Original Research

A Copula-Driven Framework for Optimizing the Performance of a Serial System with Inbuilt Redundancy

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Abstract

This study presents a new Copula-Driven Framework (CDF) aimed at improving the reliability and performance of serial system consisting of three subsystems each comprises of n units, which are essential to many industrial processes. The system is exposed to partial failure which is restored by general repair and completer or total failure which is restored by copula. Failures are assumed to follow an exponential distribution, whereas repair is assumed to follow one of the general or Gumbel-Hougaard family copulas. The system is investigated using supplementary variable and the Laplace transform. Profit analysis was derived by utilizing dependability parameters such as availability, reliability, and Mean Time to Failure (MTTF). The system has been investigated throughout the project. The computed results were displayed in a graph, and the value of the analysis was conveyed in the conclusion section.

Keywords- Reliability; Maintenance; Serial; Redundancy.

Introduction

In the realm of industrial processes, especially those involving complex systems like serial water condenser systems, the quest for enhanced reliability and performance optimization has become an imperative. These systems play a pivotal role in various industries, including power generation, chemical processing, and environmental control. The ability to ensure their reliability and optimize their performance is essential for operational efficiency, safety, and cost-effectiveness. Serial water condenser systems are designed to transfer heat from one fluid to another, often using a series of heat exchangers. Such systems are commonly employed in power plants to condense steam back into water, allowing for the reuse of valuable resources and the efficient operation of the power generation cycle. However, these systems are susceptible to various sources of uncertainty, including variations in operating conditions, material properties, and environmental factors. Addressing these uncertainties and achieving both reliability and performance targets in a synergistic manner presents a formidable challenge. Reliability and performance optimization of serial systems with inbuilt redundancy is a critical topic in engineering and operations management. Serial systems, characterized by their sequential arrangement of components, are prevalent in many industrial and commercial applications, including cooling systems, manufacturing lines, and communication networks. While their sequential configuration ensures specific operational workflows, it also poses significant challenges. A failure in any single component can lead to system-wide downtime, reducing overall reliability and performance. Incorporating redundancy—where additional components are included to take over in case of failure—offers a promising solution to mitigate these risks. However, optimizing the performance of such systems requires careful consideration of the interplay between reliability, cost, and operational efficiency. The primary challenge in optimizing serial systems with redundancy lies in accurately modelling the dependencies among components. Failures in real-world systems are rarely independent; instead, they are often correlated due to shared operational environments, similar failure mechanisms, or interactions between components. Traditional methods that assume independence among components may oversimplify these relationships, leading to suboptimal designs and performance predictions. A more sophisticated approach is needed to account for such interdependencies, especially when considering the impact of redundancy on system reliability and cost-effectiveness. Due to this rationale, various researchers have undertaken analyses of the performance of industrial systems employing diverse methodologies. Notable among these approaches are; Zhao et. al. (2021) looked into the analysis and improvement of the economic efficiency of a cold standby system. This system is prone to shocks and experiences imperfect repairs. The researchers introduce geometric process models as a method to precisely measure both the system's lifespan and the duration needed for repairs. This study significantly contributes to understanding and enhancing the economic aspects of cold standby systems, particularly in the presence of -shocks and imperfect repair scenarios. In their (2021) study, Xie and others explored and assessed the functionality of a safety system susceptible to cascading failures leading to the emergence of subsequent failures. The paper introduces an innovative approach for mitigating and preempting cascading failures. Zhang et. al. (2022) put forward an innovative condition-based activation strategy hinging on unit utilization. This strategy is designed to withstand potential shocks anticipated in the forthcoming testing interval, with the primary aim of evaluating the overall performance of the system. Yemane and Colledani (2019) presented an approach for assessing the performance of manufacturing systems characterized by instability. Their method incorporates considerations of unknown machine reliability predictions.

Niwas and Garg (2018) introduced a methodology cantered around a cost-free warranty policy, providing a framework for assessing both the reliability and profitability of industrial systems. This innovative approach not only addresses the traditional reliability aspect but also incorporates considerations of economic viability, thereby offering a comprehensive evaluation tool for industrial system performance. Ye et al. (2020) presented alternative failure models to assess the performance of serial structured, automated manufacturing systems with inadequate quality control. Chen et al. (2019) employed machine reliability as a pivotal factor for evaluating the operational status of a machine. Subsequently, they devised an integrated production scheduling model that incorporates machine reliability, offering decision-makers a valuable tool for identifying and implementing optimal scheduling strategies. Zhang et al. (2022) conducted an in-depth analysis of maintenance strategies for a k-out-of-n system characterized by partial observability and incorporated load-sharing units. The study offers a detailed examination of the maintenance considerations essential for optimizing the reliability and performance of such systems. Gokhan et al. (2022) provided a time-dependent reliability analysis specifically tailored for a repairable consecutive k-out-of-n: F system. The study examines the temporal aspects of reliability, offering insights into how the system's repairable nature influences its performance over time. Hu et al. (2022) developed a model to evaluate the reliability of cold standby k-out-of-m+n: G systems under conditions of uncertainty, specifically focusing on uncertain parameters that significantly impact the system's overall reliability. Through their study, the authors not only introduced a novel modeling approach but also conducted a comprehensive assessment that takes into account the inherent uncertainty associated with the system's parameters. Advancements in statistical modeling have introduced tools to address these challenges. Among these, copula functions provide a robust framework for capturing the dependency structure among components. Copulas allow for flexible modeling of joint failure behaviors, independent of marginal distributions, making them particularly well-suited for systems with complex failure correlations. By integrating copula-based methods into the optimization of serial systems, it becomes possible to develop more accurate and effective strategies for redundancy allocation and system design. The importance of optimizing serial systems with inbuilt redundancy extends beyond theoretical interests. In industrial applications, efficient and reliable system performance directly impacts operational costs, energy consumption, and service quality. For example, in serial chiller water systems—critical for temperature regulation in data centers and industrial processes—failures can lead to significant economic losses and disruptions. Developing methods to enhance reliability and performance in such systems contributes to the broader goals of sustainability and resilience, aligning with industry and environmental priorities.

Numerous scholars have explored repairable systems, suggesting methods to enhance reliability and contribute to fortifying complex systems through the application of Copula. Some noteworthy contributions include; Rawal et al. (2022) investigated a multi-computer system comprising n clients, employing a k-out-of-n configuration for reliability assessment. Their study focused on the G operation strategy combined with a copula repair strategy. In their study, Tyagi et al. (2021) introduced and established innovative fault coverage techniques based on Copula models specifically designed for the analysis of repairable parallel systems. This research contributes to the advancement of methods for evaluating and understanding the fault tolerance and reliability of parallel systems through the application of Copula-based approaches. Zhang et al. (2017) conducted a comprehensive study where they not only modeled but also assessed the performance of a multistage serial manufacturing system. Their analysis went beyond traditional considerations by incorporating factors such as rework and product polymorphism. By addressing these complexities, the research contributes valuable insights into the dynamics of manufacturing processes, providing a more holistic understanding of system behavior and performance in real-world scenarios. In their recent work, Singh et al. (2022) introduced a novel Copula linguistic technique designed to analyze the performance and efficacy of a redundant k-out-of-n: G system, considering multiple sequential state degradations. This innovative approach not only enhances the analytical capabilities for evaluating system reliability but also offers a linguistically-oriented perspective, contributing to a more detailed understanding of the complex dynamics associated with the degradation processes in redundant systems.

Bisht et al. (2020) looked into the characteristics of industrial systems, particularly focusing on repair processes, utilizing copula models. Their research expanded the understanding of how copula methodologies can effectively capture and explain the dynamics within industrial systems, shedding light on the details of repair procedures. By employing copula techniques, their study not only highlighted the features of these systems but also contributed to enhancing methodologies for analyzing and optimizing repair strategies in industrial settings. Isa et al. (2022) looked into the examination of the Gumbel-Hougaard Family Copula, utilizing it as a crucial tool in the comprehensive analysis of the reliability of a multi-workstation computer network organized in a series-parallel configuration. Serial chiller water systems are widely used in industrial and commercial applications, where maintaining optimal cooling performance is critical. However, these systems are prone to failures due to their sequential configuration, leading to reduced reliability and efficiency. Incorporating redundancy can enhance system reliability, but determining how to optimize performance while accounting for correlated failures and interdependencies among system components remains a significant challenge. Existing methods often fail to adequately capture the complex dependencies between components and their impact on overall system performance. Optimizing the performance of serial chiller water systems with inbuilt redundancy is essential for improving reliability, energy efficiency, and cost-effectiveness. By addressing the interdependencies between system components, this research contributes to developing sustainable and resilient cooling systems, which are critical in industries where temperature control is vital. The framework can also aid in the design and maintenance of such systems, reducing downtime and operational costs. The remainder of this paper is organized as follows: Section 2 reviews existing approaches to redundancy optimization, reliability modelling and performance optimization of serial systems. Section 3 presents the description of the system. Section 4 introduced the methodology of the study. Section 5 introduces the proposed copula-driven framework and its mathematical foundations. Section 6 presents case studies or simulations to validate the framework's applicability. Finally, Section 7 discusses the implications of the findings and suggests directions for future research, and concluding remark.

