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CAS Wavelet Method for the Numerical Solution of Boundary Integral Equations with Logarithmic Singular Kernels

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Abstract. In this paper, we present a computational method for solving boundary integral equations with logarithmic singular kernels which occur as reformulations of a boundary value problem for Laplace's equation. The method is based on the use of the Galerkin method with CAS wavelets constructed on the unit interval as basis. This approach utilizes the non-uniform Gauss-Legendre quadrature rule for approximating logarithm-like singular integrals and so reduces the solution of boundary integral equations to the solution of linear systems of algebraic equations. The properties of CAS wavelets are used to make the wavelet coefficient matrices sparse, which eventually leads to the sparsity of the coefficient matrix of the obtained system. Finally, the validity and efficiency of the new technique are demonstrated through a numerical example.

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1. Introduction

Consider the boundary value problem for Laplace's equation

$$\Delta u(\mathrm{x}) = 0, \qquad \mathrm{x} \in D \subset \mathbb{R}^2, \tag{1}$$

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with linear Robin boundary condition

$$\frac{\partial u(\mathrm{x})}{\partial n_{\mathrm{x}}} + p(\mathrm{x})u(\mathrm{x}) = g(\mathrm{x}), \qquad \mathrm{x} \in \partial D, \tag{2}$$

where D is a bounded, open, simply connected region in the plane, n_{x} is the outward unit normal on ∂D, $p(\mathrm{x})$ and $g(\mathrm{x})$ are given functions on ∂D with $p(\mathrm{x}) \geqslant 0$ but $p \not\equiv 0$ and $u(\mathrm{x})$ is the unknown function to be determined [6].

Using Green's formula, we can represent the solution $u(\mathrm{x}) \in C^1(\bar{D}) \cap C^2(D)$, for every $\mathrm{x} \in D$, by [11]

$$u(\mathrm{x}) = \frac{1}{2\pi}\int_{\partial D} u(\mathrm{y})\left(p(\mathrm{y})\ln|\mathrm{x}-\mathrm{y}| - g(\mathrm{y})\ln|\mathrm{x}-\mathrm{y}| + \frac{\partial \ln|\mathrm{x}-\mathrm{y}|}{\partial n_{\mathrm{y}}}\right)\mathrm{d}s_{\mathrm{y}}. \tag{3}$$

The boundary condition of $u(\mathrm{x})$ on ∂D satisfies the boundary integral equation of the second kind

$$u(\mathrm{x}) - \frac{1}{\pi}\int_{\partial D} u(\mathrm{y})\left(p(\mathrm{y})\ln|\mathrm{x}-\mathrm{y}| + \frac{\partial \ln|\mathrm{x}-\mathrm{y}|}{\partial n_{\mathrm{y}}}\right)\mathrm{d}s_{\mathrm{y}} = -\frac{1}{\pi}\int_{\partial D} g(\mathrm{y})\ln|\mathrm{x}-\mathrm{y}|\mathrm{d}s_{\mathrm{y}}, \tag{4}$$

for every $\mathrm{x} \in \partial D$. Let the boundary ∂D be a smooth simple closed curve with a twice continuously differentiable [6] and parameterized by

$$\mathrm{r}(t) = (\xi(t), \eta(t)), \qquad 0 \leqslant t \leqslant 1, \tag{5}$$

with $\mathrm{r} \in C^2[0,1]$ and $|\mathrm{r}'(t)| \neq 0$. We also assume the parametrization traverses ∂D in a counter-clockwise direction. Introduce the interior unit normal $\mathrm{n}(t)$ that is orthogonal to the curve ∂D at $\mathrm{r}(t)$:

$$\mathrm{n}(t) = \frac{(-\eta'(t), \xi'(t))}{\sqrt{\xi'(t)^2 + \eta'(t)^2}}, \tag{6}$$

and thus we obtain the following quantities

$$\mathrm{d}s_{\mathrm{y}} = \sqrt{\xi'(t)^2 + \eta'(t)^2}\mathrm{d}s, \tag{7}$$

and

$$\frac{\partial \ln|\mathrm{x}-\mathrm{y}|}{\partial n_{\mathrm{y}}} = \begin{cases} \frac{-\eta'(s)[\xi(t)-xi(s)]+\xi'(s)[\eta(t)-\eta(s)]}{\sqrt{\xi'(s)^2+\eta'(s)^2}([\xi(t)-\xi(s)]^2+[\eta(t)-\eta(s)]^2)}, & s \neq t, \\ \frac{-\eta'(t)\xi''(t)+\xi'(t)\eta''(t)}{2\sqrt{\xi'(s)^2+\eta'(s)^2}(\xi'(t)^2+\eta'(t)^2)}, & s = t. \end{cases} \tag{8}$$

