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Some estimates on the AM-GM inequality and its applications

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Abstract. The present paper seeks to establish an approximation of the arithmetic-geometric mean inequality (AM-GM) using a logarithmically concave function. We utilized the specific properties of this class of functions to derive modified versions of the AM-GM inequality as a specific example. These findings present a fresh perspective on the subject.

Keywords: Young inequality, arithmetic-geometric mean inequality, logarithmic mean.

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1. Introduction and preliminaries

The arithmetic-geometric mean inequality says that the arithmetic mean of non-negative real numbers is greater than or equal to the geometric mean of the same list. Also, equality holds if and only if every number in the list is the same. Mathematically, for a collection of n non-negative real numbers $a_1, \ldots, a_n$, we have $\sqrt[n]{a_1 \cdots a_n} \leq \frac{a_1+\cdots+a_n}{n}$ with equality if and only if $a_1 = \cdots = a_n$. For $n = 2$, the following simple result holds

$$\sqrt{ab} \leq \frac{a+b}{2}. \tag{1}$$

In this article, we use the succeeding symbols:

$$a\nabla_\nu b = (1-\nu)\,a + \nu b,\ \ a\sharp_\nu b = a^{1-\nu} b^\nu,\ \text{ and } a!_\nu b = \left(a^{-1}\nabla_\nu b^{-1}\right)^{-1},$$

where $0 \leq \nu \leq 1$. When $\nu = 1/2$, we will write $a\nabla b$, $a\sharp b$ and $a!b$ for brevity, respectively.

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An inequality closely related to the AM-GM inequality is the celebrated Young inequality (sometimes named weighted AM-GM inequality), which asserts that

$$a\sharp_\nu b \le a\nabla_\nu b \tag{2}$$

for $a, b \ge 0$ and $0 \le \nu \le 1$. The Young inequality has been refined and reversed in the following forms [6, 7]:

$$a\sharp_\nu b \le a\nabla_\nu b - 2r\left(a\nabla b - a\sharp b\right),$$
$$a\nabla_\nu b \le a\sharp_\nu b + 2R\left(a\nabla b - a\sharp b\right).$$

Here, $r = \min\{\nu, 1-\nu\}$ and $R = \max\{\nu, 1-\nu\}$. Several papers have been devoted to generalizing and refining the classical AM-GM inequality and its cousin, Young inequality [3, 4, 8, 11, 12]. One that attracts many researchers is the logarithmic mean given by

$$\int_0^1 \left(a\sharp_\nu b\right) d\nu = \frac{b-a}{\log b - \log a}, \quad (a \ne b), \tag{3}$$

for two positive numbers a and b (see [10], for its weighted version). It is known that

$$a\sharp b \le \int_0^1 \left(a\sharp_\nu b\right) d\nu \le a\nabla b.$$

The Hermite-Hadamard inequality (for simplicity, we write H-H inequality) asserts that if $f : [a, b] \to \mathbb{R}$ is a convex function, then

$$f\left(\frac{a+b}{2}\right) \le \frac{1}{b-a}\int_a^b f(t)\,dt \le \frac{f(a)+f(b)}{2}.$$

We refer the reader to [9, 13] to see extensions of the Hermite-Hadamard inequality in various settings, such as Hilbert space operators and matrices.

Remark 1 *[2] One of the refinements of H-H inequality says that*

$$f\left(\frac{a+b}{2}\right) \le \frac{1}{b-a}\int_a^b f(t)\,dt \le \frac{1}{2}\left(\frac{f(a)+f(b)}{2} + f\left(\frac{a+b}{2}\right)\right) \le \frac{f(a)+f(b)}{2}.$$

If we take $f(x) = e^x$ in this inequality and replace a and b by $\log a$ and $\log b$, we get

$$a\sharp b \le L(a,b) \le \frac{1}{2}\left(a\nabla b + a\sharp b\right) \le a\nabla b,$$

where $L(a,b) = \frac{b-a}{\log b - \log a}$. Meanwhile, if we take $f(x) = -\ln x$ and apply the exponential function, we infer that $a\sharp b \le (a\nabla b)\,\sharp\,(a\sharp b) \le I(a,b) \le a\nabla b$, where $I(a,b) = e\left(\frac{a^a}{b^b}\right)^{\frac{1}{b-a}}$. This remark shows how H-H inequality is related to mathematical means.