LITERATURE REVIEW

To enhance the reliability, performance, and overall efficiency of the system, redundancy optimization is regarded as an essential and strategic approach. Redundancy involves the deliberate inclusion of additional components, subsystems, or pathways within the system architecture, which act as backups or fail-safes in case of a malfunction or breakdown in the primary components. By providing multiple layers of functional duplication, redundancy ensures that the system can continue operating smoothly even if one or more elements fail, thus reducing the risk of total system failure. This method not only increases the likelihood of sustained operation under a wide range of adverse conditions, including hardware failures, unexpected surges in workload, or environmental stresses, but also bolsters the system's fault tolerance. Fault tolerance refers to the system's ability to withstand and manage failures without significant degradation in performance. By incorporating redundancy, the system becomes more resilient to disruptions, maintaining critical functionalities and minimizing downtime. In addition to improving reliability and fault tolerance, redundancy optimization also contributes to greater system resilience in enhancing the reliability and performance of the serial water condenser systems. Redundancy provides the necessary infrastructure for rapid recovery, minimizing the impact of failures on the overall system. Ultimately, redundancy optimization plays a crucial role in aligning system design with long-term reliability goals. By increasing robustness through redundancy optimization, system architects can ensure that the system meets stringent reliability benchmarks, operates at high performance levels, and maintains overall operational efficiency. This approach is particularly valuable in mission-critical applications, where system failures can lead to costly downtime, operational disruptions, or even catastrophic outcomes. Therefore, redundancy optimization is a vital component in achieving system resilience, reliability, and performance sustainability.

Literature on redundancy optimization are numerous. To cite a few, Khalili-Damghani et al. (2013) introduced a novel approach using particle swarm optimization (PSO) to tackle binary-state reliability redundancy allocation and optimization problems. Their method effectively combines penalty functions with modification strategies to enhance the optimization process, providing a robust solution to complex system reliability challenges. Khalili-Damghani and Amiri (2012) address the binary-state reliability redundancy allocation problem by employing a combined approach based on the epsilon-constraint method and data envelopment analysis (DEA). Their research provides a comprehensive framework for optimizing system reliability while balancing multiple conflicting objectives, leveraging the strengths of both techniques to achieve efficient and effective redundancy allocation. Attar et al. (2017) introduced a simulation-based optimization technique to address the complex problem of multi-dimensional availability redundancy optimization and allocation in series-parallel systems. In their study, system availability is estimated through a simulation-driven approach, with optimization achieved using both the Non-Dominated Sorting Genetic Algorithm (NSGA) and the Strength Pareto Evolutionary Algorithm (SPEA). This dual-method approach provides a robust framework for solving availability optimization challenges while efficiently allocating redundancy in highly intricate systems. Dolatshahi-Zand and Khalili-Damghani (2015) developed a comprehensive bi-objective redundancy allocation model specifically designed for Tehran's SCADA water resource management control center, aimed at maximizing system reliability while minimizing operational costs. Their approach addresses the critical trade-off between enhancing system reliability and controlling budgetary constraints, both of which are crucial for the efficient management of water resources. To solve this complex optimization problem, they applied a multi-objective particle swarm optimization (MOPSO) algorithm, offering an effective solution for achieving optimal redundancy allocation in a highly sensitive and resource-critical infrastructure. Their work contributes significantly to the field of reliability engineering and water resource management by providing a balanced, cost-efficient method to enhance system resilience.

Khalili-Damghani et al. (2014) proposed a decision support system aimed at addressing the multi-objective redundancy allocation problem for series-parallel systems. They introduced an efficient ε-constraint method to generate non-dominated solutions on the Pareto front. Zand et al. (2022) investigated the redundancy allocation problem in the SCADA (Supervisory Control and Data Acquisition) systems of reservoir stations within water transfer networks. Their study developed a mathematical model utilizing redundancy policies and reliability block diagrams, which was effectively solved using a customized hybrid approach combining dynamic NSGA-II (Non-dominated Sorting Genetic Algorithm II) with a multi-objective particle swarm optimization algorithm. Nobakhti et al. (2022) presented a novel methodology for developing a system reliability response surface based on varying operating pressures and temperatures. This approach integrates a hybrid fault tree with a fuzzy inference system. Similarly, Nobakhti et al. (2020) introduced an earlier approach that also utilized a hybrid of fault tree analysis and fuzzy inference systems to develop a reliability surface for systems under different operating pressures and temperatures. The previously reviewed literature above provided research findings regarding redundancy optimization, performance and reliability analysis in particular sequential systems, highlighting improvements in system efficiency using various techniques. However, there is a clear gap in the field of serial water systems consisting of three subsystem in which each of the subsystem consists of n identical units working as k-out-of-n redundancy, particularly when it comes to studies that use copula-based methods. This disparity stands out even more in light of the lack of thorough analyses that address performance and reliability issues in the context of serial water systems. Therefore, the current study aims to bridge the existing research gap by using copula-based techniques to conduct a thorough analysis of performance and reliability in serial water systems consisting of three subsystem in which each of the subsystem consists of n identical units working as k-out-of-n redundancy. This study aims to address these challenges by proposing a copula-driven framework for optimizing the performance of serial systems with inbuilt redundancy. The framework leverages copula models to capture dependencies among system components and integrates this understanding into the optimization process. By balancing reliability, cost, and operational constraints, the framework provides practical solutions for designing and maintaining robust serial systems. The framework:

- Captures correlated failures among components using copula functions.
- Optimizes system performance by balancing reliability, cost, and operational efficiency.
- Provides practical guidelines for designing and maintaining serial chiller water systems with redundancy to achieve sustainable and resilient operations.

While redundancy optimization focuses on ensuring reliability through the duplication of components, general and copula repair mechanisms emphasize strategic interventions and resource efficiency. By addressing both individual and correlated failures directly, these approaches can enhance system reliability and performance more sustainably and cost-effectively. Most existing literature studies focus on redundancy optimization as a primary strategy for improving the reliability of systems. This approach, which involves adding extra components or subsystems to serve as backups, has been widely recognized for its ability to mitigate failures and ensure continuous system operation. However, while redundancy optimization has been extensively analysed and its benefits well-documented, these studies often overlook the potential advantages of alternative strategies, such as general and copula repair mechanisms. In particular, the role of general and copula repair in addressing both unit-level and system-wide failures remains underexplored. These repair approaches can significantly enhance the reliability and performance of systems by directly addressing the root causes of failures and the dependencies between them. Unlike redundancy, which primarily prevents failure through duplication, repair mechanisms restore functionality post-failure and offer a dynamic means of maintaining system performance. The limited focus on these innovative repair strategies in the existing literature leaves a critical gap, underscoring the need for further research to evaluate their impact on system reliability and efficiency comprehensively. Carrying out general and copula repair mechanisms at both the unit and system failure levels offers several advantages over redundancy optimization for enhancing reliability and performance due to the following reasons:

- Repair mechanisms address specific failures rather than duplicating components, which is the focus of redundancy optimization. This leads to more efficient use of resources.
- General repair often requires fewer resources compared to adding redundant components, resulting in reduced capital and operational expenditures.
- General repairs can adapt to different failure distributions and rates, offering more versatile solutions compared to fixed redundancy schemes.
- Copulas allow modelling the dependencies between unit failures, leading to a more accurate assessment of reliability and tailored repair strategies.
- Less energy is consumed in repairing systems compared to maintaining large-scale redundancies.
- Copula models provide insights into correlated failures, helping to prevent cascading failures that might otherwise require significant redundancy to counteract.

First and foremost, the main objective is to thoroughly assess the serial water system, paying particular attention to important elements like dependability, resilience, and reliability. Critical metrics such as mean time to failure (MTTF) and sensitivity analysis of MTTF are also included in the assessment. Secondly, the paper attempts to7 give a thorough overview of how well these systems function in real-world agricultural scenarios by closely examining these performance metrics. The third goal presents modelling the complex interdependencies between different components impacting the efficiency and dependability of serial water system using the copula methodology. This statistical method provides insights that go beyond simple correlations and enables a more nuanced understanding of the relationships between various factors. Fourthly, the goal of the paper is to create a useful framework by combining the results of the copula analysis. The goal of this framework is to maximize serial water system design and performance. Through the application of theory to practical methods, the research advances the overall effectiveness and dependability of these systems. The key contribution of the present paper lies in the development of a Copula-Driven Framework (CDF) to enhance the reliability and performance of serial systems. This framework addresses the limitations of traditional methods by incorporating component interdependencies into the reliability analysis, rather than treating system components as independent. Using Copula theory, the framework models correlations between critical components, revealing hidden relationships that affect system performance and reliability. The CDF also provides a basis for identifying optimal repair policies, offering a more comprehensive and accurate approach to reliability assessment and system optimization. This contribution is significant because it introduces a holistic methodology that better captures the complexities of real-world systems.

Description of the Model

A condenser water system is crucial in many cooling setups, commonly found in air conditioning systems or industrial processes. It works to transfer heat from one place to another efficiently. This system works in a continuous loop where the water constantly cycles between the chiller and the cooling tower, allowing the chiller to remove heat from the space it's cooling and the cooling tower to dissipate that heat into the atmosphere. It's an effective way to maintain a consistent cooling process in various settings. A water condenser system is a key component in various industries, particularly in thermal power plants and refrigeration systems, where it's used to convert vapor into liquid. The water condenser system is designed to efficiently and effectively remove heat from the vapor, turning it into a liquid state. This is crucial in various applications, such as power generation or refrigeration, where the efficient conversion of steam or vapor to liquid is essential for the proper functioning of the system. A condenser water system is an integral part of large-scale air conditioning or refrigeration setups, often found in commercial buildings or industrial facilities. Its primary function is to remove heat from space by using water as a medium for heat exchange. The condenser water system described in Figure 1 is structured as a sequential system, featuring three interdependent subsystems: the cooling tower, condenser water pump, and water-cooled chiller. Within each of these subsystems, multiple units are operating in active parallel. The system's overall functionality is contingent on the collective performance of these units across the subsystems. Suppose any of these subsystems fail due to the failure of all "n" units within that subsystem or due to a common cause affecting all units simultaneously. In that case, it fails in the entire condenser water system. This system typically consists of several key components:

Subsystem A (Cooling Tower): This part of the system helps to remove heat from the condenser water. Warm water from the chiller, which has absorbed heat, is pumped to the cooling tower. The tower exposes the water to air, causing a portion of it to evaporate, effectively removing heat from the remaining water. The cooled water is then returned to the chiller to absorb more heat. Cooling towers can be of various types such as crossflow or counterflow, and they come in different sizes and designs based on the system's requirements.

