Vibration Analysis of FG Micro-Beam Based on the Third Order Shear De-formation and Modified Couple Stress Theories

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Abstract

In this paper, analysis of free and forced vibration of an FG doubly clamped micro-beam is studied based on the third order shear deformation and modified couple stress theories. The size dependent dynamic equilibrium equations along with boundary conditions are derived using the variational approach. It is assumed that all properties of the FG micro-beam follow the power law form through its thickness. The motion equations are solved employing Fourier series in conjunction with Galerkin method. Also, effects of aspect ratio, power index and dimensionless length scale parameter on the natural frequencies and frequency response curves are investigated. Findings indicate that dimensionless frequencies are strongly dependent on the values of the material length scale parameter and power index. The numerical results indicate that if the thickness of the beam is in the order of the material length scale parameter, size effects are more significant.

Keywords: Vibration, Functionally graded material, Modified couple stress, Third order shear deformation.

1- Introduction

Micro-beams have an important role in micro and nano electromechanical systems (MEMs and NEMs), e.g. micro resonators, micro mirrors, actuators, Atomic Force Microscopes (AFMs), biosensors, and micro-pumps [1-4]. The size-dependent mechanical behavior has been observed in some experiments accomplished on the micro-scale structures [5-6]. Because of inability of the classical continuum theory to interpret the experimentally-detected small-scale effects in the micro-scale systems, various non-classical theories such as the nonlocal, strain gradient, and couple stress were proposed to remove the shortcoming in dealing with micro structures. As a non-classical theory, the couple stress theory is introduced by former leading researchers, e.g. Toupin [7]. According to the theory, the couple stress tensor is taken into account in addition to the classical force stress tensor. Yang et al. [8] suggested a simple form of couple stress theory in which a new higher-order equilibrium equation, i.e. the equilibrium equation of moments of couple stresses, was considered, as well as the classical equilibrium equations. In the last decade, numerous researches include; the static,

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dynamic, and thermal analyses have been accomplished on micro-structures, using non-classical continuum mechanics theories (for instance, see these studies that are based on the non-local [9, 10], strain gradient [11, 12], modified couple stress [13, 14], theories).

Functionally graded materials (FGMs) are inhomogeneous materials in which the volume fraction of two or more materials is changed gradually as a function of position along a certain direction in the material. In recent years, applications of FG structures have been widely increased and some researchers have studied different aspects of FG structures based on non-classical and classical continuum theories. In what follows, works investigating mechanical and thermal behaviors of FG micro-structures are reviewed. In this regard, Ke and Wang [15] studied the dynamic stability of FG micro-beams based on the modified couple stress theory. In addition based on theory, formulation of nonlinear vibration of micro-beams has been developed by Ke et al. [16]. Asghari and Taati [17] developed a size-dependent formulation for mechanical analyses of FG micro-plates based on the modified couple stress theory. The plate properties can arbitrarily vary through the thickness. The governing differential equations of motion were derived for functionally graded (FG) plates with arbitrary shapes utilizing the variational approach. Moreover, the boundary conditions were provided at smooth parts and at sharp area of the plate periphery. Reddy and Kim [18] formulated a general third-order model of FG plates with microstructure-dependent length scale parameter and the von Kármán nonlinearity. Their model accounted for temperature dependent properties of the

constituents in the functionally graded material. Also, modified couple stress theory was used to study microstructural length scale parameter. Molaei et al. [19] employed the separation of variables to solve transient hyperbolic heat conduction and thermos-elastic problems in the FGM micro-slab exposed to symmetric surface heating. Symmetrical surface heating was considered as a suitable boundary condition for designing of materials in order to optimize their resistance to failure under thermal loadings. Furthermore, the physical properties were assumed to vary spatially following an exponential relation. Thai and Choi [20] developed size dependent models for bending, buckling, and vibration of functionally graded Kirchhoff and Mindlin plates utilizing a modified couple stress theory. The numerical results showed that the small scale effect leads to a reduction of the magnitude of deflection. Molaei et al. [21] provided the transient temperature and associated thermal stresses in a functionally graded micro slab symmetrically heated on both sides by separation of the variables scheme. This method was applied to the heat conduction equation in terms of heat flux for obtaining the temperature profile. Further, exponential space dependent function of physical properties was considered. Effect of the inhomogeneity parameter and the Fourier number on the distribution of temperature, displacement, and stress was discussed. Taati [22] obtained analytical solutions for the buckling and post-buckling analysis of FG micro-plates under different kinds of traction on the edges through modified couple stress theory. The static equilibrium equations of an FG rectangular micro-plate as well as the boundary conditions were derived using

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the principle of minimum total potential energy.

To the best of authors' knowledge, no work has been reported to investigate free and forced vibration analyses of FG doubly clamped micro-beams based on the non-classical theories until now. This paper tries to fulfill the gap in the literature by deriving the size dependent dynamic equilibrium equations and both the classical and non-classical boundary conditions utilizing the third order shear deformation and modified couple stress theories. In present formulation, all properties of the FG micro beam are assumed to follow a power law form through thickness. The motion equations are solved by employing Fourier series in conjunction with Galerkin method. Moreover effects of aspect ratio, power index and dimensionless length scale parameter on the natural frequencies and frequency response function curves are studied. Findings showed that dimensionless frequencies are strongly dependent on the values of the material length scale parameter and power index.

2- Background

2-1- Problem definition

Consider a functionally graded doubly-clamped micro-beam with geometric dimensions of length L, width b, and thickness of h, as shown in figure (1). Micro-beam is composed of a functionally graded material including two metal and ceramic phases, whose properties vary linearly through its thickness exponentially. The geometry of the intended beam is depicted in figure (1). In the study of forced vibrations, uniform load of $q = q_0 e^{i\omega t}$ is applied on the upper surface of the beam.
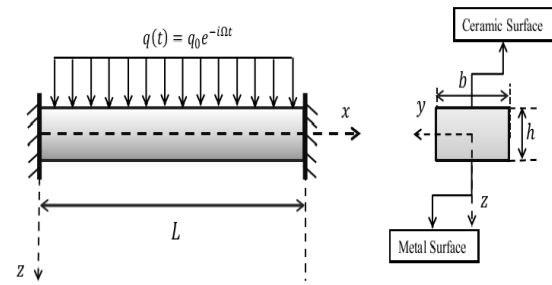

Fig. 1- Coordinate system, loading, geometric dimensions, and end conditions of FG micro-beam.

2-2- Modified Couple Stress Theory

The modified couple stress theory developed by Yang et al. [8] is employed in the present formulations. This theory is derived from the classical couple stress theory [7], which has been well established. Based on the theory, an additional equilibrium equation is considered for the moments of couple, which causes the couple stress tensor to be symmetric. Moreover, the strain energy density function is only dependent on the strain and the symmetric part of the curvature tensor, and hence, only one length scale parameter is involved in the constitutive relations. According to the theory, the variation of the strain energy U for an anisotropic linear elastic material occupying region Ω can be written as [8]:

$$\delta U = \int_\Omega (\sigma_{ij} \delta \varepsilon_{ij} + m_{ij} \delta \chi_{ij}) d\Omega \qquad (1)$$

In Equation (1), σ_{ij} and λ_{ij} denote the components of the strain tensor ε and the symmetric part of the curvature tensor χ, which are defined as:

$$\varepsilon_{ij} = \frac{1}{2}\left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i}\right) \qquad (2)$$

$$\chi_{ij} = \frac{1}{2}\left(\frac{\partial \theta_i}{\partial x_j} + \frac{\partial \theta_j}{\partial x_i}\right) \qquad (3)$$

Also, the components of the infinitesimal rotation vector $\theta = 1/2\, curl(u)$ are

introduced by θ. For linear isotropic elastic materials, constitutive relations of the symmetric part of the force stress and the deviatoric part of the couple stress tensor with the kinematic parameters are given as [8]:

$$\sigma_{ij} = \lambda tr(\varepsilon)\delta_{ij} + 2\mu\varepsilon_{ij} \tag{4}$$

$$m_{ij} = 2\mu l^2 \chi_{ij} \tag{5}$$

where σ_{ij} and m_{ij} are called the force and higher-order stresses, respectively. Furthermore, the parameters λ and μ in the constitutive equation of the classical stress σ are Lame constants. The parameter l, which appears in the constitutive equation, is the material length scale parameter [6]. It should be noticed that the Lame constants can be represented in terms of the Young's modulus E, and Poisson's ratio v as $\lambda = Ev/(1+v)(1-2v)$ and $\mu = E/2(1+v)$.