Remark 2 *It is worth mentioning here that there is no order between $I(a,b)$ and $L(a,b)$, in general. To show this, letting $a = 0.1$ and $b = 0.2$, then $L(a,b) - I(a,b) \approx -6.65$, on the other hand, taking $a = 2$ and $b = 4$, then $L(a,b) - I(a,b) \approx 2.54$.*

We will prove that if $a, b > 0$ and $0 \le t, \nu \le 1$, then

$$a\sharp_\nu b \le \frac{a\sharp_\nu b - a\nabla_\nu b}{\log(a\sharp_\nu b) - \log(a\nabla_\nu b)} \le \frac{a\nabla_{(1-t)\nu+t} b - a\nabla_{(1-t)\nu} b}{\log\left(a\nabla_{(1-t)\nu+t} b\right) - \log\left(a\nabla_{(1-t)\nu} b\right)} \le a\nabla_\nu b.$$

A nonnegative function $f : \mathbb{R} \to (0, \infty)$ is log-concave if it fulfills the inequality

$$f(a)\sharp_\nu f(b) \le f(a\nabla_\nu b)\,; \quad (0 \le \nu \le 1 \text{ and } a, b \in \mathbb{R})\,. \tag{4}$$

If f is strictly positive, this is analogous to stating that the logarithm of the function, $\log f$, is concave.

We will offer new tight (lower and upper) bounds involving these functions, which have a high-interest potential for many mathematics or theoretical physics researchers. We demonstrate how our results extend and generalize previous inequalities involving the arithmetic-geometric mean inequality and logarithmically concave functions. Our findings contribute to the growing body of mathematical research on inequalities and function theory, highlighting the versatility and applicability of logarithmically concave functions in mathematical analysis. Additionally, we provide insights into the relationships between different types of inequalities, showcasing the rich interplay between various mathematical concepts.

2. Inequalities involving log-concave functions

We start this section by refining (4) when $\nu = 1/2$.

Theorem 2.1 Let $f : \mathbb{R} \to (0, \infty)$ be a log-concave function. Then for any $a, b \in \mathbb{R}$,

$$f(a)\sharp f(b) \le \int_0^1 \left(f(a\nabla_t b)\sharp f(b\nabla_t a)\right) dt \le f(a\nabla b)\,.$$

Proof. We have for any $0 \le t \le 1$ that

$$\begin{aligned} f(a\nabla b) &= f\left((a\nabla_t b)\nabla(b\nabla_t a)\right) \\ &\ge f(a\nabla_t b)\sharp f(b\nabla_t a) \\ &\ge \left(f(a)\sharp_t f(b)\right)\sharp\left(f(b)\sharp_t f(a)\right) \\ &= f(a)\sharp f(b)\,. \end{aligned}$$

Whence, $f(a)\sharp f(b) \le f(a\nabla_t b)\sharp f(b\nabla_t a) \le f(a\nabla b)$. Taking integral over $t \in [0,1]$, we receive

$$f(a)\sharp f(b) \le \int_0^1 \left(f(a\nabla_t b)\sharp f(b\nabla_t a)\right) dt \le f(a\nabla b)\,,$$

as desired. ■

We state the following result using the same approach in [5, Theorem 2.1].

Theorem 2.2 Let $f : \mathbb{R} \to (0, \infty)$ be a log-concave function and let $0 \le t, \nu \le 1$. Then for any $a, b \in \mathbb{R}$,

$$\begin{aligned} f(a) \sharp_\nu f(b) &\le (f(a\nabla_\nu b) \sharp_t f(a)) \sharp_\nu (f(a\nabla_\nu b) \sharp_t f(b)) \\ &\le f((a\nabla_\nu b)\nabla_t a) \sharp_\nu f((a\nabla_\nu b)\nabla_t b) \\ &\le f(a\nabla_\nu b). \end{aligned}$$

Proof. One can readily check for $a, b > 0$ and $0 \le t, \nu \le 1$ that

$$a\nabla_\nu b = ((a\nabla_\nu b)\nabla_t a)\nabla_\nu ((a\nabla_\nu b)\nabla_t b),$$

holds (see e.g., [5, (2.2)]). If $f : \mathbb{R} \to (0, \infty)$ is log-concave, we have

$$\begin{aligned} f(a\nabla_\nu b) &= f(((a\nabla_\nu b)\nabla_t a)\nabla_\nu ((a\nabla_\nu b)\nabla_t b)) \\ &\ge f((a\nabla_\nu b)\nabla_t a) \sharp_\nu f((a\nabla_\nu b)\nabla_t b) \\ &\ge (f(a\nabla_\nu b) \sharp_t f(a)) \sharp_\nu (f(a\nabla_\nu b) \sharp_t f(b)) \\ &\ge ((f(a) \sharp_\nu f(b)) \sharp_t f(a)) \sharp_\nu ((f(a) \sharp_\nu f(b)) \sharp_t f(b)) \\ &= f(a) \sharp_\nu f(b). \end{aligned}$$