Using these representations and multiplying in (4) by $-\pi$, then (4) can be written as

$$-\pi u(t) + \int_0^1 K(t,s)u(s)\mathrm{d}s = f(t), \qquad 0 \leqslant t \leqslant 1, \tag{9}$$

or in the operator form

$$(-\pi + \mathcal{K})u = f, \tag{10}$$

where

$$K(t,s) = p(\mathrm{r}(s))\sqrt{\xi'(s)^2+\eta'(s)^2}\ln|\mathrm{r}(t)-\mathrm{r}(s)| + q(t,s), \tag{11}$$

with

$$q(t,s) = \begin{cases} \frac{-\eta'(s)[\xi(t)-xi(s)]+\xi'(s)[\eta(t)-\eta(s)]}{[\xi(t)-\xi(s)]^2+[\eta(t)-\eta(s)]^2}, & s \neq t, \\ \\ \frac{-\eta'(t)\xi''(t)+\xi'(t)\eta''(t)}{2(\xi'(t)^2+\eta'(t)^2)}, & s = t, \end{cases} \tag{12}$$

and

$$f(t) = \int_0^1 g(\mathrm{r}(s))\sqrt{\xi'(s)^2+\eta'(s)^2}\ln|\mathrm{r}(t)-\mathrm{r}(s)|\mathrm{d}s. \tag{13}$$

Note that in the weakly singular integral equation (9), we have used $u(t) \equiv u(\mathrm{r}(t))$ for simplicity in notation.

It is well-known that integral equations are one of the significant topics in computational mathematics and a large number of papers have presented many numerical methods for solving them [4, 17, 19, 21]. In recent years, several simple and accurate methods based on orthogonal basic functions, including wavelets, have been used to approximate the solution of integral and integro-differential equations [2, 9, 12, 20]. The main advantage of using orthogonal basis is that it reduces the problem into solving a system of algebraic equations. Overall, there are so many different families of orthogonal functions which can be used in this method so that it is sometimes difficult to select the most suitable one. Since 1991, wavelet technique has been applied to solve integral equations. Wavelets, as very well localized functions, are considerably useful for solving integral equations and provide accurate solutions. Also, the wavelet technique allows the creation of very fast algorithms when compared with the algorithms ordinarily used [5].

Several methods have been proposed for solving the Fredholm integral equation of the second kind with logarithmic singular kernel. Discrete Petrov-Galerkin methods [8] and iterated fast multiscale Galerkin methods [16] have been applied to solve Fredholm integral equations of the second kind with weakly singular kernels. Pedas and Vainikko [18] used the piecewise polynomial collocation method to solve weakly singular integral equations. In [7], a numerical solution of Fredholm integral equations of the second kind with weakly singular kernels by using the hybrid collocation method is studied. Daubechies interval wavelet [22], Legendre wavelet [1, 5] and trigonometric Hermit wavelet [13] are used to give a numerical solution of weakly singular integral equations.

The main purpose of this article is to present a numerical method for solving the weakly singular integral equation (9) by using CAS wavelets. The properties of CAS wavelets are used to convert (9) into a linear system of algebraic equations. We will notice that this wavelet make the wavelet coefficient matrices sparse and accordingly leads to the sparsity of the coefficient matrix of the final system and provide accurate solutions.

The outline of the paper is as follows: In Section 2, we review some properties of CAS wavelets and approximate the one variable function $f(x)$ and also the kernel function $K(x,y)$ by these wavelets. Section 3 is devoted to present a computational method for solving the integral equation (9) utilizing CAS wavelets. A numerical example is given in Section 4. Finally, we conclude the article in Section 5.