Subsystem B (Condenser Water Pump): The condenser water pump is responsible for circulating the water between the chiller and the cooling tower. It moves the warm water from the chiller to the cooling tower for heat dissipation and then returns the cooled water to the chiller to continue the cooling process.

Subsystem C (Water-cooled Chiller): This is the primary component that absorbs heat from a building or industrial process. It uses chilled water to extract heat from the space it's cooling. The warm water leaving the chiller goes into the cooling tower to shed the absorbed heat, and the now-cooled water is sent back to the chiller to restart the heat absorption cycle.

This sequential structure emphasizes the interconnectedness of the subsystems and their reliance on each other for the system to operate effectively. If, for example, all units within the cooling tower, condenser water pump, or water-cooled chiller fail, it directly impacts the system's ability to function. Additionally, if a common issue, such as a shared failure in a component or a systemic problem affecting all units, occurs across any of the subsystems, it fails in the entire condenser water system. The parallel operation of multiple units within each subsystem is designed to provide redundancy and enhance the system's reliability. However, the system remains vulnerable to a complete failure if all units in any of the subsystems are compromised or if a common failure affects all units in any one of the subsystems. This design consideration underscores the importance of maintaining and monitoring individual units within the subsystems to ensure the overall reliability and functionality of the condenser water system.

I. Methodology

To analyse the probabilistic behaviour of the system across various transition states, a Markov process was employed. To derive key reliability characteristics—such as steady-state behaviour, reliability function, availability, mean time to failure (MTTF), sensitivity analysis, and profit—the supplementary variable technique, in conjunction with the copula method for determining joint probability distributions, will be utilized.

1. Define the transition diagram: Start by constructing a transition diagram that represents the different states and transitions within the solar system as in Figure 2. Each state represents a particular condition or configuration of the system, and transitions denote the movement or changes between states. The diagram should capture the relevant aspects that affect reliability, mean time to failure, availability, and profit function.
2. Assign variables and parameters: Identify the variables and parameters relevant to your reliability and performance analysis. These may include factors such as component failure rates, repair rates, transition probabilities, and system configurations.
3. Formulate PDEs using transition rates: Use the transition diagram in Figure 2 to derive the transition rates between states. The transition rates represent the probabilities of moving from one state to another. Express these transition rates as functions of the variables and parameters identified in step 2. The PDEs are typically formulated based on the rate at which the state probabilities change over time.
4. Apply Laplace transformation: Apply the Laplace transformation to the PDEs obtained in step 3. The Laplace transformation converts the time-domain equations into the frequency-domain, enabling easier analysis and solution. This step involves transforming the time derivatives into algebraic expressions involving the Laplace variable, typically denoted as 's'.
5. Solve the transformed equations: Manipulate the transformed equations to solve for the state probabilities or other desired reliability and performance metrics. This step involves algebraic manipulation, including solving for unknown variables and rearranging the equations as needed.
6. Inverse Laplace transformation: After obtaining the solutions in the frequency domain, perform an inverse Laplace transformation to convert the solutions back to the time domain. This step allows you to obtain the time-dependent behavior of the system, such as the state probabilities or other relevant performance metrics.

7. Analyze and interpret the results: Once you have the solutions in the time domain, analyze and interpret the results to gain insights into the reliability, mean time to failure, availability, and profit function of the solar system. Evaluate the performance metrics and make conclusions based on the obtained probabilities and other relevant indicators.

II. Mathematical Formulation and Solution

Using probability and continuity principles, we have established a set of differential equations governing our current mathematical model. These partial differential equations, shown in Figure 2, are derived alongside their initial and boundary conditions, which are also formulated from Figure 1. To obtain state probabilities, we solve these equations using Laplace transformation.

$$\left\{\frac{\partial}{\partial q}+n\left(b_1+b_2+b_3\right)\right\}R_0\left(q\right)=\int_0^\infty v_1\left(x\right)R_1\left(x,q\right)dx+\int_0^\infty v_1\left(y\right)R_2\left(y,q\right)dy+ \int_0^\infty v_1\left(z\right)R_3\left(z,q\right)dz+\int_0^\infty v_0\left(x\right)R_{19}\left(x,q\right)dx+\int_0^\infty v_0\left(y\right)R_{20}\left(y,q\right)dy+\int_0^\infty v_0\left(z\right)R_{21}\left(z,q\right)dz \tag{1}$$

$$\left\{\frac{\partial}{\partial q}+\frac{\partial}{\partial x}+c_1b_1+nb_2+nb_3+v_1\left(x\right)\right\}R_1\left(x,q\right)=0 \tag{2}$$

$$\left\{\frac{\partial}{\partial q}+\frac{\partial}{\partial y}+c_1b_2+nb_1+nb_3+v_1\left(y\right)\right\}R_2\left(y,q\right)=0 \tag{3}$$

$$\left\{\frac{\partial}{\partial q}+\frac{\partial}{\partial z}+c_1b_3+nb_1+nb_2+v_1\left(z\right)\right\}R_3\left(z,q\right)=0 \tag{4}$$

$$\left\{\frac{\partial}{\partial q}+\frac{\partial}{\partial y}+c_1b_2+v_1\left(y\right)\right\}R_4\left(y,q\right)=0 \tag{5}$$

$$\left\{\frac{\partial}{\partial q}+\frac{\partial}{\partial z}+c_1b_3+v_1\left(z\right)\right\}R_5\left(z,q\right)=0 \tag{6}$$

$$\left\{\frac{\partial}{\partial q}+\frac{\partial}{\partial x}+c_1b_1+v_1\left(x\right)\right\}R_6\left(x,q\right)=0 \tag{7}$$

$$\left\{\frac{\partial}{\partial q}+\frac{\partial}{\partial z}+c_1b_3+v_1\left(z\right)\right\}R_7\left(z,q\right)=0 \tag{8}$$

$$\left\{\frac{\partial}{\partial q}+\frac{\partial}{\partial x}+c_1b_1+v_1\left(x\right)\right\}R_8\left(x,q\right)=0 \tag{9}$$

$$\left\{\frac{\partial}{\partial q}+\frac{\partial}{\partial y}+c_1b_2+v_1\left(y\right)\right\}R_9\left(y,q\right)=0 \tag{10}$$

$$\left\{\frac{\partial}{\partial q}+\frac{\partial}{\partial x}+c_2b_1+v_1(x)\right\}R_{10}(x,q)=0 \tag{11}$$

$$\left\{\frac{\partial}{\partial q}+\frac{\partial}{\partial y}+c_2b_2+v_1(y)\right\}R_{11}(y,q)=0 \tag{12}$$

$$\left\{\frac{\partial}{\partial q}+\frac{\partial}{\partial z}+c_2b_3+v_1(z)\right\}R_{12}(z,q)=0 \tag{13}$$

$$\left\{\frac{\partial}{\partial q}+\frac{\partial}{\partial y}+c_2b_2+v_1(y)\right\}R_{13}(y,q)=0 \tag{14}$$

$$\left\{\frac{\partial}{\partial q}+\frac{\partial}{\partial z}+c_2b_3+v_1(z)\right\}R_{14}(z,q)=0 \tag{15}$$

$$\left\{\frac{\partial}{\partial q}+\frac{\partial}{\partial x}+c_2b_1+v_1(x)\right\}R_{15}(x,q)=0 \tag{16}$$

$$\left\{\frac{\partial}{\partial q}+\frac{\partial}{\partial z}+c_2b_3+v_1(z)\right\}R_{16}(z,q)=0 \tag{17}$$

$$\left\{\frac{\partial}{\partial q}+\frac{\partial}{\partial x}+c_2b_1+v_1(x)\right\}R_{17}(x,q)=0 \tag{18}$$

$$\left\{\frac{\partial}{\partial q}+\frac{\partial}{\partial y}+c_2b_2+v_1(y)\right\}R_{18}(y,q)=0 \tag{19}$$

$$\left\{\frac{\partial}{\partial q}+\frac{\partial}{\partial x}+v_0(x)\right\}R_{19}(x,q)=0 \tag{20}$$

$$\left\{\frac{\partial}{\partial q}+\frac{\partial}{\partial y}+v_0(y)\right\}R_{20}(y,q)=0 \tag{21}$$

$$\left\{\frac{\partial}{\partial t}+\frac{\partial}{\partial z}+v_0(z)\right\}R_{21}(z,q)=0 \tag{22}$$

