot (s-a) \Theta (-1) f''_{2-gH}(a) \odot \frac{(a-s)^2}{2!} \Theta (-1) \dots$$

$$\Theta f^{(n-1)}_{2-gH}(a) \odot \frac{(a-s)^{n-1}}{(n-1)!} \Theta (-1) R_n(a, s),$$

Where

$$R_n(a, s) = \int_a^s \left(\int_a^{s_1} \dots \left(\int_a^{s_{n-1}} f^{(n)}_{2-gH}(s_n) ds_n \right) d_{s_{n-1}} \dots \right) ds_1$$

3. if $f^{(i)}$ is (1)-gH-differentiable for $i = 2k-1, k \in \mathbb{N}$, and $f^{(i)}$ are (2)-gH-differentiable for $i = 2k, k \in \mathbb{N} \cup \{0\}, then$

$$f(s) = f(a)\Theta(-1)f'_{2-gH}(a) \odot (s-a) \oplus f''_{1-gH}(a) \odot \frac{(a-s)^2}{2!}\Theta(-1)\ldots$$

$$\Theta(-1)f_{2-gH}^{(\frac{i-1}{2})}(a) \odot \frac{(a-s)^{\frac{i}{2}-1}}{(\frac{i}{2}-1)!} \oplus f_{1-gH}^{(\frac{i}{2})}(a) \odot \frac{(a-s)^{\frac{i}{2}}}{(\frac{i}{2})!}\Theta(-1)\ldots$$

$$\Theta(-1)R_n(a,s),$$

Where

$$R_n(a,s) := \int_a^s \left(\int_a^{s_1} \cdots \left(\int_a^{s_{n-1}} f_{1-gH}^{(n)}(s_n)ds_n\right) d_{s_{n-1}} \cdots\right) ds_1$$

Lemma 2.1 ([6]). Let $f: [t,s] \to R_f$ be fuzzy continuous. Then $\int_t^s f(t)dt$ exists and belongs to Rf. Furthermore,

$$[\int_t^s f(x)dx]_\alpha = [\int_t^s \underline{f}(s;\alpha)dx, \int_t^s \bar{f}(x;\alpha)dx] \quad (2.8)$$

Lemma 2.2 ([6]). Suppose that $f: [t,s] \to R_f$ is gH-differentiable and f'_{gH} is continuous on [t,s]. Then

$$\int_k^t f'_{1-gH}(x)dx = (-1) \odot \int_t^k f'_{2-gH}(x)dx. \quad (2.9)$$

Theorem 2.2 ([6, Theorem 3.1]). Suppose that $f: [t,s] \to R_f$ is gH-differentiable and the type of differentiability of f on [t,s] does not change. Then for all $t \le k \le s$, the following statements hold:

1) If f(x) is (1)-gH-differentiable then $f'_{1-gH}(x)$ is (FR)-integrable on [t,s] and

$$\int_t^k f'_{1-gH}(x)dx = f(k)\Theta_H f(t) \quad (2.10)$$

2) If f(x) is (2)-gH-differentiable then $f'_{2-gH}(x)$ is (FR)-integrable on [t,s] and

$$\int_t^k f'_{2-gH}(x)dx = (-1) \odot f(x)\Theta_H(-1) \odot f(k). \quad (2.11)$$

Lemma 2.3 ([6]). For all real numbers z, the inequality $1+z \le e^z$ holds.

3. Fuzzy Adams Methods

The general linear multi-step method can be formulated as

$$\sum_{i=0}^{k} \alpha_j y_{n+j} = h\sum_{i=0}^{k} \beta_j f_{n+j},$$

where α_j's and β_j's are constants. We assume that $\alpha_k \neq 0$ and also that both α_0 and β_0 are not zero. The Adams methods are only a subclass of linear multi-step method with

$$\alpha_k = 1, \quad \alpha_{k-1} = -1, \quad \alpha_j = 0 \quad j = 0,1,\ldots,k-2$$

In this section, we construct fuzzy Adams-Bashforth (A-B) and fuzzy Adams-Moulton (A-M) methods based on the generalized Hukuhara (gH)-differentiability. To this end, we first introduce finite backward and forward differences based on gH-differences. Using them, we go on the Newton's interpolation polynomials and finally, using Newton's forward and backward interpolation polynomials, we obtain fuzzy Adams-Bashforth and fuzzy Adams-Moulton methods. We employ the former as the predictor and the latter as the corrector.

A fuzzy polynomial interpolation of the data is a fuzzy number-valued function $p: R \to R_f$ which satisfies

1) for all $i = 1,2, \dots, n, \quad p(x_i) = f_i$

2) p is continuous.

3) If the data be crisp, then the interpolation p is a crisp polynomial.