2-3- Modeling of Functionally graded materials

For modeling non-homogeneous materials, like FGMs usually the rule of mixtures is used. Property Λ of the FG material is stated as follows, according to the rule of mixtures:

$$\Lambda = \Lambda_m V_m + \Lambda_c V_c, \tag{6}$$

Equation (6) is presented for the FG material composed of two phases. Also, the volume percentage V and the suffixes of m and c indicate metal and ceramic, respectively. In equation (6), we have:

$$V_m = \left(\frac{z + \frac{h}{2}}{h}\right)^n, V_c = 1 - V_m = 1 - \left(\frac{z + h/2}{h}\right)^n \tag{7}$$

Regarding equations (6) and (7) for FG material, we will have:

$$\Lambda = \Lambda_c + (\Lambda_m - \Lambda_c)\left(\frac{z + h/2}{h}\right)^n \tag{8}$$

2.4 Reddy's third shear deformation model

Reddy introduced the vector displacement field for shells, considering the shear deformations as follows:

$$u_x = u_0(x,y) + z\phi_x(x,y) + z^2\psi_x(x,y) + z^3\beta_x(x,y)$$

$$u_y = v_0(x,y) + z\phi_y(x,y) + z^2\psi_y(x,y) + z^3\beta_y(x,y)$$

$$u_z = w(x,y) \tag{9}$$

(u_x, u_y, u_z) are the displacement vectors. One can replace these components into equation (2), to achieve the components ε_{13} and ε_{23} of the shear strains, as follows:

$$\varepsilon_{13} = \frac{1}{2}\left(\phi_x(x,y) + 2z\psi_x(x,y) + 3z^2\beta_x(x,y) + \frac{\partial w}{\partial x}\right)$$

$$\varepsilon_{23} = \frac{1}{2}\left(\phi_y(x,y) + 2z\psi_y(x,y) + 3z^2\beta_y(x,y) + \frac{\partial w}{\partial y}\right) \tag{10}$$

By applying free stress on the top and bottom surfaces, i.e. $\varepsilon_{13} = \varepsilon_{23} = 0$, we will have:

$$(\varepsilon_{13})_{z=\frac{h}{2}} = 0 \implies \emptyset_x(x,y) + h\psi_x(x,y) + \frac{3h^2}{4}\beta_x(x,y) + \frac{\partial w}{\partial x} = 0$$

$$(\varepsilon_{13})_{z=-\frac{h}{2}} = 0 \implies \emptyset_x(x,y) - h\psi_x(x,y) + \frac{3h^2}{4}\beta_x(x,y) + \frac{\partial w}{\partial x} = 0$$

$$(\varepsilon_{23})_{z=\frac{h}{2}} = 0 \implies \emptyset_y(x,y) + h\psi_y(x,y) + \frac{3h^2}{4}\beta_y(x,y) + \frac{\partial w}{\partial y} = 0$$

$$(\varepsilon_{23})_{z=-\frac{h}{2}} = 0 \implies \emptyset_y(x,y) - h\psi_y(x,y) + \frac{3h^2}{4}\beta_y(x,y) + \frac{\partial w}{\partial y} = 0$$

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(11)

Solution of the equation (11) is:

$$\psi_x(x, y) = \psi_y(x, y) = 0$$

$$\beta_x(x, y) = -\frac{4}{3h^2}\left(\phi_x(x, y) + \frac{\partial w(x, y)}{\partial x}\right)$$

$$\beta_y(x, y) = -\frac{4}{3h^2}\left(\phi_y(x, y) + \frac{\partial w(x, y)}{\partial y}\right)$$

(12)

Regarding relation (13), components of the vector field introduced in (4) are modified as the following form:

$$u_x = u_0(x, y) + z\emptyset_x(x, y) - c_1 z^3\left(\emptyset_x + \frac{\partial w}{\partial x}\right)$$

$$u_y = v_0(x, y) + z\emptyset_y(x, y) - c_1 z^3\left(\emptyset_y + \frac{\partial w}{\partial y}\right)$$

$$u_z = w(x, y)$$

(13)

wherein, $c_1 = \frac{4}{3h^2}$

Based on Reddy's third shear deformation model, components of the displacement vector field of the beam can be written as:

$$u_x = u(x, t) + z\emptyset_x(x, t) - c_1 z^3\left(\emptyset_x + \frac{\partial w}{\partial x}\right)$$

$$u_z = w(x, t)$$

(14)

u(x,t) shows the in-plane displacement of particles along the beam axis on the mid-plane of the beam, which is perpendicular to e_3 direction. The side cross-section of the beam which is under pure bending, only have rotation around lines on the mid-plane.

3- Derivation of governing equations of dynamic equilibrium

By substitution of equation (14) into equation (2), the non-zero component of strain was calculated as follows:

$$\varepsilon_{xx} = \frac{\partial u}{\partial x} + (z - c_1 z^3)\frac{\partial \emptyset_x}{\partial x} - c_1 z^3 \frac{\partial^2 w}{\partial x^2}$$

$$2\varepsilon_{xz} = (1 - 3c_1 z^2)\left(\emptyset_x + \frac{\partial w}{\partial x}\right)$$

(15)

Non-zero components of rotation vector are as follows:

$$\theta_y = \frac{1}{2}(-(1 + 3c_1 z^2)\frac{\partial w}{\partial x} +$$

$$(1 - 3c_1 z^2)\emptyset_x).$$

(16)

Substitution of rotation component from equation (16) into relation (3) delivers the non-zero components of curvature, as:

$$\chi_{xy} = \chi_{yx} = \frac{1}{4}(-(1 + 3c_1 z^2)\frac{\partial^2 w}{\partial x^2}$$

$$+(1 + 3c_1 z^2)\frac{\partial \phi_x}{\partial x})$$

$$\chi_{yz} = \chi_{zy} = -\frac{3}{2}c_1 z\left(\frac{\partial w}{\partial x} + \phi_x\right)$$

(17)

In order to make sure that there is no axial strain ε_{zz} in the thickness direction, no constraint has been applied. On the other hand, the amount of the traction of the normal force on both top and bottom surfaces of the beam are relatively small or even zero. As a result, the amount of σ_{zz} stress in all points of the plate is not considerable in comparison to other stress components. For the same reason, along the width of the beam $\sigma_{zz} = \sigma_{yy} = 0$, which results in the following relation for the normal stress σ_{xx}.

$$\sigma_{xx} = E\varepsilon_{xx}$$

(18)

From now on, equation (18) is used for finding the σ_{xx} stress. For a plate made of FG materials, the Young's modulus E_z, shear modulus μ_z, length of structure parameter l_z, and Poisson's ratio υ_z have been considered as an arbitrary continuous function of the vertical position of z. By substituting the non-zero components of

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strain and curvature from relations (15) and (17) into the introduced constitutive equation, the non-zero stress components are obtained as follows:

$$\sigma_{xx} = E(z)\left(\frac{\partial u}{\partial x} + (z - c_1 z^3)\frac{\partial \phi_x}{\partial x} - c_1 z^3 \frac{\partial^2 w}{\partial x^2}\right)$$

$$\sigma_{xz} = \sigma_{zx} = \mu(z)(1 - 3c_1 z^2)\left(\phi_x + \frac{\partial w}{\partial x}\right)$$

$$m_{xy} = m_{yx} = \frac{\mu(z)l^2(z)}{2}(-(1 + 3c_1 z^2)\frac{\partial^2 w}{\partial x^2} + (1 - 3c_1 z^2)\frac{\partial \phi_x}{\partial x})$$

$$m_{yz} = m_{zy} = -3\mu(z)l^2(z)c_1 z\left(\frac{\partial w}{\partial x} + \phi_x\right) \quad (19)$$

Using variations of strain energy based on the modified coupled stress theory for linear elastic material in equation (1), and the relations of strain and curvature, changes of the strain energy of the micro-beam is stated as follows:

$$\delta U = \int_0^L \left\{ N_{xx}\delta(\frac{\partial u}{\partial x}) + \left(M_{xx} + \frac{M_{xy}^m}{2} - \left(P_{xx} + \frac{P_{xy}^m}{2}\right)\right)\delta\left(\frac{\partial \phi_x}{\partial x}\right) - \left(P_{xx} + \frac{P_{xy}^m + M_{xy}^m}{2}\right)\delta\left(\frac{\partial^2 w}{\partial x^2}\right) + \left(Q_X - (R_x + R_y^m)\right)\delta\left(\phi_x + \frac{\partial w}{\partial x}\right)\right\}dx, \quad (20)$$

Stress resultants in relation (20), are presented in Appendix A.1.