Boundary Conditions

$$R_1(0,q)=nb_1R_0(0) \tag{21}$$

$$R_2(0,q)=nb_2R_0(0) \tag{22}$$

$$R_3(0,q) = nb_3R_0(0) \tag{23}$$

$$R_{4,8}(0,q) = n^2b_1b_2R_0(0) \tag{24}$$

$$R_{5,9}(0,q) = n^2b_2b_3R_0(0) \tag{25}$$

$$R_{6,7}(0,q) = n^2b_1b_3R_0(0) \tag{26}$$

$$R_{10}(0,q) = nc_1b_1^2R_0(0) \tag{27}$$

$$R_{11}(0,q) = nc_1b_2^2R_0(0) \tag{28}$$

$$R_{12}(0,q) = nc_1b_3^2R_0(0) \tag{29}$$

$$R_{13}(0,q) = n^2c_1b_1b_2^2R_0(0) \tag{30}$$

$$R_{14}(0,q) = n^2c_1b_2b_3^2R_0(0) \tag{31}$$

$$R_{15}(0,q) = n^2c_1b_1^2b_3R_0(0) \tag{32}$$

$$R_{16}(0,q) = n^2c_1b_1b_3^2R_0(0) \tag{33}$$

$$R_{17}(0,q) = n^2c_1b_1^2b_2R_0(0) \tag{34}$$

$$R_{18}(0,q) = n^2c_1b_2^2b_3R_0(0) \tag{35}$$

$$R_{19}(0,q) = nc_1c_2b_1^3R_0(0) \tag{36}$$

$$R_{20}(0,q) = nc_1c_2b_2^3R_0(0) \tag{37}$$

$$R_{21}(0,q) = nc_1c_2b_3^3R_0(0) \tag{38}$$

By applying Laplace transforms to equations (1) to (22), we obtain the following transformed expressions:

$$\begin{aligned}\{s+n(b_1+b_2+b_3)\}\overline{R}_0(s) &= \int_0^\infty v_1(x)\overline{R}_1(x,s)dx + \int_0^\infty v_1(y)\overline{R}_2(y,s)dy \\ &+\int_0^\infty v_1(z)\overline{R}_3(z,s)dz + \int_0^\infty v_0(x)\overline{R}_{19}(x,s)dx + \int_0^\infty v_0(y)\overline{R}_{20}(y,s)dy + \int_0^\infty v_0(z)\overline{R}_{21}(z,s)dz\end{aligned} \tag{39}$$

$$\left\{s+\frac{\partial}{\partial x}+c_1b_1+nb_2+nb_3+v_1(x)\right\}\overline{R}_1(x,s)=0 \tag{40}$$

$$\left\{s+\frac{\partial}{\partial y}+c_1b_2+nb_1+nb_3+v_1(y)\right\}\overline{R}_2(y,s)=0 \tag{41}$$

$$\left\{s+\frac{\partial}{\partial z}+c_1b_3+nb_1+nb_2+v_1(z)\right\}\overline{R}_3(z,s)=0 \tag{42}$$

$$\left\{s+\frac{\partial}{\partial y}+c_1b_2+v_1(y)\right\}\overline{R}_4(y,s)=0 \tag{43}$$

$$\left\{s+\frac{\partial}{\partial z}+c_1b_3+v_1(z)\right\}\overline{R}_5(z,s)=0 \tag{44}$$

$$\left\{s+\frac{\partial}{\partial x}+c_1b_1+v_1(x)\right\}\overline{R}_6(x,s)=0 \tag{45}$$

$$\left\{s+\frac{\partial}{\partial z}+c_1b_3+v_1(z)\right\}\overline{R}_7(z,s)=0 \tag{46}$$

$$\left\{s+\frac{\partial}{\partial x}+c_1b_1+v_1(x)\right\}\overline{R}_8(x,s)=0 \tag{47}$$

$$\left\{s+\frac{\partial}{\partial y}+c_1b_2+v_1(y)\right\}\overline{R}_9(y,s)=0 \tag{48}$$

$$\left\{s+\frac{\partial}{\partial x}+c_2b_1+v_1(x)\right\}\overline{R}_{10}(x,s)=0 \tag{49}$$

$$\left\{s+\frac{\partial}{\partial y}+c_2b_2+v_1(y)\right\}\overline{R}_{11}(y,s)=0 \tag{50}$$

$$\left\{s+\frac{\partial}{\partial z}+c_2b_3+v_1(z)\right\}\overline{R}_{12}(z,s)=0 \tag{51}$$

$$\left\{s+\frac{\partial}{\partial y}+c_2b_2+v_1(y)\right\}\overline{R}_{13}(y,s)=0 \tag{52}$$

$$\left\{s+\frac{\partial}{\partial z}+c_2b_3+v_1(z)\right\}\overline{R}_{14}(z,s)=0 \tag{53}$$

$$\left\{s+\frac{\partial}{\partial x}+c_2b_1+v_1(x)\right\}\overline{R}_{15}(x,s)=0 \tag{54}$$

$$\left\{s+\frac{\partial}{\partial z}+c_2b_3+v_1(z)\right\}\overline{R}_{16}(z,s)=0 \tag{55}$$

$$\left\{s+\frac{\partial}{\partial x}+c_2b_1+v_1(x)\right\}\overline{R}_{17}(x,s)=0 \tag{56}$$

$$\left\{s+\frac{\partial}{\partial y}+c_2b_2+v_1(y)\right\}\overline{R}_{18}(y,s)=0 \tag{57}$$

$$\left\{s+\frac{\partial}{\partial x}+v_0(x)\right\}\overline{R}_{19}(x,s)=0 \tag{58}$$

$$\left\{s+\frac{\partial}{\partial y}+v_0(y)\right\}\overline{R}_{20}(y,s)=0 \tag{59}$$

$$\left\{s+\frac{\partial}{\partial z}+v_0(z)\right\}\overline{R}_{21}(z,s)=0 \tag{60}$$

When we perform Laplace transforms on the boundary conditions mentioned above i.e., equations (23) to (38), we derive the following transformed expressions:

$$\overline{R}_1(0,s)=nb_1\overline{R}_0(0) \tag{61}$$

$$\overline{R}_2(0,s)=nb_2\overline{R}_0(0) \tag{62}$$

$$\overline{R}_3(0,s)=nb_3\overline{R}_0(0) \tag{63}$$

$$\overline{R}_{4,8}(0,s)=n^2b_1b_2\overline{R}_0(0) \tag{64}$$

$$\overline{R}_{5,9}(0,s)=n^2b_2b_3\overline{R}_0(0) \tag{65}$$

$$\overline{R}_{6,7}(0,s)=n^2b_1b_3\overline{R}_0(0) \tag{66}$$

$$\overline{R}_{10}(0,s)=nc_1b_1^2\overline{R}_0(0) \tag{67}$$

$$\overline{R}_{11}(0,s)=nc_1b_2^2\overline{R}_0(0) \tag{68}$$

$$\overline{R}_{12}(0,s)=nc_1b_3^2\overline{R}_0(0) \tag{69}$$

$$\overline{R}_{13}(0,s) = n^2 c_1 b_1 b_2^2 \overline{R}_0(0) \tag{70}$$

$$\overline{R}_{14}(0,s) = n^2 c_1 b_2 b_3^2 \overline{R}_0(0) \tag{71}$$

$$\overline{R}_{15}(0,s) = n^2 c_1 b_1^2 b_3 \overline{R}_0(0) \tag{72}$$

$$\overline{R}_{16}(0,s) = n^2 c_1 b_1 b_3^2 \overline{R}_0(0) \tag{73}$$

$$\overline{R}_{17}(0,s) = n^2 c_1 b_1^2 b_2 \overline{R}_0(0) \tag{74}$$

$$\overline{R}_{18}(0,s) = n^2 c_1 b_2^2 b_3 \overline{R}_0(0) \tag{75}$$

$$\overline{R}_{19}(0,s) = nc_1 c_2 b_1^3 \left[1 + +nb_2 + nb_3\right] \overline{R}_0(0) \tag{76}$$

$$\overline{R}_{20}(0,s) = nc_1 c_2 b_2^3 \left[1 + +nb_1 + nb_3\right] \overline{R}_0(0) \tag{77}$$

$$\overline{R}_{21}(0,s) = nc_1 c_2 b_3^3 \left[1 + +nb_1 + nb_2\right] \overline{R}_0(0) \tag{78}$$

By utilizing the Laplace transforms of equations (39) to (60) and incorporating the transformed boundary conditions as depicted in equations (61) to (78), we arrive at the following solutions: $\overline{R}_0(s) = \dfrac{1}{U(s)}$ (79)

$$\overline{R}_1(s) = \frac{nb_1}{\overline{R}_0(s)} \left\{ \frac{1 - \overline{s}_{v_1}(s + c_1 b_1 + nb_2 + nb_3)}{s + c_1 b_1 + nb_2 + nb_3} \right\} \tag{80}$$

$$\overline{R}_2(s) = \frac{nb_2}{\overline{R}_0(s)} \left\{ \frac{1 - \overline{s}_{v_1}(s + c_1 b_2 + nb_1 + nb_3)}{s + c_1 b_2 + nb_1 + nb_3} \right\} \tag{81}$$

$$\overline{R}_3(s) = \frac{nb_3}{\overline{R}_0(s)} \left\{ \frac{1 - \overline{s}_{v_1}(s + c_1 b_3 + nb_1 + nb_2)}{s + c_1 b_3 + nb_1 + nb_2} \right\} \tag{82}$$

$$\overline{R}_4(s) = \frac{n^2 b_1 b_2}{\overline{R}_0(s)} \left\{ \frac{1 - \overline{s}_{v_1}(s + c_1 b_2)}{s + c_1 b_2} \right\} \tag{83}$$

$$\overline{R}_5(s) = \frac{n^2 b_2 b_3}{\overline{R}_0(s)} \left\{ \frac{1 - \overline{s}_{v_1}(s + c_1 b_3)}{s + c_1 b_3} \right\} \tag{84}$$

$$\bar{R}_6(s)=\frac{n^2b_1b_3}{\bar{R}_0(s)}\left\{\frac{1-\bar{s}_{v_1}(s+c_1b_1)}{s+c_1b_1}\right\} \tag{85}$$