A function p fulfilling these conditions can be constructed following.

At first, forward gH-difference between fi and f_{i+1} shown by $\Delta_{gH}$is introduced.

For each $\alpha \in [0,1], \alpha$ - level of f_i is shown by $[f_i]_\alpha$ and $[f_i]_\alpha = \left[\underline{f}_i(\alpha), \bar{f}_i(\alpha)\right]$.

3.1. Forward Finite Differences

Suppose that $f: [a, b] \to R_f$ is continuous over [a,b] and the values of f are known in $x_i = a + ih, i = 0,1, \dots, n, h = \frac{b-a}{n}$. Define,

$$\Delta_{gH} f_i(\alpha) = f_{i+1}(\alpha) \Theta_{gH} f_i(\alpha) = \begin{cases} (i) f_{i+1}(\alpha) \Theta_H f_i(\alpha), \\ (ii)(-1) \odot (f_i(\alpha) \Theta_H f_{i+1}(\alpha)). \end{cases}$$

The $\alpha - cut$of Δ_{gH} f_i is defined in following form.

$$[\Delta_{gH} f_i(\alpha)]_\alpha = [f_{i+1}(\alpha) \Theta_{gH} f_i(\alpha)]_\alpha =$$

$$\left[min\left\{\underline{f}_{i+1}(\alpha) - \underline{f}_i(\alpha), \bar{f}_{i+1}(\alpha) - \bar{f}_i(\alpha)\right\}, max\left\{\underline{f}_{i+1}(\alpha) - \underline{f}_i(\alpha), \bar{f}_{i+1}(\alpha) - \bar{f}_i(\alpha)\right\}\right]_\alpha$$

$$\Delta_{gH}^2 f_i(\alpha) = \Delta_{gH}(\Delta_{gH} f_i(\alpha)) = \Delta_{gH}(f_{i+1}(\alpha) \Theta_{gH} f_i(\alpha))$$

$$= (f_{i+2}(\alpha) \Theta_{gH} f_{i+1}(\alpha)) \Theta_{gH} (f_{i+1}(\alpha) \Theta_{gH} f_i(\alpha))$$

$$= \begin{cases} (i) f_{i+2}(\alpha) \Theta_{gH} 2 \odot f_{i+1}(\alpha) \oplus f_i(\alpha), \\ (ii)(-1) \odot (f_{i+1}(\alpha) \Theta_H f_{i+2}(\alpha)) \Theta_{gH} f_{i+1}(\alpha) \oplus f_i(\alpha) \end{cases} \quad (3.12)$$

$$= \begin{cases} (i) f_{i+2}(\alpha) \Theta_H 2 \odot f_{i+1}(\alpha) \oplus f_i(\alpha), \\ (ii)(-1) \odot (2 \odot f_{i+1}(\alpha) \Theta_H f_{i+2}(\alpha)) \oplus f_i(\alpha), \\ (iii)(-1) \odot (f_{i+1}(\alpha) \Theta_H f_{i+2}(\alpha)) \Theta_{gH} f_{i+1}(\alpha) \oplus f_i(\alpha), \\ (iv)(-1) \odot (f_{i+1}(\alpha) \Theta_H (-1) \odot (f_{i+1}(\alpha) \Theta_H f_{i+2}(\alpha))) \oplus f_i(\alpha) \end{cases}$$

The reasoning above leads us to $\Delta_{gH}^3 fi(\alpha) = \Delta_{gH}(\Delta_{gH}^2 f_i(\alpha))$ and so on. $\Delta_{gH}^k f_i, k = 1,2, \dots$ are the forward finite differences.

Now, using our new notations and concepts, we can define fuzzy interpolation polynomial with forward gH-difference as follows:

Definition 3.1. Let $\boldsymbol{\alpha} \in [\mathbf{0}, \mathbf{1}]$. Then the fuzzy interpolation polynomial p(x) with forward gH-differences can be written as

$$p(x) = f_0(\alpha) \oplus \theta \odot \Delta_{gH} f_0(\alpha) \oplus \frac{\theta(\theta-1)}{2} \odot \Delta_{gH}^2 f_0(\alpha) \oplus \ldots \oplus \frac{\theta(\theta-1)\ldots(\theta-n+1)}{n!} \odot \Delta_{gH}^n f_0(\alpha)$$

Remark 3.1. We know that θ is crisp because of $\boldsymbol{\theta} = \frac{\boldsymbol{x}-\boldsymbol{x_0}}{\boldsymbol{h}}$ and all of $\boldsymbol{x}, \boldsymbol{x_i}, \boldsymbol{i} = \mathbf{1}, \mathbf{2}, \ldots, \boldsymbol{n}$ and h are crisp.