Accordingly,

$$\begin{aligned} f(a) \sharp_\nu f(b) &\le (f(a\nabla_\nu b) \sharp_t f(a)) \sharp_\nu (f(a\nabla_\nu b) \sharp_t f(b)) \\ &\le f((a\nabla_\nu b)\nabla_t a) \sharp_\nu f((a\nabla_\nu b)\nabla_t b) \\ &\le f(a\nabla_\nu b), \end{aligned}$$

as desired. ■

3. Interpolations AM-GM Inequality

Using Theorem 2.1, we express the next corollary connecting both sides of (1).

Corollary 3.1 Let $a, b > 0$ with $a \ne b$. Then

$$a\sharp b \le \frac{(a+b)^2}{4|a-b|} \cdot \arcsin\left(\frac{|a-b|}{a+b}\right) + \frac{a\sharp b}{2} \le a\nabla b.$$

Proof. The special choice $f(x) = x$ in Theorem 2.1 and the fact that

$$\begin{aligned} (a\nabla_t b)(b\nabla_t a) &= (1-t)^2 \left(a^2\sharp b^2\right) + t(1-t)a^2 + t(1-t)b^2 + t^2\left(a^2\sharp b^2\right) \\ &= \left((1-t)^2 + t^2\right)\left(a^2\sharp b^2\right) + 2t(1-t)\left(a^2\nabla b^2\right) \end{aligned}$$

gives

$$a\sharp b \leq \int_0^1 \sqrt{\left((1-t)^2+t^2\right)(a^2\sharp b^2)+2t(1-t)(a^2\nabla b^2)}dt \leq a\nabla b.$$

By calculating the integral, we have

$$\begin{aligned}
&\int_0^1 \sqrt{\left((1-t)^2+t^2\right)(a^2\sharp b^2)+2t(1-t)(a^2\nabla b^2)}dt \\
&= \frac{(a+b)^2\arcsin\left(\frac{|a-b|(2t-1)}{a+b}\right)}{8|a-b|} + \left.\frac{(2t-1)\sqrt{\left((1-t)^2+t^2\right)(a^2\sharp b^2)+2t(1-t)(a^2\nabla b^2)}}{4}\right|_0^1 \\
&= \frac{(a+b)^2}{8|a-b|}\left(\arcsin\left(\frac{|a-b|}{a+b}\right)-\arcsin\left(-\frac{|a-b|}{a+b}\right)\right)+\frac{a\sharp b}{2}.
\end{aligned}$$

Therefore,

$$a\sharp b \leq \frac{(a+b)^2}{8|b-a|}\left(\arcsin\left(\frac{|a-b|}{a+b}\right)-\arcsin\left(-\frac{|a-b|}{a+b}\right)\right)+\frac{a\sharp b}{2} \leq a\nabla b.$$

Since $\arcsin(-y) = -\arcsin y$, we get

$$a\sharp b \leq \frac{(a+b)^2}{4|a-b|}\cdot\arcsin\left(\frac{|a-b|}{a+b}\right)+\frac{a\sharp b}{2} \leq a\nabla b.$$

This completes the proof. ■

Remark 3 *It follows from Corollary 3.1 that*

$$a\sharp b \leq \left(\frac{1}{2}\left(\left(\frac{a\nabla b}{a\sharp b}\right)\left(\frac{a+b}{|a-b|}\cdot\arcsin\left(\frac{|a-b|}{a+b}\right)\right)+1\right)\right)a\sharp b \leq a\nabla b. \tag{5}$$

The constant appeared in (5) *is always equal or greater than one, i.e.,*

$$\frac{1}{2}\left(\left(\frac{a\nabla b}{a\sharp b}\right)\left(\frac{a+b}{|a-b|}\cdot\arcsin\left(\frac{|a-b|}{a+b}\right)\right)+1\right) \geq 1.$$

Of course, the above inequality is equivalent to

$$\frac{a\sharp b}{a\nabla b} \leq \frac{a+b}{|a-b|}\cdot\arcsin\left(\frac{|a-b|}{a+b}\right), \tag{6}$$

which is an obvious inequality. Indeed, the left side of (6) *is always equal or greater than one, since* $x\arcsin\left(\frac{1}{x}\right) \geq 1$ *for* $x \geq 1$. *The right side of* (6) *is equal or smaller than one due to the arithmetic-geometric mean inequality* (1).