$$\bar{R}_7(s)=\frac{n^2b_1b_3}{\bar{R}_0(s)}\left\{\frac{1-\bar{s}_{v_1}(s+c_1b_3)}{s+c_1b_3}\right\} \tag{86}$$

$$\bar{R}_8(s)=\frac{n^2b_1b_2}{\bar{R}_0(s)}\left\{\frac{1-\bar{s}_{v_1}(s+c_1b_1)}{s+c_1b_1}\right\} \tag{88}$$

$$\bar{R}_9(s)=\frac{n^2b_2b_3}{\bar{R}_0(s)}\left\{\frac{1-\bar{s}_{v_1}(s+c_1b_2)}{s+c_1b_2}\right\} \tag{89}$$

$$\bar{R}_{10}(s)=\frac{nc_1b_1^2}{\bar{R}_0(s)}\left\{\frac{1-\bar{s}_{v_1}(s+c_2b_1)}{s+c_2b_1}\right\} \tag{90}$$

$$\bar{R}_{11}(s)=\frac{nc_1b_2^2}{\bar{R}_0(s)}\left\{\frac{1-\bar{s}_{v_1}(s+c_2b_2)}{s+c_2b_2}\right\} \tag{91}$$

$$\bar{R}_{12}(s)=\frac{nc_1b_3^2}{\bar{R}_0(s)}\left\{\frac{1-\bar{s}_{v_1}(s+c_2b_3)}{s+c_2b_3}\right\} \tag{92}$$

$$\bar{R}_{13}(s)=\frac{n^2c_1b_1b_2^2}{\bar{R}_0(s)}\left\{\frac{1-\bar{s}_{v_1}(s+c_2b_2)}{s+c_2b_2}\right\} \tag{93}$$

$$\bar{R}_{14}(s)=\frac{n^2c_1b_2b_3^2}{\bar{R}_0(s)}\left\{\frac{1-\bar{s}_{v_1}(s+c_2b_3)}{s+c_2b_3}\right\} \tag{94}$$

$$\bar{R}_{15}(s)=\frac{n^2c_1b_1^2b_3}{\bar{R}_0(s)}\left\{\frac{1-\bar{s}_{v_1}(s+c_2b_1)}{s+c_2b_1}\right\} \tag{95}$$

$$\bar{R}_{16}(s)=\frac{n^2c_1b_1b_3^2}{\bar{R}_0(s)}\left\{\frac{1-\bar{s}_{v_1}(s+c_2b_3)}{s+c_2b_3}\right\} \tag{96}$$

$$\bar{R}_{17}(s)=\frac{n^2c_1b_1^2b_2}{\bar{R}_0(s)}\left\{\frac{1-\bar{s}_{v_1}(s+c_2b_1)}{s+c_2b_1}\right\} \tag{97}$$

$$\bar{R}_{18}(s)=\frac{n^2c_1b_2^2b_3}{\bar{R}_0(s)}\left\{\frac{1-\bar{s}_{v_1}(s+c_2b_2)}{s+c_2b_2}\right\} \tag{98}$$

$$\bar{R}_{19}(s)=\frac{nc_1c_2b_1^3[1++nb_2+nb_3]}{\bar{R}_0(s)}\left\{\frac{1-\bar{s}_{v_0}(s)}{s}\right\} \tag{99}$$

$$\bar{R}_{20}(s)=\frac{nc_1c_2b_2^3[1++nb_1+nb_3]}{\bar{R}_0(s)}\left\{\frac{1-\bar{s}_{v_0}(s)}{s}\right\} \tag{100}$$

$$\bar{R}_{21}(s)=\frac{nc_1c_2b_3^3[1++nb_1+nb_2]}{\bar{R}_0(s)}\left\{\frac{1-\bar{s}_{v_0}(s)}{s}\right\} \tag{101}$$

$$U(s)=\{s+n(b_1+b_2+b_3)\}-(N_3+N_4),$$

To determine the overall probability of the system being operational, we sum up the individual probabilities related to its uptime and we obtain:

$$\bar{R}_{up}(s)=\sum_{i=0}^{18}\bar{R}_i(s) \tag{102}$$

The probability of the system being in a state of failure or non-operational status can be represented as follows:

$$\bar{R}_{down}(s)=1-\sum_{i=0}^{18}\bar{R}_i(s) \tag{103}$$

Theorem: In the domain of multi-state serial systems employing a J-out-of-n redundancy scheme, as comprehensively described by assumptions (1) to (5),

$$\bar{R}_{up}(s)=\sum_{i=0}^{n}\bar{R}_i(s).$$

The theorem explores the distinct properties that elaborately characterize the performance measures within this system. These properties are as follows:

Property 1: In this multi-state serial system, regardless of the chosen repair policy—be it Copula or General—the system's availability demonstrates a notable and consistent declination to decrease over time.

III. Justification

The availability of the system over time can be represented as a real-valued function A(t), where t represents time. The property states that $\lim_{t\to\infty} A(t) < \lim_{t\to a} A(t)$ for any finite value a, indicating a consistent declination over time.

Property 2: The system's reliability, conversely, exhibits a diminishing trend, signifying a trade-off between availability and reliability as defined by this particular configuration.

IV. Justification

Reliability is the complement of the failure rate, and in real analysis, one can analyze the diminishing trend of reliability by considering the limit as t approaches infinity. The property implies that $\lim_{t\to\infty} R(t) < \lim_{t\to a} R(t)$ for any finite value a, where R(t) represents the reliability function. This signifies a trade-off between availability and reliability, whereas time progresses, reliability decreases, indicating that the system becomes less reliable over time.

Property 3: Both the Mean Time To Failure (MTTF) and sensitivity metrics display a concurrent decrease within this multi-state serial system, reflecting the intricate relationship between these measures and the inherent redundancy scheme.

V. Justification

In real analysis, the relationship between the Mean Time To Failure (MTTF) and sensitivity metrics can be analyzed using mathematical functions and their properties. If both MTTF and sensitivity metrics decrease concurrently, this suggests that there is a consistent mathematical relationship between these measures and the redundancy scheme. This relationship can be explored using mathematical analysis techniques.

Property 4: The cost function, whether governed by the Copula repair policy or the General repair policy, invariably experiences an escalation, underscoring the economic implications of maintaining and operating this system.

VI. Justification

Economic implications can be analyzed using real-valued functions and their behavior over time. If the cost function consistently increases, this can be examined using real analysis to understand the financial implications of the repair policies. One can analyze the limit of the cost function as time goes to infinity to confirm the escalation.

Property 5: As the value of n, representing the redundancy scheme, increases, the system's availability exhibits a correlated augmentation. This property showcases how system configuration parameters influence its availability performance, offering valuable insights into the scalability of the system.

- **Analysis of the Mathematical Model for Specific Scenarios**

In the context of validating models derived from partial differential equations and transition diagrams for reliability, availability, mean time to failure (MTTF), and profit, the validation process ensures that the mathematical solutions align with real-world or simulated system behavior.

Sensitivity Analysis: Perform sensitivity analysis to check how sensitive the model is to various parameters (e.g., failure rates, repair rates). This ensures that the model responds appropriately to realistic changes in system behavior. In the model, parameters such as failure rates, repair rates, and costs are often estimated or derived from data. Sensitivity analysis involves varying these parameters to test how the model responds to changes.

- **Failure Rate Changes**: By increasing or decreasing the failure rate, you can check if the model reacts appropriately in terms of decreased reliability and availability.
- **Repair Time Variation**: Changing the repair rate should reflect changes in availability and MTTF as expected.
- **Profit Impact**: Varying costs and system performance should directly affect the profit metric, with the model showing logical increases or decreases in profitability based on system uptime and downtime.

In this section, we present the expressions and numerical results for specific scenarios that have been included in our study. These scenarios encompass the following:

a. *System Availability*

In this part, we provide an analysis of availability in two distinct manners, aiming to differentiate between two categories of repair methods. Specifically, we examine scenarios where repairs are executed through the Copula approach as well as those employing the General approach. By adopting this approach, we can draw a clear contrast between the outcomes and implications of these two repair strategies.

I. System Availability for Copula Repair

In this situation, we set $S_{v_0}(s) = \bar{S}_{exp[x^\theta+\{log\, v(x)\}^\theta]^{1/\theta}}(s) = \frac{exp[x^\theta+\{log\, v(x)\}^\theta]^{1/\theta}}{s+exp[x^\theta+\{log\, v(x)\}^\theta]^{1/\theta}}, \bar{S}_v(s) = \frac{v}{s+v}$, the failure rates are set at various values, such as $b_1 = 0.001, b_2 = 0.002, b_3 = 0.003, n = 50, j = 15, v = x = y = z = 1$ and all the repair rates are set equal to 1 i.e. $v(x) = v(y) = v(z) = v_0(x) = v_0(y) = v_0(z) = 1$ in equation (102). Now applying the inverse Laplace transform, we can derive the availability equation for Copula repair policy as:

$$R_{up}(t) = \begin{Bmatrix} -0.006822e^{-1.102000000t} - 0.003653e^{-1.068000000t} - 0.006822e^{-1.102000000t} \\ -0.001072e^{-1.034000000t} - 0.008630e^{-1.070000000t} - 0.008355e^{-1.105000000t} \\ -0.006108e^{-1.035000000t} + 0.001024e^{-2.72135904t} - 0.069162e^{-1.511920600t} \\ -0.000087e^{-1.281311461t} - 0.000096e^{-1.263272243t} - 1.102962e^{-0.050436655548t} \end{Bmatrix} \quad (104)$$

Table 2 illustrates how the system's availability parameters are influenced by variations in time intervals, ranging from 0 to 20, within the context of the Copula repair policy.