2. Properties of CAS Wavelets

2.1 *CAS Wavelets*

Wavelets consist of a family of functions constructed from dilation and translation of a single function called the mother wavelet. When the dilation parameter a and the translation parameter b vary continuously, we have the following family of continuous wavelets [10]

$$\psi_{a,b}(t) = |a|^{\frac{-1}{2}}\psi(\frac{t-b}{a}), \qquad a, b \in \mathbb{R}, a \neq 0. \tag{14}$$

If we restrict the parameters a and b to discrete values $a = a_0^{-k}$, $b = nb_0a_0^{-k}$, $a_0 > 1, b_0 > 0$ where n and k are positive integers, then we have the following family of discrete wavelets

$$\psi_{k,n}(t) = |a_0|^{\frac{k}{2}}\psi(a_0^k t - nb_0), \tag{15}$$

where $\psi_{k,n}(t)$ form a wavelet basis for $L^2(\mathbb{R})$. In particular, when $a_0 = 2, b_0 = 1$ then $\psi_{k,n}(t)$ forms an orthonormal basis [10, 23].

The CAS wavelets, $\psi_{nm}(t) = \psi(k, n, m, t)$ have four arguments; $n = 1, 2, ..., 2^k$, k is any non-negative integer, m is any integer and t is the normalized time. The orthonormal CAS wavelets are defined on the interval $[0, 1)$ by [1, 23]

$$\psi_{nm}(t) = \begin{cases} 2^{k/2}\mathrm{CAS}_m(2^k t - n + 1), & \frac{n-1}{2^k} \leqslant t < \frac{n}{2^k}, \\ 0, & \text{otherwise}, \end{cases} \tag{16}$$

where

$$\mathrm{CAS}_m(t) = \cos(2m\pi t) + \sin(2m\pi t). \tag{17}$$

Remark 1 Note that for $m = 0$, the CAS wavelets have the following form [14]

$$\psi_{n0}(t) = 2^{k/2}B_n(t) = 2^{k/2}\begin{cases} 1, & \frac{n-1}{2^k} \leqslant t < \frac{n}{2^k}, \\ 0, & \text{otherwise}, \end{cases} \tag{18}$$

where $\{B_n(t)\}_{n=1}^{2^k}$ are a basis set that are called the block pulse functions (BPFs) over $[0, 1)$.

2.2 *Function Approximation*

A function $f(x) \in L^2[0, 1]$ may be expanded as

$$f(x) = \sum_{n=1}^{\infty}\sum_{m\in\mathbb{Z}} c_{nm}\psi_{nm}(x), \tag{19}$$

where

$$c_{nm} =< f(x), \psi_{nm}(x) >= \int_0^1 f(x)\psi_{nm}(x)\mathrm{d}x, \tag{20}$$

in which $<.,.>$ denotes the inner product. The series (19) is truncated as

$$f(x) \simeq P_{k,M}(f(x)) = \sum_{n=1}^{2^k} \sum_{m=-M}^{M} c_{nm}\psi_{nm}(x) = C^t\Psi(x), \tag{21}$$

where C and $\Psi(x)$ are two vectors given by

$$\begin{aligned} C &= [c_{1(-M)}, c_{1(-M+1)}, ..., c_{1M}, c_{2(-M)}, ..., c_{2M}, ..., c_{(2^k)(-M)}, ..., c_{(2^k)M}]^t \\ &= [c_1, c_2, ..., c_{2^k(2M+1)}]^t, \end{aligned} \tag{22}$$

and

$$\begin{aligned} \Psi(x) = [&\psi_{1(-M)}(x), \psi_{1(-M+1)}(x), ..., \psi_{1M}(x), \psi_{2(-M)}(x), ..., \psi_{2M}(x), \\ &..., \psi_{(2^k)(-M)}(x), ..., \psi_{(2^k)M}(x)]^t \\ = [&\psi_1(x), \psi_2(x), ..., \psi_{2^k(2M+1)}(x)]^t. \end{aligned} \tag{23}$$

Based on the above formulations, we can present the following theorem from [1]:

THEOREM 2.1 *A function $f(x) \in L^2[0,1]$, with bounded second derivative, say $|f''(x)| \leqslant \gamma$, can be expanded as an infinite sum of the CAS wavelets, and the series converges uniformly to $f(x)$, that is*

$$f(x) = \sum_{n=1}^{\infty} \sum_{m\in\mathbb{Z}} c_{nm}\psi_{nm}(x). \tag{24}$$