The next result improves the harmonic-geometric mean inequality.

Corollary 3.2 Let $a, b > 0$ with $a \neq b$. Then

$$a!b \leq \frac{a\sharp b}{\frac{1}{2}\left(\left(\frac{a\nabla b}{a\sharp b}\right)\left(\frac{a+b}{|a-b|}\cdot\arcsin\left(\frac{|a-b|}{a+b}\right)\right)+1\right)} \leq a\sharp b.$$

Proof. substitute a and b by a^{-1} and b^{-1} in (5), respectively, we get

$$\begin{aligned}
(a\sharp b)^{-1} &= a^{-1}\sharp b^{-1} \\
&\leq \left(\frac{1}{2}\left(\left(\frac{a^{-1}\nabla b^{-1}}{a^{-1}\sharp b^{-1}}\right)\left(\frac{a^{-1}+b^{-1}}{|a^{-1}-b^{-1}|}\cdot\arcsin\left(\frac{|a^{-1}-b^{-1}|}{a^{-1}+b^{-1}}\right)\right)+1\right)\right)\left(a^{-1}\sharp b^{-1}\right) \\
&= \left(\frac{1}{2}\left(\left(\frac{a\nabla b}{a\sharp b}\right)\left(\frac{a+b}{|a-b|}\cdot\arcsin\left(\frac{|a-b|}{a+b}\right)\right)+1\right)\right)(a\sharp b)^{-1} \\
&\leq a^{-1}\nabla b^{-1},
\end{aligned}$$

i.e.,

$$(a\sharp b)^{-1} \leq \left(\frac{1}{2}\left(\left(\frac{a\nabla b}{a\sharp b}\right)\left(\frac{a+b}{|a-b|}\cdot\arcsin\left(\frac{|a-b|}{a+b}\right)\right)+1\right)\right)(a\sharp b)^{-1} \leq a^{-1}\nabla b^{-1}.$$

Now, by taking the inverse, we reach the desired inequality. ■

As an immediate product of Theorem 2.2, we introduce an improvement of the Young inequality (2).

Corollary 3.3 Let $a, b > 0$ with $a \neq b$ and $0 \leq \nu \leq 1$. Then

$$\begin{aligned}
a\sharp_\nu b &\leq \frac{a\sharp_\nu b - a\nabla_\nu b}{\log(a\sharp_\nu b) - \log(a\nabla_\nu b)} \\
&\leq \frac{a\nabla_{(1-t)\nu+t} b - a\nabla_{(1-t)\nu} b}{\log\left(a\nabla_{(1-t)\nu+t} b\right) - \log\left(a\nabla_{(1-t)\nu} b\right)} \\
&\leq a\nabla_\nu b.
\end{aligned}$$

Proof. It observes from Theorem 2.2 that

$$\begin{aligned}
a\sharp_\nu b &\leq ((a\nabla_\nu b)\sharp_t a)\sharp_\nu((a\nabla_\nu b)\sharp_t b) \\
&\leq ((a\nabla_\nu b)\nabla_t a)\sharp_\nu((a\nabla_\nu b)\nabla_t b) \\
&\leq a\nabla_\nu b.
\end{aligned} \tag{7}$$

Notice that

$$\begin{aligned}
((a\nabla_\nu b)\nabla_t a)\sharp_\nu((a\nabla_\nu b)\nabla_t b) &= ((a\nabla_\nu b)\nabla_t(a\nabla_0 b))\sharp_\nu((a\nabla_\nu b)\nabla_t(a\nabla_1 b)) \\
&= \left(a\nabla_{(1-t)\nu} b\right)\sharp_\nu\left(a\nabla_{(1-t)\nu+t} b\right).
\end{aligned}$$