3.2. Backward Finite Differences

Similarly, the backward finite differences are defined in the following form.

$$\nabla_{gH} f_i(\alpha) = f_i(\alpha) \Theta_{gH} f_{i-1}(\alpha) = \begin{cases} (i) f_i(\alpha) \Theta_H f_{i-1}(\alpha), \\ (ii)(-1) \odot (f_{i-1}(\alpha) \Theta_H f_i(\alpha)). \end{cases}$$

$$[\nabla_{gH} f_i(\alpha)]_\alpha = [f_i(\alpha) \Theta_{gH} f_{i-1}(\alpha)]_\alpha$$

$$= \left[min\left\{\underline{f_i}(\alpha) - \underline{f}_{i-1}(\alpha), \bar{f}_i(\alpha) - \bar{f}_{i-1}(\alpha)\right\}, max\left\{\underline{f_i}(\alpha) - \underline{f}_{i-1}(\alpha), \bar{f}_i(\alpha) - \bar{f}_{i-1}(\alpha)\right\}\right]_\alpha$$

$$\nabla_{gH}^2 f_i(\alpha) = \nabla_{gH}(\nabla_{gH} f_i(\alpha)) = \nabla_{gH}(f_i(\alpha) \Theta_{gH} f_{i-1}(\alpha)) = (f_i(\alpha) \Theta_{gH} f_{i-1}(\alpha)) \Theta_{gH} (f_i - 1(\alpha)) \Theta_{gH} f_{i-2}(\alpha)) = \begin{cases} (i) f_i(\alpha) \Theta_{gH} 2 \odot f_{i-1}(\alpha) \oplus f_{i-2}(\alpha), \\ (ii)(-1) \odot (f_{i-1}(\alpha) c_H f_i(\alpha)) \Theta_{gH} f_{i-1}(\alpha) \oplus f_{i-2}(\alpha) \end{cases} \quad (3.13)$$

$$= \begin{cases} (i) f_i(\alpha) \Theta_H 2 \odot f_{i-1}(\alpha) \oplus f_{i-2}(\alpha), \\ (ii)(-1) \odot (2 \odot f_{i-1}(\alpha) \Theta_H f_i(\alpha)) \oplus f_{i-2}(\alpha), \\ (iii)(-1) \odot (f_{i-1}(\alpha) \Theta_H f_i(\alpha)) \Theta_H f_{i-1}(\alpha) \oplus f_{i-2}(\alpha), \\ (iv)(-1) \odot (f_{i-1}(\alpha) \Theta_H (-1) \odot (f_{i-1}(\alpha) \Theta_H f_i(\alpha))) \oplus f_{i-2}(\alpha) \end{cases}$$

The reasoning above leads us to $\nabla_{gH}^3 fi(\alpha) = \nabla_{gH}(\nabla_{gH}^2 f_i(\alpha))$ and so on.

Now, using our new notations and concepts, we can define fuzzy interpolation polynomial with backward gH-difference as follows:

Definition 3.2. Let $\boldsymbol{\alpha} \in [\mathbf{0}, \mathbf{1}]$. Then the fuzzy interpolation polynomial p(x) with forward gH-differences can be written as

$$p(x) = f_n(\alpha) \oplus \theta \odot \nabla_{gH} f_n(\alpha) \oplus \frac{\theta(\theta+1)}{2} \odot \nabla_{gH}^2 f_n(\alpha) \oplus \ldots \oplus \frac{\theta(\theta+1)\ldots(\theta+n-1)}{n!} \odot \nabla_{gH}^n f_n(\alpha)$$

Remark 3.2. We know that θ is crisp because of $\theta = \frac{x-x_n}{h}$ and all of $x, x_i, i = 0,1, \ldots, n-1$ and h are crisp.

3.3. A-B Two-Step Method

To solve FIVP

$$\begin{cases} u'_{gH}(x) = f(x, u(x)), & x_0 \le x \le T \\ u(x_0) = u_0 \end{cases} \quad (3.14)$$

by A-B two-step method, suppose that the fuzzy initial values are $u(x_{i-1}), u(x_i)$. $i.e$ $f(x_{i-1}, u(x_{i-1}), f(x_i, u(x_i))$ are fuzzy numbers.