Taking part by part integration on equation (20), results in:

$$\delta U = \int_0^L \left\{ -\left(\frac{\partial N_{xx}}{\partial x}\right)\delta u(x,t) - \left(\frac{\partial}{\partial x}\left[M_{xx} + \frac{M_{xy}^m}{2} - \left(P_{xx} + \frac{P_{xy}^m}{2}\right)\right] + (R_x + r_y^m) - Q_x\right)\delta\phi_x(x,t) - \left(\frac{\partial^2}{\partial x^2}\left[P_{xx} + \frac{P_{xy}^m + M_{xy}^m}{2}\right] + \frac{\partial}{\partial x}\left[Q_x - (R_x + R_y^m)\right]\right)\delta w(x,t)\right\}dx$$

$$+(N_{11}\delta u(x,t))\Big|_{x=0}^{x=L} + \left(\left(M_{xx}\frac{M_{xy}^m}{2} - \left(P_{xx} + \frac{P_{xy}^m}{2}\right)\right)\delta\phi_x(x,t)\right)\Bigg|_{x=0}^{x=L}$$

$$+\left(\left(\frac{\partial}{\partial x}\left[P_{xx} + \frac{P_{xy}^m + M_{xy}^m}{2}\right] + Q_x - (R_x + R_y^m)\right)\delta w(x,t)\right)\Bigg|_{x=0}^{xL}$$

$$-\left(\left(P_{xx} + \frac{P_{xy}^m + M_{xy}^m}{2}\right)\delta(\frac{\partial w}{\partial x})\right)\Bigg|_{x=0}^{x=L} \quad (21)$$

Kinetic energy can be calculated as:

$$T = \frac{1}{2}\int_0^L [I_0(\dot{u}^2 + \dot{w}^2) + 2I_1(\dot{u}\dot{\phi}_x) + I_2(\dot{\phi}_x)^2 - 2c_1 I_3 \dot{u}(\dot{\phi}_x + \frac{\partial \dot{w}}{\partial x}) - 2c_1 I_4 \dot{\phi}_x(\dot{\phi}_x + \frac{\partial \dot{w}}{\partial x})^2 + (c_1{}^2)I_6(\dot{\phi}_x + \frac{\partial \dot{w}}{\partial x})^2]dx, \quad (22)$$

The parameter I_i is defined as:

$$I_i = \int_A \rho(z)z^i dA \quad (23)$$

Variation of kinetic energy after simplification is stated as follows:

$$\delta T = \int_A \left\{ -(I_0\ddot{u} + (I_1 - c_1 I_3)\ddot{\phi}_x - c_1 I_3 \frac{\partial \ddot{w}}{\partial x})\delta u + \frac{\partial}{\partial t}\left[\left(I_0\dot{u} + (I_1 - c_1 I_3)\dot{\phi}_x - c_1 I_3 \frac{\partial \dot{w}}{\partial x}\right)\delta u\right] \right.$$

$$-\left((I_1 - c_1 I_3)\ddot{u} + (I_2 - 2c_1 I_4 + (c_1)^2 I_6)\ddot{\phi}_x - c_1(I_4 - c_1 I_6)\frac{\partial \ddot{w}}{\partial x}\right)\delta\phi_x$$

$$+\frac{\partial}{\partial t}\left[\left((I_1 - c_1 I_3)\dot{u} + (I_2 - 2c_1 I_4 + (c_1)^2 I_6)\dot{\phi}_x - c_1(I_4 - c_1 I_6)\frac{\partial \dot{w}}{\partial x}\right)\delta\phi_x\right]$$

$$-(I_0\ddot{w})\delta w + \frac{\partial}{\partial t}[(I_0\dot{w})\delta w] - \left(c_1 I_3 \frac{\partial \ddot{u}}{\partial x} + c_1(I_4 - c_1 I_6)\frac{\partial \ddot{w}}{\partial x} - (c_1)^2 I_6 \frac{\partial \ddot{w}}{\partial x}\right)\delta w$$

$$+\frac{\partial}{\partial t}\left[\left(c_1 I_3 \dot{u}_0 + c_1(I_4 - c_1 I_6)\dot{\phi}_x - (c_1)^2 I_6 \frac{\partial \dot{w}}{\partial x}\right)\delta(\frac{\partial w}{\partial x})\right]\right\}dA - \left(c_1 I_3 \dot{u}_0 + c_1(I_4 - c_1 I_6)\dot{\phi}_x - (c_1)^2 I_6 \frac{\partial \dot{w}}{\partial x}\right)\delta\dot{w} \quad (24)$$

Variation of the work done by the transverse force on the surface unit $q(x,t)$ is achieved as:

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$$\delta W = \int_0^L q_0 e^{-i\Omega t}\, \delta w(x,t)\,dx \qquad (25)$$

Based on Hamilton principle, we have:

$$\int_{t_1}^{t_2} (\delta T + \delta W - \delta U)\,dt = 0 \qquad (26)$$

By replacement of variations of strain energy, kinetic energy and the work done by external force in Hamilton principle, differential equation governing the dynamic equilibrium is achieved as:

$$k_1 \frac{\partial^2 u}{\partial x^2} + k_2 \frac{\partial^2 \phi_x}{\partial x^2} + k_3 \frac{\partial^3 w}{\partial x^3} - I_0 \ddot{u} -$$
$$(I_1 - c_1 I_3)\ddot{\phi}_x + c_1 I_3 \frac{\partial \ddot{w}}{\partial x} = 0, \qquad (27)$$

$$k_4 \frac{\partial^2 u}{\partial x^2} + k_5 \frac{\partial^2 \phi_x}{\partial x^2} + k_6 \frac{\partial^3 w}{\partial x^3}$$
$$+ k_7 \left(\phi_x + \frac{\partial w}{\partial x}\right) - (I_1 - c_1 I_3)\ddot{u} -$$
$$(I_2 - 2c_1 I_4 + (c_1)^2 I_6)\ddot{\phi}_x$$
$$+ c_1 (I_4 - c_1 I_6)\frac{\partial \ddot{w}}{\partial x} = 0 \qquad (28)$$

$$k_8 \frac{\partial^3 u}{\partial x^3} + k_9 \frac{\partial^3 \phi_x}{\partial x^3} + k_{10} \frac{\partial^4 w}{\partial x^4} + k_{11}\left(\frac{\partial \phi_x}{\partial x} + \frac{\partial^2 w}{\partial x^2}\right) - I_0 \ddot{w} - c_1 I_3 \frac{\partial \ddot{u}}{\partial x} - c_1 (I_4 - c_1 I_6)\frac{\partial \ddot{\phi}_x}{\partial x} + (c_1)^2 I_6 \frac{\partial^2 \ddot{w}}{\partial x^2} + q_0 e^{-i\Omega t} = 0 \qquad (29)$$

k_i coefficients in above equations have been specified in the Appendix A.2.