On the other hand,

$$\int_0^1 \left(a\nabla_{(1-t)\nu}b\right)\sharp_\nu\left(a\nabla_{(1-t)\nu+t}b\right)dt=\frac{a\nabla_{(1-t)\nu+t}b-a\nabla_{(1-t)\nu}b}{\log\left(a\nabla_{(1-t)\nu+t}b\right)-\log\left(a\nabla_{(1-t)\nu}b\right)}$$

$$\int_0^1 \left(\left(\left(a\nabla_\nu b\right)\sharp_t a\right)\sharp_\nu\left(\left(a\nabla_\nu b\right)\sharp_t b\right)\right)dt=\int_0^1\left(\left(a\nabla_\nu b\right)\sharp_t\left(a\sharp_\nu b\right)\right)dt=\frac{a\sharp_\nu b-a\nabla_\nu b}{\log\left(a\sharp_\nu b\right)-\log\left(a\nabla_\nu b\right)},$$

due to (3). For these reasons,

$$a\sharp_\nu b\le\frac{a\sharp_\nu b-a\nabla_\nu b}{\log\left(a\sharp_\nu b\right)-\log\left(a\nabla_\nu b\right)}\le\frac{a\nabla_{(1-t)\nu+t}b-a\nabla_{(1-t)\nu}b}{\log\left(a\nabla_{(1-t)\nu+t}b\right)-\log\left(a\nabla_{(1-t)\nu}b\right)}\le a\nabla_\nu b,$$

as desired. ■

Remark 4 *We consider λ-logarithmic function $\ln_\lambda x=\frac{x^\lambda-1}{\lambda}$ for $x,\lambda>0$. Notice that $\ln_\lambda x\le x-1$ for $0<\lambda\le 1$ [14, Lemma 1]. This inequality can be improved by Theorem 2.2. It follows from (7) that*

$$x^\lambda\le\left(1-\lambda+\lambda x\right)\sharp_t x^\lambda\le\left(\left(1-\lambda+\lambda x\right)\nabla_t 1\right)\sharp_\lambda\left(\left(1-\lambda+\lambda x\right)\nabla_t x\right)\le 1-\lambda+\lambda x$$

for $0<\lambda\le 1$ and $0\le t\le 1$. Whence

$$\ln_\lambda x\le\frac{\left(1-\lambda+\lambda x\right)\sharp_t x^\lambda-1}{\lambda}\le\frac{\left(\left(1-\lambda+\lambda x\right)\nabla_t 1\right)\sharp_\lambda\left(\left(1-\lambda+\lambda x\right)\nabla_t x\right)-1}{\lambda}\le x-1.$$

4. Further improvements

We understand that if $f:I\to\mathbb{R}$ is a concave function, then

$$f\left(a\right)\nabla_\nu f\left(b\right)\le f\left(a\nabla_\nu b\right)-2r\left(f\left(a\nabla b\right)-f\left(a\right)\nabla f\left(b\right)\right)\tag{8}$$

$$f\left(a\nabla_\nu b\right)-2R\left(f\left(a\nabla b\right)-f\left(a\right)\nabla f\left(b\right)\right)\le f\left(a\right)\nabla_\nu f\left(b\right)\tag{9}$$

for any $a,b\in I$, where $r=\min\{\nu,1-\nu\}$, $R=\max\{\nu,1-\nu\}$, and $0\le\nu\le 1$ [1]. We initiate this subsection by improving the first inequality in Theorem 2.1.

Theorem 4.1 Let $f:\mathbb{R}\to(0,\infty)$ be a log-concave function. Then

$$\frac{f\left(a\nabla b\right)-f\left(a\right)\sharp f\left(b\right)}{\ln\left(\frac{f(a\nabla b)}{f(a)\sharp f(b)}\right)^{f(a)\sharp f(b)}}\cdot\left(f\left(a\right)\sharp f\left(b\right)\right)\le\int_0^1\left(f\left(a\nabla_t b\right)\sharp f\left(b\nabla_t a\right)\right)dt\le f\left(a\nabla b\right).$$

Proof. As stated overhead, the function $f:\mathbb{R}\to(0,\infty)$ is log-concave if $\log f$ is concave. Consequently, from the inequality (8), we acquire

$$\log\left(f\left(a\right)\sharp_t f\left(b\right)\right)\le\log\frac{f\left(a\nabla_t b\right)}{\left(\frac{f(a\nabla b)}{f(a)\sharp f(b)}\right)^{2r}},\tag{10}$$

where $r = \min\{t, 1-t\}$ and $0 \le t \le 1$. Applying exp from both sides of (10) gives

$$f(a)\sharp_t f(b) \le \left(\frac{f(a)\sharp f(b)}{f(a\nabla b)}\right)^{2r} f(a\nabla_t b). \tag{11}$$