Also let $\nabla_{gH} f_i = f_i \Theta_{gH} f_{i-1}, \theta = \frac{x-x_i}{h}$ |. The fuzzy interpolation polynomial for fuzzy numbers use of $f(x_{i-1}), u(x_{i-1}), f(x_i, u(x_i))$, can be written as.

$$p(x, u(x)) = f_i \oplus \theta \odot \nabla_{gH} f_i.$$

(i) If u(x) is (1)-gH-differentiable and $\boldsymbol{f(x, u(x))}$ is replaced with $\boldsymbol{p(x, u)})$, then,

$$\int_{x_i}^{x_{i+1}} u'_{1-gH}(x)dx = \int_{x_i}^{x_{i+1}} f(x, u(x))dx$$

$$u_{i+1} = u_i \oplus \int_{x_i}^{x_{i+1}} (f_i \oplus \theta \odot \nabla_{gH} f_i)dx = u_i \oplus h \odot \int_0^1 (f_i \oplus \theta \odot \nabla_{gH} f_i)d\theta$$

$$= u_i \oplus h \odot [f_i \odot \theta \oplus \frac{\theta^2}{2} \odot \nabla_{gH} f_i]_0^1 =$$

$$u_i \oplus h \odot [f_i \oplus \frac{1}{2} \odot (f_i \Theta_{gH} f_{i-1})] = u_i \oplus \frac{h}{2} \odot (3 \odot f_i \Theta_{gH} f_{i-1})$$

Therefore,

$$u_{i+1} = u_i \oplus \frac{h}{2} \odot (3 \odot f_i \Theta_{gH} f_{i-1}). \tag{3.15}$$

Then, using the properties of gH-difference, all different forms of A-B two-step method of (I) can be written as follows.

1) $u_{i+1} = u_i \oplus \frac{h}{2} \odot (3 \odot f_i \Theta_H f_{i-1}),$

2) $u_{i+1} = u_i \oplus \frac{-h}{2} \odot (f_{i-1} \Theta_H 3 \odot f_i).$

(II) If u(x) is (2)-gH-differentiable, we have

$$\int_{x_i}^{x_i+1} u'_{2-gh}(x)dx = \int_{x_i}^{x_i+1} f(x, u(x))dx.$$

Similarly, replacing $f(x, u(x))$ with $p(x, u(x)))$ and at last using the properties of gH-difference, the following result can be obtained.

$$u_{i+1} = u_i \Theta_H (-1) \odot \frac{h}{2} \odot (3 \odot f_i \Theta_{gH} f_{i-1}). \tag{3.16}$$

Therefore,

1) $u_{i+1} = u_i \Theta_H (-1) \odot \frac{h}{2} \odot (3 \odot f_i \Theta_H f_{i-1}),$

2) $u_{i+1} = u_i \Theta_H (-1) \odot \frac{-h}{2} \odot (f_{i-1} \Theta_H 3 \odot f_i).$

3.4. A-M Three-Step Method

For A-M three-step method suppose $u(x_i), u(x_{i+1}), u(x_{i+2})$, be the fuzzy initial values, i.e $f(x_i, u(x_i)), f(x_{i+1}, u(x_{i+1})), f(x_{i+2}, u(x_{i+2})) f(x_{i+3}, u(x_{i+3}))$ are de_ned. Then, consider the following identity,

$$(I)\int_{x_{i+2}}^{x_{i+3}} u'_{1-gH}(x)dx = \int_{x_{i+2}}^{x_{i+3}} f(x,u(x))dx.$$

when u(x) is (1)-gH-differentiable. Now, replace $f(x,u(x))$ with p(x,u(x)) in $(x_i,f_i),(x_{i+1},f_{i+1}),(x_{i+2},f_{i+2}),(x_{i+3},f_{i+3})$.

$$u_{i+3} = u_{i+2} \oplus \int_{x_{i+2}}^{x_{i+3}} f(x,u(x))dx$$

$$u_{i+2} \oplus \int_{x_{i+2}}^{x_{i+3}} (f_i \oplus \theta \odot \Delta_{gH} f_i \oplus \frac{\theta(\theta-1)}{2} \odot \Delta_{gH}^2 f_i$$

$$\oplus \frac{\theta(\theta-1)(\theta-2)}{6} \odot \Delta_{gH}^3 f_i)dx$$

$$u_{i+2} \oplus h \odot \int_2^3 (f_i \oplus \theta \odot \Delta_{gH} f_i \oplus \frac{\theta(\theta-1)}{2} \odot \Delta_{gH}^2 f_i$$

$$\oplus \frac{\theta(\theta-1)(\theta-2)}{6} \odot \Delta_{gH}^3 f_i)dx$$

$$u_{i+2} \oplus h \odot [f_i \odot \theta \oplus \frac{\theta^2}{2} \odot \Delta_{gH} f_i \oplus (\frac{1}{6}\theta^3 - \frac{1}{4}\theta^2) \odot \Delta_{gH}^2 f_i$$

$$\oplus (\frac{1}{24}\theta^4 - \frac{\theta^3}{6} + \frac{\theta^2}{6}) \odot \Delta_{gH}^3 f_i]|_2^3$$