Similarly, for the boundary conditions at both ends of the beam, one can write:

$$k_1 \frac{\partial u}{\partial x} + k_2 \frac{\partial \phi_x}{\partial x} + k_3 \frac{\partial^2 w}{\partial x^2} = 0, \quad or \quad \delta u(x,t) = 0$$

$$k_4 \frac{\partial u}{\partial x} + k_5 \frac{\partial \phi_x}{\partial x} + k_6 \frac{\partial^2 w}{\partial x^2} = 0, \quad or \quad \delta \phi_x(x,t) = 0$$

$$k_8 \frac{\partial u}{\partial x} + k_9 \frac{\partial \phi_x}{\partial x} + k_{10} \frac{\partial^2 w}{\partial x^2} = 0,$$
$$or \quad \delta\left(\frac{\partial w(x,t)}{\partial x}\right) = 0$$

$$k_8 \frac{\partial^2 u}{\partial x^2} + k_9 \frac{\partial^2 \phi_x}{\partial x^2} + k_{10} \frac{\partial^3 w}{\partial x^3} + k_{11}\left(\phi_x + \frac{\partial w}{\partial x}\right)$$
$$= -(c_1 I_3 \ddot{u}$$
$$+ c_1(I_4 - c_1 I_6)\ddot{\phi} - (c_1)^2 I_6 \frac{\partial \ddot{w}}{\partial x}) \quad or \quad \delta w(x,t) = 0 \qquad (30)$$

4- Solution of governing equations

In this section, governing equations for two analyses of free and forced vibrations are solved by means of Galerkin's semi-analytical method. The transverse mode shapes of the doubly-clamped beam assumed as follows:

$$\psi_n(x) = \left\{\cos\left(\frac{\alpha_n x}{L}\right) - \cosh\left(\frac{\alpha_n x}{L}\right) - \frac{\cos(\alpha_n) - \cosh(\alpha_n)}{\sin(\alpha_n) - \sinh(\alpha_n)}\left(\sin\left(\frac{\alpha_n x}{L}\right) - \sinh\left(\frac{\alpha_n x}{L}\right)\right)\right\} \qquad (31)$$

4-1- Analysis of free vibrations

Considering boundary conditions of doubly clamped ends, kinematic parameters of the free vibration of the micro-beam are:

$$u(x,t) = \sum_{m=1} U_m \sin\left(\frac{m\pi}{L}x\right)\exp(j\omega_m t),$$

$$\phi_x(x,t) = \sum_{m=1} \Phi_m \sin\left(\frac{m\pi}{L}x\right)\exp(j\omega_m t),$$

$$w(x,t) = \sum_{m=1} W_m \psi_n(x)\exp(j\omega_m t). \qquad (32)$$

Based on the Galerkin's method, the integral form of the equations is stated as:

$$\int_0^L [k_1 \frac{\partial^2 u}{\partial x^2} + k_2 \frac{\partial^2 \phi_x}{\partial x^2} + k_3 \frac{\partial^3 w}{\partial x^3} - I_0 \ddot{u} - (I_1 - c_1 I_3)\ddot{\phi}_x + c_1 I_3 \frac{\partial \ddot{w}}{\partial x}]\sin\left(\frac{m\pi}{L}x\right)dx = 0, \qquad (33)$$

$$\int_0^L \left[k_4 \frac{\partial^2 u}{\partial x^2} + k_5 \frac{\partial^2 \phi_x}{\partial x^2} + k_6 \frac{\partial^3 w}{\partial x^3} + \right.$$
$$k_7 \left(\phi_x + \frac{\partial w}{\partial x} \right) - (I_1 - c_1 I_3) \ddot{u}$$
$$- (I_2 - 2c_1 I_4 + (c_1)^2 I_6) \ddot{\phi}_x + c_1 (I_4$$
$$\left. - c_1 I_6) \frac{\partial \ddot{w}}{\partial x} \right] \sin\left(\frac{m\pi}{L} x \right) dx = 0, \qquad (34)$$

$$\int_0^L \left[k_8 \frac{\partial^3 u}{\partial x^3} + k_9 \frac{\partial^3 \phi_x}{\partial x^3} + k_{10} \frac{\partial^4 w}{\partial x^4} + \right.$$
$$k_{11} \left(\frac{\partial \phi_x}{\partial x} + \frac{\partial^2 w}{\partial x^2} \right)$$
$$- I_0 \ddot{w} - c_1 I_3 \frac{\partial \ddot{u}}{\partial x} - c_1 (I_4 - c_1 I_6) \frac{\partial \ddot{\phi}_x}{\partial x} \qquad (35)$$
$$\left. + (c_1)^2 I_6 \frac{\partial^2 \ddot{w}}{\partial x^2} \right] \psi_n(x) dx$$
$$= 0$$

Substituting the displacement components, the following set of linear equations is obtained.

$$\begin{bmatrix} g_1 + g_2 \omega_m^2 & g_3 + g_4 \omega_m^2 & g_5 - g_6 \omega_m^2 \\ g_7 + g_8 \omega_m^2 & g_9 + g_{10} \omega_m^2 & g_{11} - g_{12} \omega_m^2 \\ g_{13} + g_{14} \omega_m^2 & g_{15} + g_{16} \omega_m^2 & g_{17} - g_{18} \omega_m^2 \end{bmatrix} \begin{Bmatrix} U_m \\ \Phi_m \\ W_m \end{Bmatrix}$$
$$= 0 \qquad (36)$$

g_i parameters are defined in the Appendix A.3.

Equating the determinant of the coefficients matrix to zero, the characteristic equation is obtained as:

$$a_1 \omega_m^6 + a_2 \omega_m^4 + a_3 \omega_m^2 + a_4 = 0 \qquad (37)$$

in equation (37):

$$a_1 = g_2(g_{12}g_{16} - g_{10}g_{18}) + g_4(g_8 g_{18} - g_{12}g_{14}) - g_6(g_8 g_{16} - g_{10}g_{14}),$$

$$a_2 = g_1(g_{12}g_{16} - g_{10}g_{18})$$
$$+ g_2(g_{10}g_{17} - g_9 g_{18} + g_{15}g_{12} - g_{11}g_{16}) +$$
$$g_3(g_8 g_{18} - g_{12}g_{14})$$
$$+ g_4(g_{11}g_{14} - g_{13}g_{12} + g_7 g_{18} - g_8 g_{17})$$
$$+ g_5(g_8 g_{16} - g_{10}g_{14})$$
$$- g_6(g_7 g_{16} + g_8 g_{15} - g_{13}g_{10} - g_9 g_{14})$$

$$a_3 = g_2(g_9 g_{17} - g_{15}g_{11})$$
$$+ g_1(g_{10}g_{17} - g_9 g_{18} + g_{15}g_{12} - g_{11}g_{16})$$
$$+ g_4(g_{11}g_{13} - g_7 g_{17})$$
$$+ g_3(g_{11}g_{14} - g_{13}g_{12} + g_7 g_{18} - g_8 g_{17})$$

$$- g_6(g_7 g_{15} - g_9 g_{13})$$
$$+ g_5(g_7 g_{16} + g_8 g_{15} - g_{13}g_{10} - g_9 g_{14})$$

$$a_4 = g_1(g_9 g_{17} - g_{15}g_{11}) + g_3(g_{11}g_{13} - g_7 g_{17})$$
$$+ g_5(g_7 g_{15} - g_9 g_{13}), \qquad (38)$$

4-2- Analysis of forced vibrations

As mentioned earlier, external force is assumed to be a distributed harmonic type. The kinematic parameters for the analysis of forced vibrations can be stated as:

$$u(x,t) = \sum_{m=1} U_m \sin(\frac{m\pi}{L} x) e^{i\Omega t}$$
$$\phi_x(x,t) = \sum_{m=1} \Phi_m \sin(\frac{m\pi}{L} x) e^{i\Omega t} \qquad (39)$$
$$w(x,t) = \sum_{m=1} W_m \psi_n(x) e^{i\Omega t}.$$