From the inequality (11), we have

$$\begin{aligned} f(a\nabla b) &= f((a\nabla_t b)\nabla(b\nabla_t a)) \\ &\ge f(a\nabla_t b)\sharp f(b\nabla_t a) \\ &\ge \left(\frac{f(a\nabla b)}{f(a)\sharp f(b)}\right)^{2r}((f(a)\sharp_t f(b))\sharp(f(b)\sharp_t f(a))) \\ &= \left(\frac{f(a\nabla b)}{f(a)\sharp f(b)}\right)^{2r}(f(a)\sharp f(b)) \end{aligned}$$

i.e.,

$$\left(\frac{f(a\nabla b)}{f(a)\sharp f(b)}\right)^{1-|2t-1|}(f(a)\sharp f(b)) \le f(a\nabla_t b)\sharp f(b\nabla_t a) \le f(a\nabla b). \tag{12}$$

Here we used the fact that $\min(a,b) = \frac{a+b-|a-b|}{2}$ for $a, b \in \mathbb{R}$. If we take the integral over $0 \le t \le 1$ in the inequality (12), we conclude that

$$\int_0^1 \left(\frac{f(a\nabla b)}{f(a)\sharp f(b)}\right)^{1-|2t-1|} dt\,(f(a)\sharp f(b)) \le \int_0^1 (f(a\nabla_t b)\sharp f(b\nabla_t a))\,dt \le f(a\nabla b). \tag{13}$$

Since

$$\begin{aligned} \int_0^1 \left(\frac{f(a\nabla b)}{f(a)\sharp f(b)}\right)^{1-|2t-1|} dt &= \frac{\frac{f(a\nabla b)}{f(a)\sharp f(b)} - 1}{\ln\left(\frac{f(a\nabla b)}{f(a)\sharp f(b)}\right)} \\ &= \frac{f(a\nabla b) - f(a)\sharp f(b)}{(f(a)\sharp f(b))\ln\left(\frac{f(a\nabla b)}{f(a)\sharp f(b)}\right)} \\ &= \frac{f(a\nabla b) - f(a)\sharp f(b)}{\ln\left(\frac{f(a\nabla b)}{f(a)\sharp f(b)}\right)^{f(a)\sharp f(b)}}, \end{aligned}$$

we can write (13) in the following arrangement

$$\frac{f(a\nabla b) - f(a)\sharp f(b)}{\ln\left(\frac{f(a\nabla b)}{f(a)\sharp f(b)}\right)^{f(a)\sharp f(b)}} \cdot (f(a)\sharp f(b)) \le \int_0^1 (f(a\nabla_t b)\sharp f(b\nabla_t a))\,dt \le f(a\nabla b).$$

This concludes the proof of the theorem. ∎

Remark 5 *Note that,* $\frac{u-1}{\ln u} \geq 1$ *when* $u \geq 1$. *This signifies that*

$$\frac{f(a\nabla b) - f(a)\sharp f(b)}{\ln\left(\frac{f(a\nabla b)}{f(a)\sharp f(b)}\right)^{f(a)\sharp f(b)}} \geq 1.$$

The reverse of the first inequality in Theorem 2.1 is shown in the next result.

Theorem 4.2 Let $f : \mathbb{R} \to (0, \infty)$ be a log-concave function. Then

$$\frac{\ln\left(\frac{f(a\nabla b)}{f(a)\sharp f(b)}\right)^{\frac{(f(a)\sharp f(b))^2}{f(a\nabla b)}}}{f(a\nabla b) - f(a)\sharp f(b)} \int_0^1 (f(a\nabla_t b)\sharp f(b\nabla_t a))\, dt \leq f(a)\sharp f(b).$$

Proof. We can write from (9) that

$$\log \frac{f(a\nabla_t b)}{\left(\frac{f(a\nabla b)}{f(a)\sharp f(b)}\right)^{2R}} \leq \log\left(f(a)\sharp_t f(b)\right),$$

where $R = \max\{t, 1-t\}$ and $0 \leq t \leq 1$. Consequently,

$$\left(\frac{f(a)\sharp f(b)}{f(a\nabla b)}\right)^{2R} \cdot f(a\nabla_t b) \leq f(a)\sharp_t f(b). \tag{14}$$