TABLE 1
EFFECT OF TIME ON SYSTEM AVAILABILITY FOR COUPLA REPAIR

Time	**0**	**2**	**4**	**6**	**8**	**10**	**12**	**14**	**16**	**18**	**20**
$R_{up}(t)$	1.0000	0.9897	0.9008	0.8149	0.7368	0.6661	0.6022	0.5444	0.4921	0.4449	0.4022
$R_{down}(t)$	0.0000	0.0103	0.0992	0.1851	0.2632	0.3339	0.3978	0.4556	0.5079	0.5551	0.5978

II. System Availability for Broad/General Repair

Putting $\bar{S}_\nu(s) = \frac{\nu}{s+\nu}$ in equation (102) and assigning various values as $b_1 = 0.001, b_2 = 0.002, b_3 = 0.003, n = 50, j = 15$, and $\varphi = 1, \nu = 1$, then taking inverse Laplace transform, one may get availability expression for General repair policy as:

$$R_{up}(t) = \begin{Bmatrix} -0.008442e^{-1.105000000t} - 0.006342e^{-1.035000000t} - 0.068323e^{-1.513690310t} \\ -0.000084e^{-1.281314364t} - 0.000093e^{-1.263275377t} + 0.002485e^{-1.001354004t} \\ +1.101309e^{-0.05036594566t} - 0.008781e^{-1.070000000t} - 0.003719e^{-1.068000000t} \\ -0.006896e^{-1.102000000t} - 0.001115e^{-1.034000000t} \end{Bmatrix} \quad (105)$$

Table 2 provides an overview of how the Copula repair policy impacts the system's availability parameters across a range of time intervals, spanning from 0 to 20.

TABLE 2
EFFECT OF TIME ON SYSTEM AVAILIBILITY FOR GENERAL REPAIR

Time	**0**	**2**	**4**	**6**	**8**	**10**	**12**	**14**	**16**	**18**	**20**
$R_{up}(t)$	1.0000	0.9887	0.8998	0.8140	0.7361	0.6655	0.6018	0.5441	0.4920	0.4448	0.4022
$R_{down}(t)$	0.0000	0.0113	0.1002	0.1860	0.2639	0.3345	0.3982	0.4559	0.5080	0.5552	0.5978

TABLE 3
EFFECT OF TIME ON SYSTEM AVAILIBILITY FOR COUPLA REPAIR

Time	0	2	4	6	8	10	12	14	16	18	20
$R_{up}(t), j=15$	1.0000	0.9897	0.9008	0.8149	0.7368	0.6661	0.6022	0.5444	0.4921	0.4449	0.4022
$R_{up}(t), j=20$	1.0000	0.9892	0.9028	0.8190	0.7426	0.6733	0.6104	0.5534	0.5018	0.4549	0.4125
$R_{up}(t), j=25$	1.0000	0.9889	0.9051	0.8237	0.7492	0.6814	0.6197	0.5636	0.5125	0.4661	0.4239
$R_{up}(t), j=30$	1.0000	0.9886	0.9078	0.8289	0.7564	0.6903	0.6299	0.5748	0.5245	0.4786	0.4368
$R_{up}(t), j=35$	1.0000	0.9886	0.9108	0.8346	0.7644	0.7001	0.6411	0.5872	0.5377	0.4925	0.4510
$R_{up}(t), j=40$	1.0000	0.9886	0.9142	0.8409	0.7732	0.7108	0.6535	0.6008	0.5523	0.5078	0.4668
$R_{up}(t), j=45$	1.0000	0.9886	0.9179	0.8478	0.7826	0.7225	0.6669	0.6157	0.5683	0.5246	0.4843

b. System Reliability

Here, there are no actions taken to rectify the malfunctioning component of the system. Consequently, all the repair-related variables specified in Equation (102) are assigned a value of zero. Employing the inverse Laplace transform, we proceed to derive the reliability expression as outlined below.

$$R(t)=\left\{\begin{array}{l} -0.086957e^{-0.07000000000t}+0.091477e^{-0.1020000000t}+3.333333e^{-0.2550000000t} \\ +0.008224e^{-0.03400000000t}-9.385419e^{-0.3000000000t}+0.036207e^{-0.06800000000t} \\ +3.333333e^{-0.2700000000t}+0.047170e^{-0.035000000000t}+0.115385e^{-0.1050000000t} \\ +3.333333e^{-0.2850000000t} \end{array}\right\} \tag{106}$$

Using (106), one may obtain various values of reliability for different values of time t units ranging from 0 to 20 as presented in table 4.

TABLE 4
EFFECT OF TIME ON SYSTEM *RELIABILITY*

Time	0	2	4	6	8	10	12	14	16	18	20
$R(t)$	1.0000	1.0054	0.8513	0.6701	0.5102	0.3838	0.2891	0.2199	0.1700	0.1339	0.1076

c. Mean Time To Failure of the System

The Mean Time To Failure (MTTF) for the system is derived by a process that involves eliminating all repair operations from equation (102) and then applying a limit as $s \to 0$.

Table 5 showcases the Mean Time To Failure (MTTF) of the system. This analysis encompasses fixed values for failure rates (b_1 = 0.001, b_2 = 0.002, b_3 = 0.003), spans various time intervals (t = 0, 2, 4, 6, 8, 10, 12, 14, 16, 18, 20), and systematically explores the parameter variations, including b_1, b_2 and b_3, across a spectrum of values (ranging from 0.001 to 0.10).

TABLE 5
COMPUTATION OF MTTF IN RELATION TO DIFFERENT FAILURE RATE VALUES

Failure rates	**MTTF w.r.t** b_1	**MTTF w.r.t** b_2	**MTTF w.r.t** b_3
0.001	11.1888	12.6596	14.6591
0.002	10.1699	11.1888	12.5387
0.003	9.4451	10.2010	11.1888
0.004	8.8979	9.4826	10.2393
0.005	8.4670	8.9318	9.5287
0.006	8.1171	8.4934	8.9739
0.007	7.8262	8.1349	8.5272
0.008	7.5799	7.8353	8.1590
0.009	7.3683	7.5807	7.8497
0.010	7.1842	7.3614	7.5861

d. Sensitivity Analysis

The sensitivity analysis in this study involves calculating the partial derivatives of the MTTF concerning each parameter. Table 6 presents the system's sensitivity analysis. This analysis is conducted under fixed values of failure rates (b_1 = 0.001, b_2 = 0.002, b_3 = 0.003), various time points (t = 0, 2, 4, 6, 8, 10, 12, 14, 16, 18, 20), and where each of the parameters (b_1 , b_2 and b_3) is systematically varied across a range of values (0.001, 0.002, 0.003, 0.004, 0.005, 0.006, 0.007, 0.008, 0.009, 0.10).

TABLE 6
SENSIVITY AGAINST FAILURE RATE

Failure rates	**Sensitivity w.r.t** b_1	**Sensitivity w.r.t** b_2	**Sensitivity w.r.t** b_3
0.001	-1231.3629	-1842.8905	-2750.5247
0.002	-845.1748	-1231.3629	-1750.2104
0.003	-622.6042	-892.6523	-1231.3629
0.004	-481.5980	-679.3209	-916.6572
0.005	-385.8916	-533.3451	-707.2048
0.006	-317.4886	-427.6552	-559.0727
0.007	-266.6193	-347.9672	-449.7025
0.008	-227.5881	-286.0253	-366.2971
0.009	-196.8778	-236.7178	-301.0552
0.010	-172.2119	-196.7073	-248.9572

e. Cost Analysis

The cost function is presented in a manner similar to availability, offering two distinct approaches. The formula below can be employed to evaluate the cost or benefit derived from the system within the interval [0, t) when the service facility remains consistently accessible or open. In the time interval [0, t), P_1 and P_2 represent the income generated and the service cost incurred per unit of time, respectively.

I. Cost Analysis for Copula Repair Policy

The cost function for repair, utilizing the Copula approach, is delineated below, employing the same input parameters as those given in Equation (104).

$$R_{up}(t) = P_1 \begin{Bmatrix} -0.008065e^{-1.07000000000t} + 0.007561e^{-1.105000000t} - 0.000376e^{-2.721359041t} \\ +0.045745e^{-1.511920600t} + 0.000068e^{-1.281311461t} + 0.000076e^{-1.263272243t} \\ -21.868255e^{-0.05043665548t} + 0.001037e^{-1.034000000t} + 0.006190e^{-1.1020000000t} \\ +0.003420e^{-1.068000000t} + 0.005902e^{-1.035000000t} + 21.790567 \end{Bmatrix} - P_2 t \quad (107)$$

By utilizing a range of time values, including 0, 2, 4, 6, 8, 10, 12, 14, 16, 18, and 20, and subsequently applying the inverse Laplace transform to Equation (107), we can generate Table 7. These results showcase a variety of cost benefits under conditions where the generated income is fixed at 1 and the service costs are adjusted to 0.1, 0.2, 0.3, 0.4, 0.5, and 0.6.