The integral form of equations of forced vibrations is like the previous state, except that the following term associated with external force appears in the third equation as the non-homogeneous part:

$$\int_0^L \left[\begin{array}{c} k_8 \frac{\partial^3 u}{\partial x^3} + k_9 \frac{\partial^3 \phi_x}{\partial x^3} + k_{10} \frac{\partial^4 w}{\partial x^4} + k_{11} \left(\frac{\partial \phi_x}{\partial x} + \frac{\partial^2 w}{\partial x^2} \right) \\ - I_0 \ddot{w} - c_1 I_3 \frac{\partial \ddot{u}}{\partial x} - c_1 (I_4 - c_1 I_6) \frac{\partial \ddot{\phi}_x}{\partial X} + \\ (c_1)^2 I_6 \frac{\partial^2 \ddot{w}}{\partial x^2} \end{array} \right] \psi_m(x(dx)$$
$$= - \int_0^L q_0 e^{-i\Omega t} \psi_m(x) dx$$

$$(40)$$

Substitution of displacement components into the governing equations, delivers the set of algebraic equations, as follows:

$$\begin{bmatrix} g_1 + g_2 \Omega^2 & g_3 + g_4 \Omega^2 & g_5 - g_6 \Omega^2 \\ g_7 + g_8 \Omega^2 & g_9 + g_{10} \Omega^2 & g_{11} - g_{12} \Omega^2 \\ g_{13} + g_{14} \Omega^2 & g_{15} + g_{16} \Omega^2 & g_{17} - g_{18} \Omega^2 \end{bmatrix} \begin{Bmatrix} U_m \\ \Phi_m \\ W_m \end{Bmatrix}$$
$$= \begin{Bmatrix} 0 \\ 0 \\ Q_m \end{Bmatrix} \qquad (41)$$

where, in above we have:

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$$Q_m = -q_0 \int_0^L \psi_m(x)dx \qquad (42)$$

By means of Cramer's rule, solution of the above set of equations is written as:

U_m

$$= Q_m \frac{\begin{vmatrix} g_3 + g_4\Omega^2 & g_5 - g_6\Omega^2 \\ g_9 + g_{10}\Omega^2 & g_{11} - g_{12}\Omega^2 \end{vmatrix}}{\begin{vmatrix} g_1 + g_2\Omega^2 & g_3 + g_4\Omega^2 & g_5 - g_6\Omega^2 \\ g_7 + g_8\Omega^2 & g_9 + g_{10}\Omega^2 & g_{11} - g_{12}\Omega^2 \\ g_{13} + g_{14}\Omega^2 & g_{15} + g_{16}\Omega_m^2 & g_{17} - g_{18}\Omega^2 \end{vmatrix}}$$

$$= Q_m \frac{a_5\Omega^4 + a_6\Omega^2 + a_7}{a_1\Omega^6 + a_2\Omega^4 + a_3\Omega^2 + \alpha_4}$$

Φ_m

$$= -Q_m \frac{\begin{vmatrix} g_1 + g_2\Omega^2 & g_5 - g_6\Omega^2 \\ g_7 + g_8\Omega^2 & g_{11} - g_{12}\Omega^2 \end{vmatrix}}{\begin{vmatrix} g_1 + g_2\Omega^2 & g_3 + g_4\Omega^2 & g_5 - g_6\Omega^2 \\ g_7 + g_8\Omega^2 & g_9 + g_{10}\Omega^2 & g_{11} - g_{12}\Omega^2 \\ g_{13} + g_{14}\Omega^2 & g_{15} + g_{16}\Omega_m^2 & g_{17} - g_{18}\Omega^2 \end{vmatrix}}$$

$$= Q_m \frac{a_8\Omega^4 + a_9\Omega^2 + a_{10}}{a_1\Omega^6 + a_2\Omega^4 + a_3\Omega^2 + a_4},$$

W_m

$$= Q_m \frac{\begin{vmatrix} g_1 + g_2\Omega^2 & g_3 + g_4\Omega^2 \\ g_7 + g_8\Omega^2 & g_9 + g_{10}\Omega^2 \end{vmatrix}}{\begin{vmatrix} g_1 + g_2\Omega^2 & g_3 + g_4\Omega^2 & g_5 - g_6\Omega^2 \\ g_7 + g_8\Omega^2 & g_9 + g_{10}\Omega^2 & g_{11} - g_{12}\Omega^2 \\ g_{13} + g_{14}\Omega^2 & g_{15} + g_{16}\Omega_m^2 & g_{17} - g_{18}\Omega^2 \end{vmatrix}}$$

$$= Q_m \frac{a_{11}\Omega^4 + a_{12}\Omega^2 + a_{13}}{a_1\Omega^6 + a_2\Omega^4 + a_3\Omega^2 + a_4}$$

$$(43)$$

a_i parameters have been introduced in the Appendix A.4.

5- Numerical Results

In this part, free and forced vibration of a micro-beam with height *h=1e-6*, width $b = 3 \times h$, length $L = 5 \times h$, and $b = 0.5 \times h$, have been presented. Also, mechanical properties of the desired FG material are shown in Table 1.

Table1- mechanical properties of the micro-beam.

ρ_m (kg / m3)	ρ_c (kg / m3)	E_c (N/m^2)	E_m (N/m^2)	V_c	V_m
2700	3960	393×10^9	70×10^9	0.252	0.346

5-1- Free vibration

In order to validate the method, simple case of a homogeneous metallic doubly clamped beam with no length scale parameter effect is considered. This can be achieved by setting n=0 and l/h=0 in the developed program. Attained value for the first resonance frequency coincides with the classical beam vibration theory, i.e. $\frac{\omega}{\omega^*} = 1$. First normalized resonance frequency of ω/ω^* is shown versus L/h by changing the value of *n* in figure 2. As it is seen, by the increase of the value of *n*, the normalized frequency of ω/ω^* is decreased due to higher volume percentage of ceramic. The rate of decrement is much more in lower values of L/h.

Also, variations of the normalized frequency of ω/ω^* versus *l/h* with changes of the value of *n* and by keeping the value of *L/h=5* as a constant is shown in figure 3. It is clear that by increasing the value of *n* the value of ω/ω^* has a rising trend.

Changes of the normalized frequency of ω/ω^* with keeping *n=1* as a constant is shown in figure 4. As we increase the value of l/h the value of the frequency increases.

To investigate the effect of both volume percentage and length scale effect in more detail, first resonance frequency was calculated for a wide range of these two parameters. Table (2) summarizes the results.
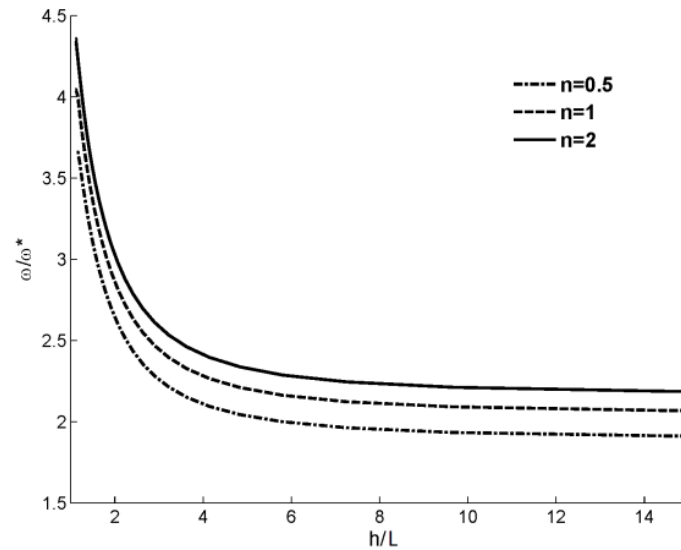
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Fig. 2- Variations of ω/ω^* versus L/h with respect to variations of n.
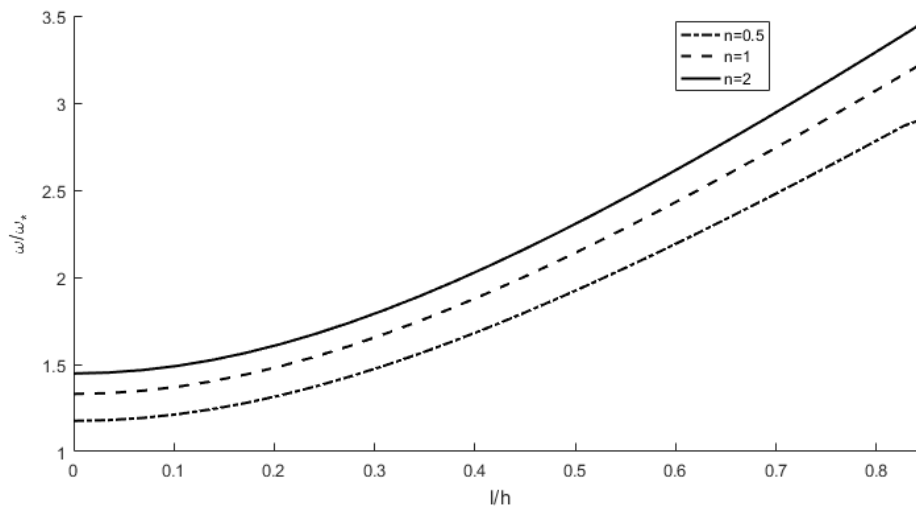
Fig. 3- Variations of ω/ω^* versus l/h with changes of n.