$$= u_{i+2} \oplus h \odot (f_i \oplus \frac{5}{2} \odot \Delta_{gH} f_i \oplus \frac{23}{12} \odot \Delta_{gH}^2 f_i \oplus \frac{3}{8} \odot \Delta_{gH}^3 f_i)$$

$$= u_{i+2} \oplus h \odot (f_i \oplus \frac{5}{2} \odot (f_{i+1} \Theta_{gH} f_i) + \frac{23}{12} \odot (f_{i+2} \Theta_{gH} 2 f_{i+1} \oplus f_i)$$

$$\oplus \frac{3}{8} \odot (f_{i+3} \Theta_{gH} 3 \odot f_{i+2} \oplus 3 \odot f_{i+1} \Theta_{gH} f_i))$$

$$= u_{i+2} \oplus h \odot (\frac{3}{8} \odot f_{i+3} \oplus \frac{19}{24} \odot f_{i+2} \Theta_{gH} \frac{5}{24} \odot f_{i+1} \oplus \frac{1}{24} \odot f_i)$$

$$= u_{i+2} \oplus \frac{h}{24} \odot (9 \odot f_{i+3} \oplus 19 \odot f_{i+2} \Theta_{gH} 5 \odot f_{i+1} \oplus f_i).$$

Therefore, all cases of A-M three-step method can be written as follows.

1) $u_{i+3} = u_{i+2} \oplus \frac{h}{24} \odot [9 \odot f_{i+3} \oplus 19 \odot f_{i+2} \Theta_H 5 \odot f_{i+1} \oplus f_i)]$,

2) $u_{i+3} = u_{i+2} \oplus \frac{h}{24} \odot [9 \odot f_{i+3} \oplus (-1) \odot (5 \odot f_{i+1} \Theta_H 19 \odot f_{i+2}) \oplus f_i)]$. (3.17)

(II) when u(x) is (2)-gH-differentiable.

Similarly, all cases of A-M three-step method in the second case can be written as

1) $u_{i+3} = u_{i+2} \Theta_H (-1) \odot \frac{h}{24} \odot [9 \odot f_{i+3} \oplus 19 \odot f_{i+2} \Theta_H 5 \odot f_{i+1} \oplus f_i]$,

2) $u_{i+3} = u_{i+2}\Theta_H(-1) \odot \frac{h}{24} \odot [9 \odot f_{i+3} \oplus (-1) \odot (5 \odot f_{i+1}\Theta_H 19 \odot f_{i+2}) \oplus f_i)]$.

3.5. Predictor-Corrector Methods

Suppose that we want to solve FIVP by using an implicit linear K-step method. For example, consider one of the cases of three-step A-M method (3.17).

1) $\boldsymbol{u_{i+3} = u_{i+2} \oplus \frac{h}{24} \odot [9 \odot f_{i+3} \oplus 19 \odot f_{i+2}\Theta_H 5 \odot f_{i+1} \oplus f_i}]$.

At each step, the equation in which only $u_{j+1}, f_{j+1}, j = 0,1,2$ are known, are solved. In general, this equation is nonlinear.

Using the theorem of a unique solution existence of the IVP, we find that a unique solution exists for u_{i+3} and it can be approached arbitrarily close by the iteration

$$u_{i+3}^{[s+1]} = u_{i+2} \oplus \frac{h}{24} \odot [9 \odot f(x_{i+3}, u_{i+3}^{[s]}) \oplus 19 \odot f_{i+2}\Theta_H 5 \odot f_{i+1} \oplus f_i] \quad s = 0,1,2,.. \quad (3.18)$$

Where $u_{i+3}^{[0]}$is arbitrary.

Each step of the iteration (3.18) involves an evaluation of $f(x_{i+3}, u_{i+3}^{[s]})$. Thus, we are concerned to keep the number of times the iteration (3.18) is applied minimum, specially whenever the evaluation of f is time-consuming at the given values of its arguments.

Therefore, we would like to accurate the initial guess $u_{i+3}^{[0]}$as much as possible. A separate explicit method can be used to estimate $u_{i+3}^{[0]}$which can be used to the initial guess of $u_{i+3}^{[0]}$. The explicit method "A-B" and the implicit method (3.18) "A-M" are the predictor and the corrector, respectively.

Let P, C, and E indicate an application of the predictor, a single application of the corrector, and an evaluation of a function f in terms of the known values of its arguments. If we compute $u_{i+3}^{[0]}$from the predictor, evaluate $\int_{i+3}^{[0]} \equiv f(x_{i+3}, u_{i+3}^{[0]}0$,and apply the corrector once to get $u_{i+3}^{[1]}$, the series of calculations done so far is denoted by PEC.