TABLE 7
COST-BENEFIT ANALYSIS VIA COUPLA REPAIR POLICY

Time	$R_{up}(t)$ $P_2 = 0.06$	$R_{up}(t)$ $P_2 = 0.05$	$R_{up}(t)$ $P_2 = 0.04$	$R_{up}(t)$ $P_2 = 0.03$	$R_{up}(t)$ $P_2 = 0.02$	$R_{up}(t)$ $P_2 = 0.01$
0	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000
2	1.9066	1.9266	1.9466	1.9666	1.9866	2.0066
4	3.6781	3.7181	3.7581	3.7981	3.8381	3.8781
6	5.2726	5.3326	5.3926	5.4526	5.5126	5.5726
8	6.7030	6.7830	6.8630	6.9430	7.0230	7.1030
10	7.9846	8.0846	8.1846	8.2846	8.3846	8.4846
12	9.1317	9.2517	9.3717	9.4917	9.6117	9.7317
14	10.1573	10.2973	10.4373	10.5773	10.7173	10.8573
16	11.0729	11.2329	11.3929	11.5529	11.7129	11.8729
18	11.8892	12.0692	12.2492	12.4292	12.6092	12.7892
20	12.6156	12.8156	13.0156	13.2156	13.4156	13.6156

II. Cost Analysis for General Repair Policy

The cost function for the repair process, implemented through the General approach, is presented below. It utilizes very much input parameters outlined in Equation (108), ensuring consistency and continuity in the analysis.

$$R_{up}(t) = P_1 \begin{Bmatrix} -0.008206e^{-1.07000000000t} + 0.007640e^{-1.105000000t} + 0.001078e^{-1.034000000t} \\ +0.006258e^{-1.102000000t} + 0.045137e^{-1.513690310t} + 0.000066e^{-1.281314364t} \\ 0.000074e^{-1.263275377t} - 0.002482e^{-1.0001354004t} - 21.866153e^{-0.05036594566t} \\ +0.003482e^{-1.068000000t} + 0.006127e^{-1.035000000t} + 21.790567 \end{Bmatrix} - P_2 t \quad (108)$$

By varying the time variable across a range of values, including 0, 2, 4, 6, 8, 10, 12, 14, 16, 18, and 20, and subsequently applying the inverse Laplace transform to Equation (108), we can derive the data presented in Table 6. This table showcases the diverse cost-benefit scenarios when the revenue generated remains constant at 1, while service expenses are adjusted to different levels, specifically 0.1, 0.2, 0.3, 0.4, 0.5, and 0.6.

TABLE *8*
COST-BENEFIT ANALYSIS VIA GENERAL REPAIR POLICY

Time	$R_{up}(t)$ $P_2 = 0.06$	$R_{up}(t)$ $P_2 = 0.05$	$R_{up}(t)$ $P_2 = 0.04$	$R_{up}(t)$ $P_2 = 0.03$	$R_{up}(t)$ $P_2 = 0.02$	$R_{up}(t)$ $P_2 = 0.01$
0	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000
2	1.9054	1.9254	1.9454	1.9654	1.9854	2.0054
4	3.6748	3.7148	3.7548	3.7948	3.8348	3.8748
6	5.2673	5.3273	5.3873	5.4473	5.5073	5.5673
8	6.6961	6.7761	6.8561	6.9361	7.0161	7.0961
10	7.9765	8.0765	8.1765	8.2765	8.3765	8.4765
12	9.1228	9.2428	9.3628	9.4828	9.6028	9.7228
14	10.1476	10.2876	10.4276	10.5676	10.7076	10.8476
16	11.0628	11.2228	11.3828	11.5428	11.7028	11.8628
18	11.8788	12.0588	12.2388	12.4188	12.5988	12.7788
20	12.6051	12.8051	13.0051	13.2051	13.4051	13.6051

RECOMMENDATIOM AND CONCLUSION

To gain a comprehensive understanding of this study, we have presented a detailed analysis of the results obtained in this section. This discussion aims to provide a deeper insight into the implications and significance of the research findings. Table 1 offers valuable insights into the impact of time on system availability when repairs are executed according to the Copula repair policy. It is noteworthy that as time progresses, there is a noticeable decline in the system's availability. This decline in system availability as time elapses underscores a crucial aspect of the Copula repair policy's performance. It suggests that over an extended period, the system tends to become less reliable or accessible, which can have significant implications for maintenance and operational planning. The data presented in Table 1 serves as compelling visual and quantitative evidence supporting this observation. Table 2 provides insights into how the passage of time impacts the availability of a system, particularly in the context of General repairs. What we can discern from our observations is that as time progresses, system availability experiences a decrease. However, it is noteworthy that when repairs are conducted according to a Copula repair strategy, the system's availability tends to exhibit a notably higher level compared to instances where repairs follow a General repair policy. This analysis outlines the effectiveness of the Copula repair approach in significantly improving system availability compared to the standard General repair policy. In essence, Copula repairs emerge as a superior strategy for enhancing system availability over time.

By examining the information presented in Table 3 a clear trend emerges: as the parameter "n" increases, there is a corresponding rise in system availability. This compelling observation shows a fundamental principle in system design and reliability analysis - the augmentation/increase of system availability through the incorporation of additional units. In essence, the data illustrates that by introducing a greater number of units or components into the system, we have the potential to substantially bolster its overall availability. This principle is at the core of redundancy strategies, where redundancy involves the deliberate introduction of duplicate or backup components to ensure uninterrupted system operation in the event of failures. The positive correlation between "n" and system availability offers practical insights for engineers and decision-makers, emphasizing the significance of scalability and redundancy as strategies to enhance the robustness and reliability of systems in various domains, from manufacturing to information technology. Table 4 serves as comprehensive illustrations of how time exerts its influence on system reliability. Notably, as time progresses, the system's reliability experiences a marked decline. This decline in reliability can be attributed to a crucial factor — the absence of any system repairs or maintenance measures. This analysis has gone a long way in justifying the fact that when systems are left unattended over time, their reliability tends to diminish, resulting in increased failure rates and decreased overall performance. This insight carries substantial implications for industries and fields where system reliability is of paramount importance, such as manufacturing, infrastructure, and technology, advocating for proactive maintenance practices to maintain optimal system performance over extended periods.

Table 5 presents a detailed analysis of the Mean Time to Failure (MTTF) in relation to different failure rates, shedding light on a significant trend. This trend reveals that as the value of each individual failure rate increases, the corresponding MTTF exhibits a noticeable decrease. Fundamentally, a higher failure rate translates to a reduced expected time until failure, highlighting the inverse relationship between failure rates and system reliability. Furthermore, a noteworthy comparison arises when considering the MTTF values concerning the failure rates of various subsystems. It becomes evident that the MTTF associated with the failure rate of subsystem 3 surpasses that of the other subsystems. This observation emphasizes the superior reliability of subsystem 1 when compared to the rest, emphasizing the critical role of failure rates in assessing and optimizing the reliability of individual components within a larger system. Such insights hold significant implications for system designers and engineers, guiding them in making informed decisions to bolster the overall reliability of complex systems.

Sensitivity analysis serves as a valuable tool for discerning the most vulnerable component or unit within a given system. In this study, the findings from our sensitivity analysis are meticulously detailed in Table 6. Our examination reveals a noteworthy trend: as the failure rates of individual components increase, their respective sensitivities decrease. Interestingly, our investigation unveils a distinct pattern - the sensitivity associated with the failure rate of subsystem 3 surpasses that of the other subsystems. This observation substantiates our assertion regarding the Mean Time To Failure (MTTF) and highlights the superior reliability of subsystem 3 when compared to its other subsystems. Similar to the approach used to assess system availability, the cost function is analyzed from two distinct perspectives: one when repairs align with the Copula repair policy, and the other when they adhere to the General repair policy. This dual study provides us with a comprehensive view of the system's financial aspects.

Specifically, when repairs are carried out in accordance with the Copula repair policy, we have presented the outcomes in Table 7, showcasing the expected profit. Conversely, when repairs follow the General repair policy, our results are presented in Table 7, shedding light on the expected profit under this policy. A salient observation emerges from these analyses: irrespective of the repair policy adopted, the expected profit displays a consistent upward trajectory with the passage of time. The anticipated profit under the Copula repair policy appears to surpass that achieved under the General repair policy. Furthermore, it's evident that the expected profit attains its apex when service costs reach their maximum, and conversely, it reaches its low point when service costs are minimized. These findings substantiate our earlier assertion that the Copula repair policy significantly enhances system performance compared to the General repair policy, as it consistently leads to higher expected profits across varying service cost scenarios. From the results presented in Table 1 to Table 8, it is evident that the present study have shown that beside redundancy optimization and copula repair in enhancing the reliability and system performance, incorporating equal units in the form of k-out-of-n redundancy in each subsystem plays a vital role in system reliability, performance and efficiency which is not captured in the related studies in this paper.

CONFLICTOF INTEREST

The Authors declare that there is no conflict of interest within this manuscript.

REFRENCES

[1] Attar, A., Raissi, S and Khalili-Damghani, K. (2017). A simulation-based optimization approach for free distributed repairable multi-state availability-redundancy allocation problems, Reliability Engineering & System Safety 157, 177-191. https://doi.org/10.1016/j.ress.2016.09.006

[2] Bisht, S., Kumar, A., Goyal, N. and Ram, M. Reliability Characteristics of Industrial System Using Copula in Repair", 2020 8th International Conference on Reliability, Infocom Technologies and Optimization (Trends and Future Directions) (ICRITO), Noida, India, 2020, pp. 1265-1267, doi:10.1109/ICRITO48877.2020.9197876.

[3] Chen, L., Wang, J and Xu, X. An energy-client single machine scheduling problem with machine reliability constraints. Computers & Industrial Engineering 137(4):106072, DOI: 10.1016/j.cie.2019.106072

[4] Cekyay B. Reliability of mission-based k-out-of-n systems with exponential phase durations and component lifetimes. Proc Inst Mech Eng O-J Ris Reliab 235(3), (2020), 446–457. https://doi.org/10.1177/1748006X20969300

[5] Dolatshahi-Zand, A and Khalili-Damghani, K. (2015). Design of SCADA water resource management control center by a bi-objective redundancy allocation problem and particle swarm optimization, Reliability Engineering & System Safety 133, 11-21.https://doi.org/10.1016/j.ress.2014.07.020

[6] Gökdere, G and Tony, H,K Ng (2022) Time-dependent reliability analysis for repairable consecutive -k-out-of-n: F system, Statistical Theory and Related Fields, 6:2, 139-147, https://doi.org/10.1080/24754269.2021.1971489 I.S., & Ryzhik, I.M. (2007). Table of Integrals, Series, and Products, 7th edn. Academic Press, MA, USA

[7] Hu, L., Bai, Z and Yang, X. Reliability modeling and evaluation of uncertain random cold standby k-out-of-m + n: G systems. J Ambient Intell Human Compute, (2022). https://doi.org/10.1007/s12652-022-04075-6.