Fig. 4- Variations of ω/ω^* versus L/h with changes of l/h.

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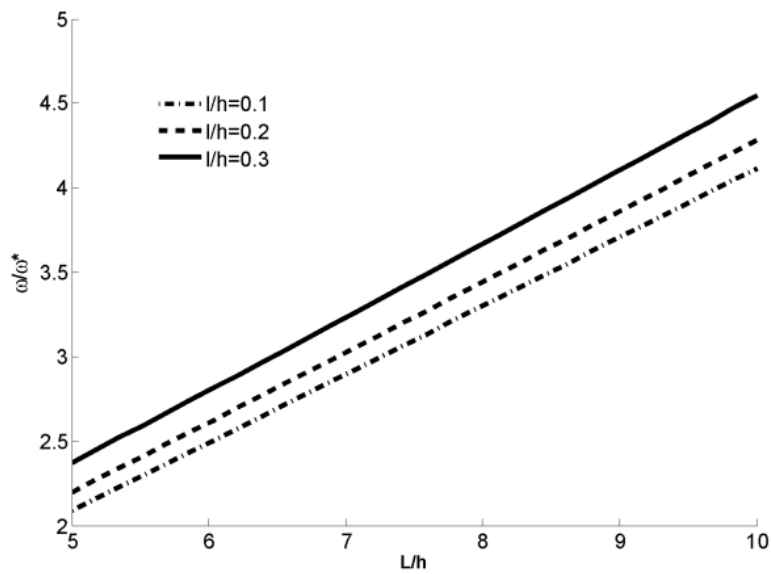
5-2- Forced vibration

The ratio of maximum value of displacement to the force amplitude W_{max}/q_0 in terms of the dimensionless frequency for the non-dimensional normalized length scale parameter (l/h) has been shown in figures 5 and 6, for L/h=5, respectively. Variations of W_{max}/q_0 versus Ω/ω^* with l/h=0.3.

Figure (5) depicts the frequency responses for various l/h ratio with the fixed values of n=0.5 and L/h=10. As can be seen, there is a big frequency shift in fist and third modes while the second mode seems less sensitive to this ratio. It can be concluded (at least for this case) that symmetric modes are more affected by the l/h ratio.

In figure (6), n and l/h are assumed to be 0.5 and 0.2 respectively. The figure shows frequency responses for different L/h values. By increasing the ratio, resonance frequencies decrease. It is not so surprising, since with the increase in L/h, the beam gets more flexible and its resonance frequencies decrease consequently.

Figure (7) shows the frequency responses for different values of n, while l/h and L/h are kept fixed at 0.2 and 20 respectively. As n increases, all resonance frequencies increase considerably. It could be predicted because of increasing the stiffness with higher proportion of ceramic in the mixture. The main point in the graph is that the responses are approximately the same in lower frequencies. This implies that the effect of n is more considerable in higher frequencies.

Non-dimensional natural frequencies for different volume percentage and length scale ratio are shown in Table 2.

Fig. 5- Variations of W_{max}/q_0 versus Ω/ω^* with l/h=0.5.

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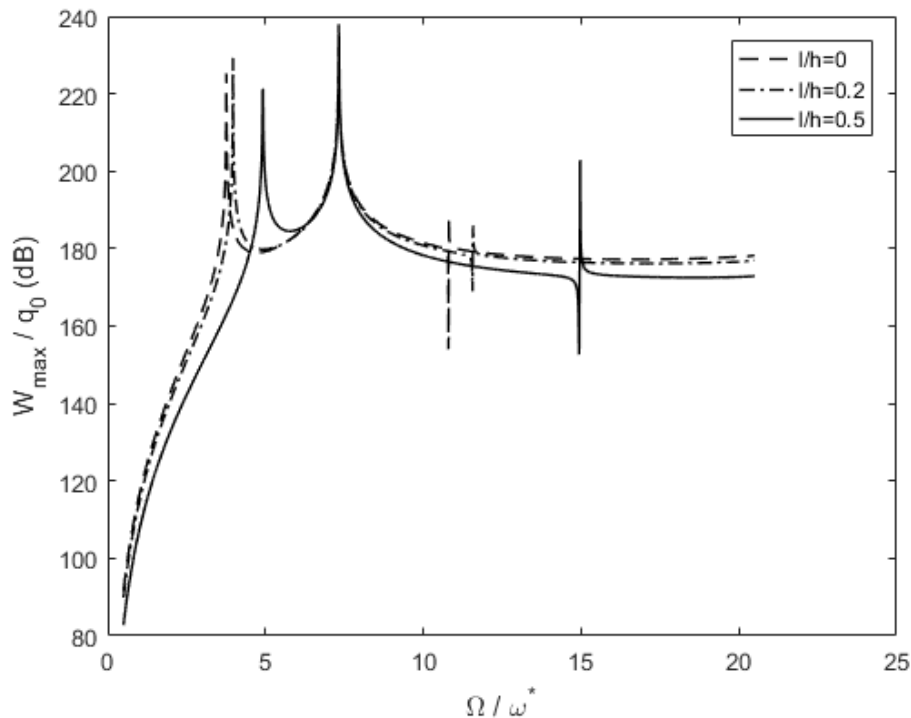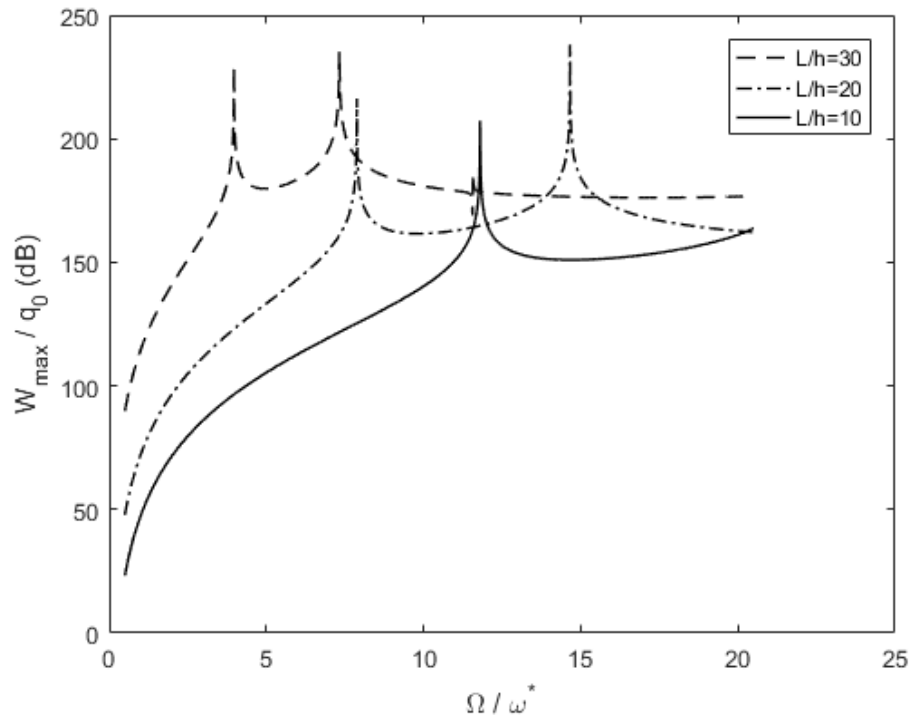
Fig. 6- Variations of W_{max}/q_0 versus Ω/ω^* with L/h=6.