A further evaluation of $\int_{i+3}^{[1]} \equiv f(x_{i+3}, u_{i+3}^{[1]})$which is followed by a second application of the corrector, yields $u_{i+3}^{[2]}$, and the calculation is now denoted by PECEC, or $P(EC)^2$. Applying the corrector m times is similarly denoted by $P(EC)^m$. Since m is fixed, we accept $u_{i+3}^{[m]}$, as the numerical solution at x_{i+3}. At this stage, the last computed value we have for $f_{i+3}$is $f_{i+3}^{[m-1]} \equiv f(x_{i+3}, u_{i+3}^{[m-1]})$, and we have a further decision to make, namely, whether or not to evaluate $f_{i+3}^{[m]} \equiv f(x_{i+3}, u_{i+3}^{[m]})$. If this final evaluation is done, we denote the mode by $P(EC)^m$ E, and if not, by $P(EC)^m$.

This choice clearly affects the next step of the calculation since both predicted and corrected values for ui+4 will depend on whether $f_{i+3}$is taken to be $f_{i+3}^{[m]}$or $f_{i+3}^{[m-1]}$. Note that for a given m, both $P(EC)^m$ E and $P(EC)^m$ modes apply the corrector the same number of times; but the former calls for one more function evaluation per step than the latter.

3.6. Local Truncation Error

Theorem 3.1. Consider that $u'''_{gH}(x)$ exists and f(x,u(x)) satisfies the Lipschitz condition on the set $(x, u(x))|x \in [0, p], u \in \bar{B}(u_0, q), p, q > 0$. Then, A-B and A-M methods converge to the solution of the FIVP (1.1).

To see the proof of Theorem 3.1, the reader is referred to [11].

4. Stability

A method is said to be stable whenever small perturbations in the initial values will only cause small changes in the solutions.

Definition 4.1. Let $u_{k+1}, k+1 \geq 0$be the fuzzy solution of the fuzzy p-c method with initial condition $u_0 \in R_f$and let z_{k+1} be the solution of the same numerical method with a perturbed fuzzy initial condition $z_0 \in R_f$ such that $z_0 = u_0 \oplus \delta_0$. The fuzzy p-c method is called stable if there exist positive constants $\tilde{h}$ and κ such that

$$D(z_{k+1}, u_{k+1}) \leq \kappa\delta \qquad \forall (k+1)h \leq T, k \leq N-1, h \in (0, \tilde{h})$$

Whenever $D(\delta_0, 0) \leq \delta$.

A-B two-step and A-M three-step methods are stable. To see this in details, the reader is referred to [11].

5. Numerical Algorithm for Solving FIIVP

The numerical algorithm for solving impulsive fuzzy initial value problems is different from the numerical algorithm for solving fuzzy initial value problems only at the pulse point, where we have to apply the operators concern with the particular point. For this reason, at first, we consider fuzzy initial value problem in the form

$$\begin{cases} y'(t) = f(t, y(t)), \\ y(t_0) = y_0, \end{cases} \tag{5.19}$$

where y is a fuzzy function of t, f(t,y) is a fuzzy function of the crisp variable t and the fuzzy variable y, y' is the fuzzy derivative of y and $y(t_0) = y_0$is a triangular fuzzy number. Therefore, we have a fuzzy Cauchy problem [20].

In this section, we present an algorithm for solving the first order impulsive fuzzy differential equations with initial value. If the function y(t) is a solution of the impulsive fuzzy differential Equation (5.19), then by the numerical algorithm for solving impulsive fuzzy differential equations, it is possible to obtain values $y(t_z)$ for a fixed value of t_z of parameter t_0, where $t_z > t_0$, at the moment t = tz. The algorithm includes following steps:

Step One: At $t = t_0$, apply the Adams-Bashforth method for y by considering $y = y_0$ from initial

condition (5.19). The algorithm applies until the first pulse point.

Step Two: At the pulse point $t = t_k$, the impulsive fuzzy operator I_k brings rapidly changes to the function y that moment is $J(t_k)$; Where

$$\mathrm{J}(\mathrm{t_k}) = \mathrm{I_k}(\mathrm{y}(\mathrm{t_k}) + \sum_{\mathrm{t_0}<\mathrm{t_a}<\mathrm{t_k}} \mathrm{J}(\mathrm{t_s})).$$

Step Three: We solve the function y of argument t by taking it from the half-segment $(\mathrm{t_k}, \mathrm{t_{k+1}}]$ by the Adams-Bashforth method.

Step Four: We repeat Steps Two and Three until we encounter with the desired $y(t_z)$ that has to be found.