[8] Isa, M.S, Yusuf I., Ali, U.A., Suleiman, K., Yusuf, B., and Ismail A.L. Reliability Analysis of Multi-Workstation Computer Network Configured as Series-Parallel System via Gumbel - Hougaard Family Copula, International Journal of Operations Research, 19(1): (2022), 13-26. DOI: 10.6886/IJOR.202203_19(1).0002

[9] Khalili-Damghani, K and Amiri, M. (2012). Solving binary-state multi-objective reliability redundancy allocation series-parallel problem using efficient epsilon-constraint, multi-start partial bound enumeration algorithm, and DEA,Reliability Engineering & System Safety 103, 35-44, https://doi.org/10.1016/j.ress.2012.03.006

[10] Khalili-Damghani, K., Abtahi, A.R., and Tavana, M. (2013). A new multi-objective particle swarm optimization method for solving reliability redundancy allocation problems, Reliability Engineering & System Safety 111, 58-75. https://doi.org/10.1016/j.ress.2012.10.009

[11] Khalili-Damghani, K., Abtahi, A.R and Tavana, M. (2014). A decision support system for solving multi-objective redundancy allocation problems, Quality and Reliability Engineering International 30 (8), 1249-1262. https://doi.org/10.1002/qre.1545

[12] Nobakhti, A., Raissi, S., Soltani, R and Khalili Damghani, K. (2020). Dynamic system reliability assessment for flare gas recovery safety units subject to covariates, Invertis Journal of Renewable Energy 10 (2), 96-113. DOI: 10.5958/2454-7611.2020.00009.0

[13] Nobakhti, A., Raissi, S., Khalili Damghani, K and Soltani, R. (2022).Assessing the reliability of dynamic systems for safety units recovering flare gas, considering covariates, , Journal of Industrial Engineering International 5 (4). DOI:10.30495/jiei.2023.1989424.1312.

[14] Farahani, A., Tohidi, H., Shoja, A. (2019). An integrated optimization of quality control chart parameters and preventive maintenance using Markov chain. Advances in Production Engineering & Management. 14(1), pp 5–14. https://doi.org/10.14743/apem2019.1.307

[15] Rawal, D.K., Sahani, S.K and Singh, V.V. Reliability assessment of multi-computer system consisting n clients and the k-out-of-n: G operation scheme with copula repair policy. Life Cycle Reliab Saf Eng 11, (2022), 163–175. https://doi.org/10.1007/s41872-022-00192-5

[16] Singh, V.V., Ismail, A.L., Chand, U. and Maiti, S.S. Performance assessment of complex system under the k-out-of-n: G type configuration with k consecutive degraded states through the copula repair approach', International Journal of Reliability, Quality and Safety Engineering, Vol. 29, No. 2, (2022) , p.2150047, https://doi.org/10.1142/S0218539321500479.

[17] Tohidi, H. & Jabbari, M.M. (2012). Measuring organizational learning capability. Procedia - Social and Behavioral Sciences, vol. 31, p.p. 428-432, 2012, https:// doi:10.1016/j.sbspro.2011.12.079

[18] Tyagi, V., Arora, R., Ram, M and Triantafyllou, I. S. Copula based Measures of Repairable Parallel System with Fault Coverage. International Journal of Mathematical, Engineering and Management Sciences. (2021). DOI:10.33889/IJMEMS.2021.6.1.021.

[19] Xie, L., Lundteigen, M.A., Liu, Y., Kassa, E and Zhu, S. Performance Assessment of Safety-instrumented Systems Subject to Cascading Failures in High-demand Mode, Proceedings of the 29th European Safety and Reliability Conference (ESREL), (2019), DOI: 10.3850/978-981-11-2724-3_0318-cd.

[20] Yemane, A and Colledani, M. Performance analysis of unreliable manufacturing systems with uncertain reliability parameters estimated from production data, International Journal of Computer Integrated Manufacturing, (2019), DOI: 10.1080/0951192X.2019.1644535.

[21] Ye, Z., Cai, Z., Si, S., Zhang, S and Yang, H. Competing Failure Modeling for Performance Analysis of Automated Manufacturing Systems with Serial Structures and Imperfect Quality Inspection, IEEE Transactions on Industrial Informatics, vol. 16, no. 10, (2020), 6476-6486. doi: 10.1109/TII.2020.2967030.

[22] Zand, A.D., Khalili-Damghani, K and Raissi, S. (2022). An evolutionary approach with reliability priority to design Scada systems for water reservoirs, Evolving Systems 13 (3), 499-517. https://doi.org/10.1007/s12530-022-09438-0

[23] Zhang, N., Tian, S., Li,L., Wang, Z and Zhang, J. Maintenance analysis of a partial observable K-out-of-N system with load sharing units, Proceedings of the Institution of Mechanical Engineers, Part O: Journal of Risk and Reliability, (2022), https://doi.org/10.1177/1748006X221109351.

[24] Zhang, A., Hao, S., Li, P., Xie, M and Liu, Y. Performance modeling for condition-based activation of the redundant safety system subject to harmful tests, Reliability Engineering and System Safety, Volume 226, (2022), 108649. https://doi.org/10.1016/j.ress.2022.108649.

[25] Zhang, H, Li., A.P and Liu, X.M. Modeling and performance evaluation of a multistage serial manufacturing system considering rework and product polymorphism. Chin J Mech Eng; 53: (2017). 191–201.

[26] Zhao,B., Yue, D., Liao, H., Liu, Y and Zhang, X. Performance analysis and optimization of a cold standby system subject to δ-shocks and imperfect repairs, Reliability Engineering and System Safety,2021, Volume 208. https://doi.org/10.1016/j.ress.2020.107330.

Appendix

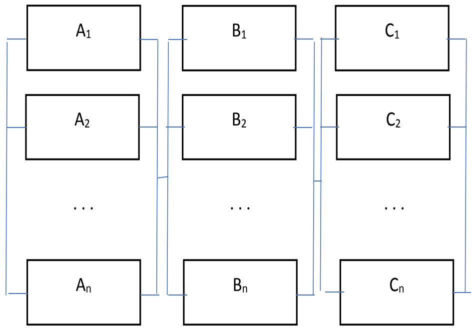

FIGURE 1
RELIABILITY BLOCK DIAGARM OF THE SYSTEM

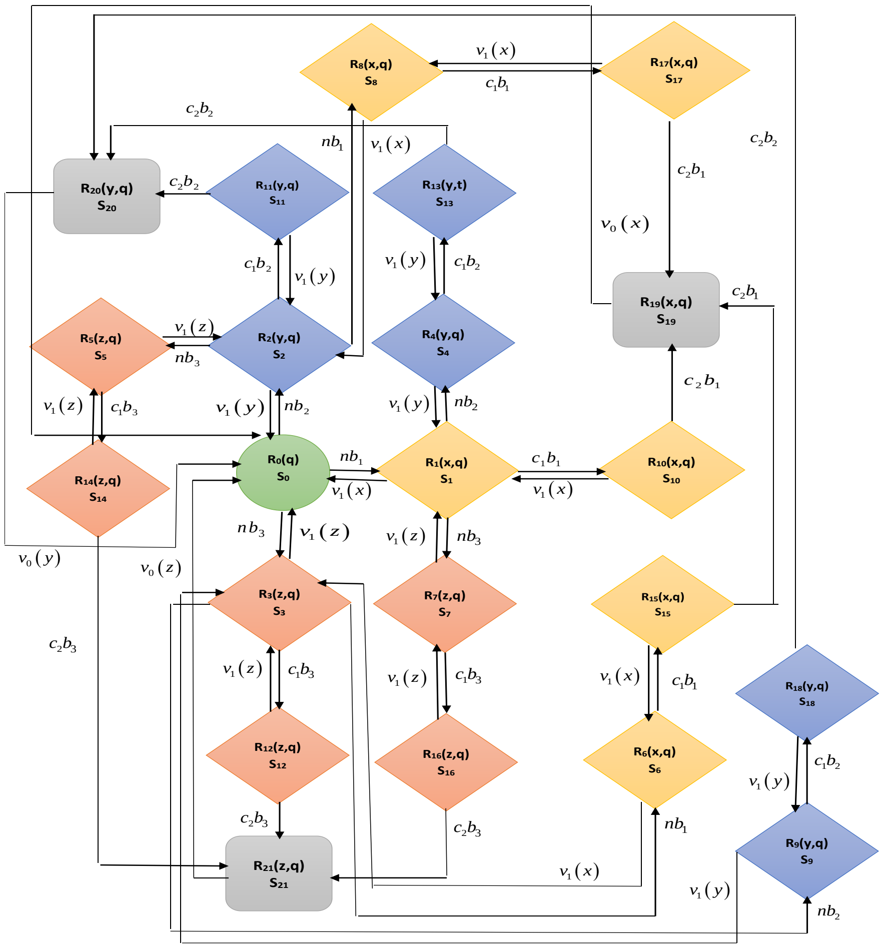

FIGURE 2
TRANSITION DIAGRAM OF CONDENSER WATER SYSTEM