Fig. 7- Variations of W_{max}/q_0 versus Ω/ω^* with L/h=8.

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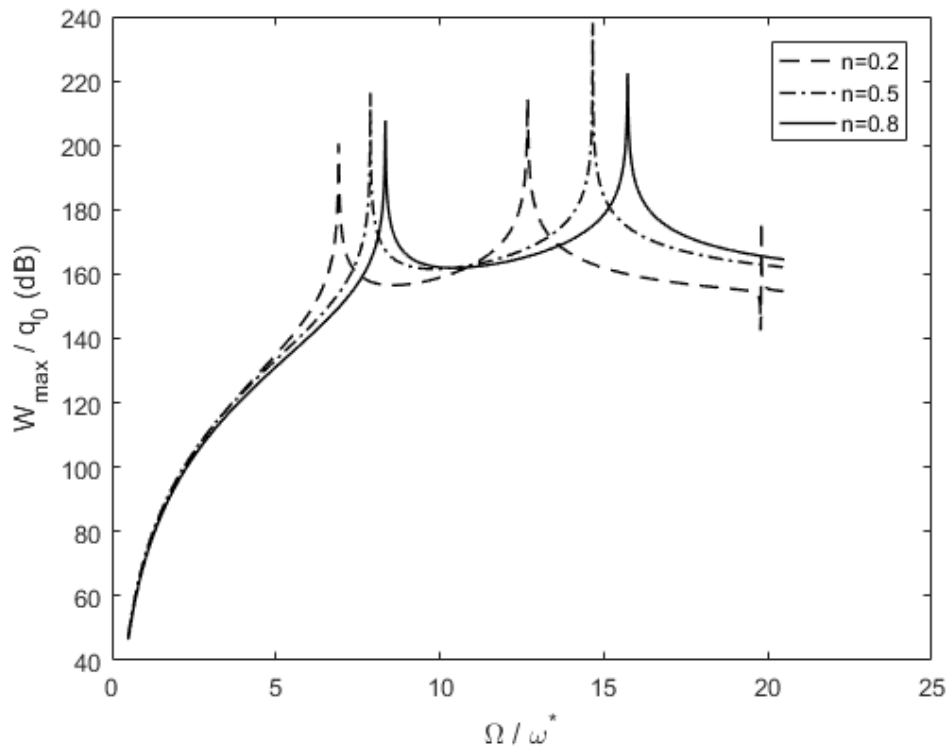
Table 2- non-dimensional natural frequencies for different volume percentage and length scale ratio

	l/h=0	l/h=0.05	l/h=0.1	l/h=0.15	l/h=0.2	l/h=0.25
n=0	1	1.008	1.034	1.076	1.134	1.208
n=0.2	1.089	1.101	1.135	1.192	1.270	1.367
n=0.4	1.107	1.119	1.158	1.221	1.307	1.415
n=0.6	1.205	1.218	1.259	1.325	1.416	1.530
n=0.8	1.274	1.288	1.330	1.399	1.493	1.611
n=1	1.337	1.351	1.394	1.464	1.560	1.681

6- Conclusions

In this study, free and forced vibration of micro-beams composed of functionally graded materials has been investigated based on the modified coupled stress and the third order shear deformation theory. The derived equations have been solved by Galerkin's method. The results show that with a fixed value of *n*, the value of ω/ω^* decreases with respect to the *h/l* parameter. Also, the normalized frequency ω/ω^* increases with respect to *n* for constant *l/h*. Also, by the increase of the value of *n*, the value of W_{max}/q_0 parameter decreases in different values of Ω/ω^*. Taking into account the results of forced vibration results, it is clear that by the increase of the value of *n*, the value of amplitude parameter W_{max}/q_0 decreases in different values of Ω/ω^*.

The frequency responses are approximately the same in lower frequencies with fixed length parameters. This implies that the effect of n is more considerable in higher frequencies.

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Appendix A

Appendix A.1 Stress resultants calculation

$$N_{xx} = (EA)_{eq}\frac{\partial u}{\partial x} + \left((EQ)_{eq} - c_1(EP)_{eq}\right)\frac{\partial \phi_x}{\partial x} - c_1(EP)_{eq}\frac{\partial^2 w}{\partial x^2}$$

$$M_{xx} = (EQ)_{eq}\frac{\partial u}{\partial x} + \left((EI)_{eq} - c_1(EH)_{eq}\right)\frac{\partial \phi_x}{\partial x}$$
$$-c_1(EH)_{eq}\frac{\partial^2 w}{\partial x^2}$$

$$P_{xx} = c_1\left((EP)_{eq}\frac{\partial u}{\partial x} + \left((EH)_{eq} - c_1(ES)_{eq}\right)\frac{\partial \phi_x}{\partial x} - c_1(ES)_{eq}\frac{\partial^2 w}{\partial x^2}\right)$$

$$Q_x = \left((\mu A)_{eq} - 3c_1(\mu I)_{eq}\right)\left(\phi_x + \frac{\partial w}{\partial z}\right)$$

$$R_x = 3c_1\left((\mu I)_{eq} - 3c_1(\mu H)_{eq}\right)\left(\phi_x + \frac{\partial w}{\partial z}\right)$$

$$M_{xy}^m = -\frac{1}{2}\left(\left((\beta A)_{eq} + 3c_1(\beta I)_{eq}\right)\frac{\partial^2 w}{\partial x^2} + \left(-(\beta A)_{eq} + 3c_1(\beta I)_{eq}\right)\frac{\partial \phi_x}{\partial x}\right)$$

$$P_{xy}^m = -\frac{3c_1}{2}\left(\left((\beta I)_{eq} + 3c_1(\beta H)_{eq}\right)\frac{\partial^2 w}{\partial x^2} + \left(-(\beta I)_{eq} + 3c_1(\beta H)_{eq}\right)\frac{\partial \phi_x}{\partial x}\right)$$

$$R_y^m = -9c_1^2(\beta I)_{eq}\left(\frac{\partial w}{\partial x} + \phi_x\right) \qquad (A.1)$$

in which

$$\left[(EA)_{eq}, (EQ)_{eq}, (EI)_{eq}, (EP)_{eq}, (EH)_{eq}, (ES)_{eq}\right] = \int_A E(z)[1, z, z^2, z^3, z^4, z^6]dA$$

$$\left[(\mu A)_{eq}, (\mu I)_{eq}, (\mu H)_{eq}\right] = \int_A E(z)[1, z^2, z^4]dA$$

$$\left[(\beta A)_{eq}, (\beta I)_{eq}, (\beta H)_{eq}\right]$$

$$= \int_A \mu(z)l^2(z)[1, z^2, z^4]dA \qquad (A.2)$$

Appendix A.2 definition of k_i coefficients in equations of motion

$$k_1 = (EA)_{eq}$$

$$k_2 = (EQ)_{eq} - c_1(EP)_{eq}$$

$$k_3 = -c_1(EP)_{eq}$$

$$k_4 = (EQ)_{eq} - c_1(EP)_{eq}$$

$$k_5 = (EI)_{eq} - 2c_1(EH)_{eq} + (c_1)^2(ES)_{eq}$$
$$+\frac{1}{4}\left((\beta A)_{eq} - 3c_1(\beta I)_{eq}\right)$$
$$+\frac{3c_1}{4}\left(-(\beta I)_{eq} + 3c_1(\beta H)_{eq}\right)$$