Step Five: Add the summation of all pulses to the function y.

$$Y(t_z) := y(t_z) + \sum_{t_0<t_k<t_z} J(t_k)$$

$[y(t_z)_r = [y_1(t_z, r), y_2(t_z, r)]$ is the approximated solution and $[J((t_k)]_r = [J_1(t_k, r), J_2(t_k, r)]$ is the pulse in point t_k. We have

$Y[y + \sum_{t_0<t_k<t_z} J(t_k)]_r = [y_1(t_k, r) + \sum_{t_0<t_k<t_z} J_1(t_k, r), y_2(t_z, r + \sum_{t_0<t_k<t_z} J_2(t_k, r))]$.
We apply Adams method on each half-segment $(t_k, t_{k+1}]$.

6. Error of Fuzzy Numerical Algorithm

By assumption first order impulsive fuzzy differential equations (1.1), (1.2), (1.3)|

$$\begin{cases} y'(t) = f(t, y(t)), & t \in J = [0, T], t \neq t_k, k = 1, \ldots, m, \\ y(t_k^+) = I_k(y(t_k^-)), & k = 1, \ldots, m \\ y(t_0) = y_0, \end{cases} \tag{6.20}$$

we get the analysis error of numerical algorithm in section 3 based on fuzzy predictor-corrector method. We import small perturbations δ_3 on the right-hand side of expression (1.3), then it creates perturbations δ_1, δ_2, on the right-hand side of expressions (1.1), (1.2), and we will have the solution as following form

$$Y = y \oplus \delta \odot y. \tag{6.21}$$

We got analysis this perturbation, whether we will have a stable numerical algorithm, and we will have converged? By perturbation, we achieve to first order perturbation impulsive fuzzy differential equation as following form

$$\begin{cases} Y'_{gH}(t) = f(t, y(t)) \oplus [\delta_1 \odot f(t, y(t))], \\ t \in J = [0, T], t \neq t_k, k = 1, \ldots, m. \end{cases} \tag{6.22}$$

$$\begin{cases} \Delta Y|_{t=t_k} = Y(t_k^+) \Theta_{gH} Y(t_k^-) \\ = I_k(y(t_k^-)) \oplus [\delta_2 \odot I_k(y(t_k^-))], (t = t_k), \qquad k = 1, \ldots, m \end{cases} \tag{2.24}$$

$$Y(t_0) = y(t_0) \oplus \delta_3 \odot y(t_0). \tag{6.24}$$

By assumption expression (6.23) and using expression (6.21) we conclude

$Y(t_k^+) \Theta_{gH} Y(t_k^-) = [y(t_k^+) \Theta_{gH} y(t_k^-)] \oplus \delta_2 \odot I_k(y(t_k^-)),$

$([y(t_k^+) \oplus \delta \odot y(t_k^+) \Theta_{gH}(y(t_k^-) \oplus \delta \odot y(t_k^-)]) \Theta_{gH}[y(t_k^+) \Theta_{gH} y(t_k^-)] = \delta_2 \odot I_k(y(t_k))$

$\delta \odot [y(t_k^+)\Theta_{gH}y(t_k^-)] = \delta_2 \odot I_k(y(t_k))$.

By using Property 3 of the Hausdorff distance

$$\delta \odot [y(t_k^+)\Theta_{gH}y(t_k^-)] \leq \delta_2 \odot I_k(y(t_k)).$$

Finally, by considering Definition 2.1 we have

$$\delta \leq \frac{I_k(y(t_k))}{[y(t_k^+)\Theta_{gH}y(t_k^-)]} \odot \delta_2.$$

Without reduce of generality, we only consider fuzzy Adamz Bashforth method (3.15), for expression (6.21) and (6.23) we obtain

$$\begin{cases} y_0 = \delta_3 \odot y(t_0). \\ y_{k+1} = y_k \oplus \frac{h}{2} \odot [\delta_1(3 \odot f(t_k, y(t_k))\Theta_{gH}f(t_{k-1}, y(t_{k-1})))] \end{cases} \quad (6.25)$$

By setting k = 1, in (6.25), we obtain

$$Y_2 = \delta_3 y_1 \oplus \frac{h}{2}[\delta_1(3 \odot f(t_1, y_1)\Theta_{gH}f(t_0, y_0))].$$

According to the expression (6.21), we obtain

$$y_2 \oplus \delta \odot y_2 = \delta_3 y_1 \oplus \frac{h}{2} \odot [\delta_1(3 \odot f(t_1, y_1)\Theta_{gH}f(t_0, y_0))].$$

By using property 3 of the Hausdorff distance once more

$$\delta \leq \frac{y_1\delta_3 \oplus \frac{h}{2} \odot \delta_1(3 \odot f(t_1, y_1)\Theta_{gH}f(t_0, y_0))}{y_2}$$

By letting $\delta_3 \to 0$ and $\delta_1 \to 0$, we obtain $\delta \to 0$, which means that the numerical algorithm is stable and converges to the exact solution.