From the inequality (14), we arrive

$$f(a)\sharp f(b) = (f(a)\sharp_t f(b))\sharp(f(b)\sharp_t f(a)) \geq \left(\frac{f(a)\sharp f(b)}{f(a\nabla b)}\right)^{2R}(f(a\nabla_t b)\sharp f(b\nabla_t a))$$

or

$$f(a\nabla_t b)\sharp f(b\nabla_t a) \leq \left(\frac{f(a\nabla b)}{f(a)\sharp f(b)}\right)^{1+|2t-1|}(f(a)\sharp f(b)) \tag{15}$$

since $\max(a,b) = \frac{a+b+|a-b|}{2}$ for $a, b \in \mathbb{R}$. Taking integral over $0 \leq t \leq 1$ in the inequality (15), we receive

$$\int_0^1 (f(a\nabla_t b)\sharp f(b\nabla_t a))\, dt \leq \int_0^1 \left(\frac{f(a\nabla b)}{f(a)\sharp f(b)}\right)^{1+|2t-1|} dt\,(f(a)\sharp f(b))$$

due to

$$\int_0^1 \left(\frac{f(a\nabla b)}{f(a)\sharp f(b)}\right)^{1+|2t-1|} dt = \frac{\frac{f(a\nabla b)}{f(a)\sharp f(b)}\left(\frac{f(a\nabla b)}{f(a)\sharp f(b)}-1\right)}{\ln\left(\frac{f(a\nabla b)}{f(a)\sharp f(b)}\right)} = \frac{\frac{f(a\nabla b)}{(f(a)\sharp f(b))^2}\left(f(a\nabla b)-f(a)\sharp f(b)\right)}{\ln\left(\frac{f(a\nabla b)}{f(a)\sharp f(b)}\right)}$$

$$= \frac{f(a\nabla b)-f(a)\sharp f(b)}{\frac{(f(a)\sharp f(b))^2}{f(a\nabla b)}\ln\left(\frac{f(a\nabla b)}{f(a)\sharp f(b)}\right)} = \frac{f(a\nabla b)-f(a)\sharp f(b)}{\ln\left(\frac{f(a\nabla b)}{f(a)\sharp f(b)}\right)^{\frac{(f(a)\sharp f(b))^2}{f(a\nabla b)}}}.$$

■

Remark 6 *Notice that* $\frac{(u-1)u}{\ln u} \geq 1$ *for* $u \geq 1$*. Accordingly, Theorem 4.2 provides a reverse for the first inequality in Theorem 2.1.*

By [1, (1.4)], we know that if f is a non-negative log-concave function and a, b are in the domain of f, then

$$\sqrt{f(a)\,f(b)} \leq \exp\left(\frac{1}{b-a}\int_a^b \log f(t)\,dt\right) \leq f\left(\frac{a+b}{2}\right).$$

From the above inequality, we have

$$\sqrt{f(0)\,f(1)} \leq \exp\left(\int_0^1 \log f(t)\,dt\right) \leq f\left(\frac{1}{2}\right). \tag{16}$$

Notice that if f is a log-concave on $[0,1]$, then $f(t) = g((1-t)a + tb)$ is also log-concave on $[0,1]$. Indeed,

$$\begin{aligned} f\left(\frac{s+t}{2}\right) &= g\left(\left(1-\frac{s+t}{2}\right)a + \frac{s+t}{2}b\right) \\ &= g\left(\frac{(1-s)a + sb + (1-t)a + tb}{2}\right) \\ &\geq \sqrt{g((1-s)a+sb)\,g((1-t)a+tb)} \\ &= \sqrt{f(s)\,f(t)}. \end{aligned}$$

Thus, from (16), we get

$$\sqrt{g(a)\,g(b)} \leq \exp\left(\int_0^1 \log g(a\nabla_t b)\,dt\right) \leq g\left(\frac{a+b}{2}\right).$$

We give refinement and a reverse for the above mentioned first inequality in the following result.

Theorem 4.3 Let $f : \mathbb{R} \to (0, \infty)$ be a log-concave function. Then

$$\sqrt{f(a)\, f(b)} \sqrt{\frac{f\left(\frac{a+b}{2}\right)}{\sqrt{f(a)\, f(b)}}} \le \exp\left(\int_0^1 \log f(a \nabla_t b)\, dt\right),$$

and

$$\exp\left(\int_0^1 \log f(a \nabla_t b)\, dt\right) \le \left(\frac{f\left(\frac{a+b}{2}\right)}{\sqrt{f(a)\, f(b)}}\right)^{\frac{3}{2}} \sqrt{f(a)\, f(b)}.$$