$$k_6 = -c_1(EH)_{eq} - (c_1)^2(ES)_{eq}$$
$$-\frac{1}{4}\left((\beta A)_{eq} - 3c_1(\beta I)_{eq}\right)$$
$$+\frac{3c_1}{4}\left((\beta I)_{eq} + 3c_1(\beta H)_{eq}\right)$$

$$k_7 = 3c_1\left((\mu I)_{eq} - 3c_1(\mu H)_{eq}\right) - 9c_1^2(\beta I)_{eq}$$
$$-\left((\mu A)_{eq} - 3c_1(\mu I)_{eq}\right)$$

$$k_8 = c_1(EP)_{eq}$$

$$k_9 = c_1\left((EH)_{eq} - c_1(ES)_{eq}\right)$$

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$$-\frac{1}{4}\left(-(\beta A)_{eq} + 3c_1(\beta I)_{eq}\right)$$

$$-\frac{3c_1}{4}\left(-(\beta I)_{eq} + 3c_1(\beta H)_{eq}\right)$$

$$k_{10} = -(c_1)^2(ES)_{eq} - \frac{1}{4}\left((\beta A)_{eq} + 3c_1(\beta I)_{eq}\right)$$

$$-\frac{3c_1}{4}\left((\beta I)_{eq} + 3c_1(\beta H)_{eq}\right)$$

$$k_{11} = \left((\mu A)_{eq} - 3c_1(\mu I)_{eq}\right)$$
$$-3c_1\left((\mu I)_{eq} - 3c_1(\mu H)_{eq}\right) + 9c_1^2(\beta I)_{eq}$$

$$(A.3)$$

Appendix A.3 Definition of free vibration equations parameters

$$g_1 = -k_1\left(\frac{m\pi}{L}\right)^2 \int_0^L \sin^2\left(\frac{m\pi}{L}x\right)dx = -\frac{k_1}{2L}(m\pi)^2$$

$$g_2 = I_0 \int_0^L \sin^2\left(\frac{m\pi}{L}x\right)dx = I_0\frac{L}{2}$$

$$g_3 = -k_2\left(\frac{m\pi}{L}\right)^2 \int_0^L \sin^2\left(\frac{m\pi}{L}x\right)dx = -\frac{k_2}{2L}(m\pi)^2$$

$$g_4 = (I_1 - c_1 I_3)\int_0^L \sin^2\left(\frac{m\pi}{L}x\right)dx$$
$$= (I_1 - c_1 I_3)\frac{L}{2}$$

$$g_5 = k_3 \int_0^L \frac{d^3\psi_m}{dx^3}\sin\left(\frac{m\pi}{L}x\right)dx$$

$$g_6 = c_1 I_3 \int_0^L \frac{d\psi_m}{dx}\sin\left(\frac{m\pi}{L}x\right)dx$$

$$g_7 = -k_2\left(\frac{m\pi}{L}\right)^2 \int_0^L \sin^2\left(\frac{m\pi}{L}x\right)dx$$
$$= -\frac{k_4}{2L}(m\pi)^2$$

$$g_8 = (I_1 - c_1 I_3)\int_0^L \sin^2\left(\frac{m\pi}{L}x\right)dx$$
$$= (I_1 - c_1 I_3)\frac{L}{2}$$

$$g_9 = \left(k_7 - k_5\left(\frac{m\pi}{L}\right)^2\right)\int_0^L \sin^2\left(\frac{m\pi}{L}x\right)dx$$
$$= \left(k_7 - k_5\left(\frac{m\pi}{L}\right)^2\right)\frac{L}{2}$$

$$g_{10} = (I_2 - 2c_1 I_4 + (c_1)^2 I_6)\int_0^L \sin^2\left(\frac{m\pi}{L}x\right)dx$$
$$= (I_2 - 2c_1 I_4 + (c_1)^2 I_6)\frac{L}{2}$$

$$g_{11} = \int_0^L \left(k_7 \frac{d\psi_m}{dx} + k_6 \frac{d^3\psi_m}{dx^3}\right)\sin\left(\frac{m\pi}{L}x\right)dx$$

$$g_{12} = c_1(I_4 - c_1 I_6)\int_0^L \sin\left(\frac{m\pi}{L}x\right)dx$$

$$g_{13} = -k_8\left(\frac{m\pi}{L}\right)^3 \int_0^L \cos\left(\frac{m\pi}{L}x\right)\psi_m(x)dx$$

$$g_{14} = c_1 I_3\left(\frac{m\pi}{L}\right)\int_0^L \cos\left(\frac{m\pi}{L}x\right)\psi_m(x)dx$$

$$g_{15} = \frac{m\pi}{L}\left(k_{11} - k_9\left(\frac{m\pi}{L}\right)^2\right)\int_0^L \cos\left(\frac{m\pi}{L}x\right)\psi_m(x)dx$$

$$g_{16} = c_1(I_4 - c_1 I_6)\left(\frac{m\pi}{L}\right)\int_0^L \cos\left(\frac{m\pi}{L}x\right)\psi_m(x)dx$$

$$g_{17} = \int_0^L \left(k_{11}\frac{d^2\psi_m}{dx^2} + k_{10}\frac{d^4\psi_m}{dx^4}\right)\psi_m(x)dx$$

$$g_{18} = \int_0^L \left((c_1)^2 I_6 \frac{d^2\psi_m}{dx^2} - I_0\psi_m\right)\psi_m(x)dx$$

$$(A.4)$$

Appendix A.4 Definition of forced vibration equations parameters

$$a_1 = g_2(g_{12}g_{16} - g_{10}g_{18}) + g_4(g_8g_{18} - g_{12}g_{14}) - g_6(g_8g_{16} - g_{10}g_{14})$$

$$a_2 = g_1(g_{12}g_{16} - g_{10}g_{18})$$
$$+g_2(g_{10}g_{17} - g_9g_{18} + g_{15}g_{12} - g_{11}g_{16})$$
$$+g_3(g_8g_{18} - g_{12}g_{14})$$
$$+g_4(g_{11}g_{14} - g_{13}g_{12} + g_7g_{18} - g_8g_{17})$$
$$+g_5(g_8g_{16} - g_{10}g_{14})$$
$$-g_6(g_7g_{16} + g_8g_{15} - g_{13}g_{10} - g_9g_{14})$$

$$a_3 = g_2(g_9g_{17} - g_{15}g_{11})$$
$$+g_1(g_{10}g_{17} - g_9g_{18} + g_{15}g_{12} - g_{11}g_{16})$$
$$+g_4(g_{11}g_{13} - g_7g_{17})$$
$$+g_3(g_{11}g_{14} - g_{13}g_{12} + g_7g_{18} - g_8g_{17})$$
$$+g_6(g_7g_{15} - g_9g_{13})$$
$$-g_5(g_7g_{16} + g_8g_{15} - g_{13}g_{10} - g_9g_{14})$$

$$a_4 = g_1(g_9g_{17} - g_{15}g_{11}) + g_3(g_{11}g_{13} - g_7g_{17}) - g_5(g_7g_{15} - g_9g_{13})$$

$$a_5 = g_6g_{10} - g_4g_{12} , a_6 = g_{11}g_4 - g_3g_{12} + g_6g_9 - g_5g_{10} , a_7 = g_{11}g_3 - g_5g_9$$

$$a_8 = g_2g_{12} - g_6g_8 , a_9 = -g_{11}g_2 + g_1g_{12} - g_6g_7 + g_5g_8 , a_{10} = g_5g_7 - g_{11}g_1$$

$$a_{11} = g_2g_{10} - g_5g_8 , a_{12} = g_9g_2 + g_1g_{10} - g_4g_7 + g_3g_8 , a_{13} = g_1g_9 - g_3g_7$$

$$(A.5)$$