Without reducing of generality, since the case for k = 3, …, z is the same as for k = 2, we only consider the case k = 2 and according to the expression (6.21), we obtain

$$y_3 \oplus \delta \odot y_3 = (y_2(t_k^+) \oplus \delta \odot y_2(t_k^+)) \oplus \frac{h}{2}[\delta_1(3 \odot f(t_2, y_2)\Theta_{gH}f(t_1, y_1))].$$

By using Definition 2.1

$$y_3 \oplus \delta \odot y_3 \leq y_2(t_k^+) \oplus \delta \odot y_2(t_k^+) \oplus [\frac{h}{2} \odot (\delta_1(3 \odot f(t_2, y_2)\Theta_{gH}f(t_1, y_1)))].$$

Finally, we obtain

$$\delta \leq \frac{y_2(t_k^+) \oplus \delta \odot y_2(t_k^+)}{y_3} + \frac{\frac{h}{2} \odot (\delta_1(3 \odot f(t_2, y_2)\Theta_{gH}f(t_1, y_1)))]}{y_3}$$

Hence

$$\delta \leq \frac{\delta_3 y_1 + \frac{h}{2} \odot \delta_1(3 \odot f(t_1, y_1)\Theta_{gH}f(t_0, y_0))}{y_3} \odot \frac{I_k(y(t_k)) \odot \delta_2}{(y(t_k^+)\Theta_{gH}y(t_k^-))}. \quad (6.26)$$

The upper bound in (6.26) shows that to have the stable numerical algorithm that converges to the exact solution, we need imported perturbations in right side of the expressions (6.22), (6.23), (6.24) near to zero, too, i.e.,$\delta_1 \to 0, \delta_2 \to 0$ and $\delta_3 \to 0$. Under this condition, we can say that for $k = 2, \ldots, z$, the numerical algorithm introduced is stable and converges to the exact solution.

7. Numerical Results

In this section we present a numerical example, in order to see the accuracy of our numerical solution. The numerical results show convergence of these methods.

Example 7.1. Consider the first order impulsive fuzzy initial value problem,

$$u'_{gH}(x) = u(x), 0 \le x \le 1$$

$$u(x_k^+) = 0.01u(x_k^-),$$

$$u(0) = (0.75 + 0.25r, 1.125 - 0.125r)$$

Where $0 \le r \le 1$, and by the considering $x_z = 1$. This example was solved by using the Adams-Bashforth two step method with N = 10, N = 100 and N = 200 in the cases that u(x) was $[(i) - gh]$ - differentiable.

These results in x = 1 has been showed in table 1 and plot of u(x) in $0 \le x \le 1$ has been showed in figure 1.

8. Conclusion

In this paper, a new method was introduced in the fuzzy numerical analysis based on gH-differentiabil- ity namely the fuzzy Adams-Bashforth and the fuzzy Adams-Moulton methods for solving (1.1), (1.2), (1.3). In final one FIIVP was solved, according to the type of gH-differentiability with their's related methods. Obtained results demonstrate the efficiency of these methods. The obtained result indicates that by decreasing step size, the approximated solution tends to the exact solution.

α	$h = 0.1$		$h = 0.01$		$h = 0.05$	
	$\underline{u}$	$\overline{u}$	$\underline{u}$	$\overline{u}$	$\underline{u}$	$\overline{u}$
0	0.018300215395	0.027450323093	0.020182431	0.03027364799	0.0202849706	0.03042745597
0.1	0.01891022257	0.02714531950	0.0208551797	0.029937274	0.020961136	0.030089373
0.2	0.019520229	0.02684031591	0.0215279274659	0.029600900265	0.0216373020	0.029751290
0.3	0.02013023693	0.0265353123	0.022200675199	0.0292645263990	0.02231346	0.0294132074
0.4	0.02074024411	0.02623030873	0.02287342293256	0.028928152532	0.022989633	0.0290751246
0.5	0.02135025129	0.02592530514	0.023546170665	0.0285917786657	0.023665799	0.02873704175
0.6	0.021960258474	0.025620301553	0.02421891839918	0.02825540479	0.0243419647	0.0283989589
0.7	0.0225702656542	0.02531529796	0.024891666132	0.027919030932	0.02501813047	0.0280608760
0.8	0.023180272834	0.0250102943	0.0255644138658	0.027582657065	0.025694296159	0.0277227932
0:9	0.0237902800139	0.024705290783	0.026237161599	0.027246283199	0.02637046184	0.02738471038
1	0.02440028719	0.02440028719	0.0269099093324	0.0269099093324	0.0270466275	0.0270466275

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