Proof. Since $2\min\{t, 1-t\} = 1 - |2t-1|$, from (8), we get

$$\frac{f(a) + f(b)}{2} \le \int_0^1 f(a \nabla_t b)\, dt - \frac{1}{2}\left(f\left(\frac{a+b}{2}\right) - \frac{f(a) + f(b)}{2}\right).$$

Furthermore, due to the fact that $2\max\{t, 1-t\} = 1 + |2t-1|$, we can write from (9) that

$$\int_0^1 f(a \nabla_t b)\, dt - \frac{3}{2}\left(f\left(\frac{a+b}{2}\right) - \frac{f(a) + f(b)}{2}\right) \le \frac{f(a) + f(b)}{2}.$$

If f is log-concave, then $\log f$ is also concave. Thus, from the two inequalities in the above, we reach

$$\log \sqrt{\frac{f\left(\frac{a+b}{2}\right)}{\sqrt{f(a)\, f(b)}}} \sqrt{f(a)\, f(b)} \le \int_0^1 \log f(a \nabla_t b)\, dt,$$

and

$$\int_0^1 \log f(a \nabla_t b)\, dt \le \log \left(\frac{f\left(\frac{a+b}{2}\right)}{\sqrt{f(a)\, f(b)}}\right)^{\frac{3}{2}} \sqrt{f(a)\, f(b)}.$$

Taking the exponential gives the desired inequalities. ■

Remark 7 *One can see that for concave function f and for any $0 \le t, \nu \le 1$,*

$$\begin{aligned} f(a)\, \nabla_\nu f(b) &\le (f(a \nabla_\nu b)\, \nabla_t f(a))\, \nabla_\nu\, (f(a \nabla_\nu b)\, \nabla_t f(b)) \\ &\le f((a \nabla_\nu b)\, \nabla_t a)\, \nabla_\nu f((a \nabla_\nu b)\, \nabla_t b) \\ &\le f(a \nabla_\nu b)\,. \end{aligned}$$

Taking integral over $0 \le t \le 1$*, we get*

$$\begin{aligned} f(a)\nabla_\nu f(b) &\le \left(\int_0^1 \left(f(a\nabla_\nu b)\nabla_t f(a)\right)dt\right)\nabla_\nu\left(\int_0^1 \left(f(a\nabla_\nu b)\nabla_t f(b)\right)dt\right)\\ &\le \left(\int_0^1 f\left((a\nabla_\nu b)\nabla_t a\right)dt\right)\nabla_\nu\left(\int_0^1 f\left((a\nabla_\nu b)\nabla_t b\right)dt\right)\\ &\le f(a\nabla_\nu b). \end{aligned}$$

If f *is log-concave, we have*

$$\begin{aligned} \log\left(f(a)\sharp_\nu f(b)\right) &\le \left(\int_0^1 \left(\log\left(f(a\nabla_\nu b)\sharp_t f(a)\right)\right)dt\right)\nabla_\nu\left(\int_0^1 \left(\log\left(f(a\nabla_\nu b)\sharp_t f(b)\right)\right)dt\right)\\ &\le \left(\int_0^1 \log f\left((a\nabla_\nu b)\nabla_t a\right)dt\right)\nabla_\nu\left(\int_0^1 \log f\left((a\nabla_\nu b)\nabla_t b\right)dt\right)\\ &\le \log f(a\nabla_\nu b), \end{aligned}$$

which easily implies that

$$\begin{aligned} f(a)\sharp_\nu f(b) &\le \exp\left(\left(\int_0^1 \left(\log\left(f(a\nabla_\nu b)\sharp_t f(a)\right)\right)dt\right)\nabla_\nu\left(\int_0^1 \left(\log\left(f(a\nabla_\nu b)\sharp_t f(b)\right)\right)dt\right)\right)\\ &\le \exp\left(\left(\int_0^1 \log f\left((a\nabla_\nu b)\nabla_t a\right)dt\right)\nabla_\nu\left(\int_0^1 \log f\left((a\nabla_\nu b)\nabla_t b\right)dt\right)\right)\\ &\le f(a\nabla_\nu b). \end{aligned}$$

Conclusion

In this research, we have demonstrated the versatility and power of logarithmically concave functions in establishing refined versions of the arithmetic-geometric means inequality. Our results further highlight the depth and breadth of applications for these functions in various mathematical inequalities. The similarities between the inequalities for logarithmically concave functions and the Hermite-Hadamard inequality suggest promising new avenues for exploration and future research. Our findings contribute to the ongoing development and enrichment of mathematical inequalities theory.

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