Furthermore, we have

$$\|P_{k,M}f - f\|_\infty \leqslant \frac{\gamma}{\pi^2} \sum_{n=2^k+1}^{\infty} \sum_{m=M+1}^{\infty} \frac{1}{n^{\frac{5}{2}}m^2}, \quad x \in [0,1]. \tag{25}$$

Similarly, by considering $i = n(2M+1) - M + m$ and $j = n'(2M+1) - M + m'$, we approximate $K(x,y) \in L^2([0,1] \times [0,1])$ as

$$K(x,y) \simeq \sum_{i=1}^{2^k(2M+1)} \sum_{j=1}^{2^k(2M+1)} K_{ij}\psi_i(x)\psi_j(y), \tag{26}$$

or in the matrix form

$$K(x,y) \simeq \Psi^t(x)\mathbf{K}\Psi(y), \tag{27}$$

where $\mathbf{K} = [K_{ij}]_{1\leqslant i,j \leqslant 2^k(2M+1)}$ with the entries

$$K_{ij} =< \psi_i(x), < K(x,y), \psi_j(y) >>= \int_0^1 \int_0^1 \psi_i(x)\psi_j(y)K(x,y)\mathrm{d}x\mathrm{d}y. \tag{28}$$

Remark 2 Note that the weakly singular logarithmic kernels are in $L^2([0,1] \times [0,1])$

and consonantly define the compact integral operators on $L^2[0,1]$ as [6]

$$\mathcal{K}u(x) = \int_0^1 K(x,y)u(y)\mathrm{d}y.$$

3. Solution of Boundary Integral Equations

In this section, the CAS wavelet method is used for solving boundary integral equations with logarithmic singular kernels of the second kind in the form (9). For this aim, we approximate the functions $f(x)$, $u(x)$ and $K(x,y)$ in the matrix forms:

$$f(t) \simeq F^t\Psi(t), \tag{29}$$

$$u(t) \simeq U^t\Psi(t), \tag{30}$$

$$K(t,s) \simeq \Psi^t(t)\mathbf{K}\Psi(s), \tag{31}$$

By substituting (29), (30) and (31) into (9), we obtain

$$\pi\Psi^t(t)U \simeq \int_0^1 \Psi^t(t)\mathbf{K}\Psi(s)\Psi^t(s)U\mathrm{d}s - \Psi^t(t)F \tag{32}$$

$$= \Psi^t(t)\mathbf{K}\left(\int_0^1 \Psi(s)\Psi^t(s)\mathrm{d}s\right)U - \Psi^t(t)F. \tag{33}$$

Now, we define the residual $R_{k,M}(t)$ as

$$R_{k,M}(t) = \pi\Psi^t(t)U - \Psi^t(t)\mathbf{K}\left(\int_0^1 \Psi(s)\Psi^t(s)\mathrm{d}s\right)U + \Psi^t(t)F, \tag{34}$$

By using the orthonormality of the CAS wavelets on $[0,1]$ implies that

$$\int_0^1 \Psi(s)\Psi^t(s)\mathrm{d}s = \mathbf{I}, \tag{35}$$

where $\mathbf{I}_{2^k(2M+1)\times 2^k(2M+1)}$ is the identity matrix. So, we have

$$R_{k,M}(t) = \pi\Psi^t(t)U - \Psi^t(t)\mathbf{K}U + \Psi^t(t)F. \tag{36}$$

Our aim is to compute $u_1, u_2, ..., u_{2^k(2M+1)}$ such that $R_{k,M}(x) \equiv 0$, but in general, it is not possible to choose such $u_i,\ i = 1, 2, ..., 2^k(2M+1)$. In this work, utilizing the Galarkin method, $R_{k,M}(x)$ is made as small as possible such that

$$< \psi_{mn}(t), R_{k,M}(t) >= 0, \tag{37}$$

where $n = 1, 2, ..., 2k$ and $m = -M, -M+1, ..., M$.
Now by taking inner product $< \Psi(x), . >$ upon both sides of (36) and using (35), we obtain

$$(\mathbf{K} - \pi I)U = F. \tag{38}$$

Remark 1 As a conclusion from the property of the sparsity of matrix $\mathbf{K}$ presented in [1], when i or $j \to \infty$ then $|K_{ij}| \to 0$. Accordingly, by increasing k or M, we can make $\mathbf{K}$ sparse. For this purpose, we choose a threshold ε_0 and define

$$\bar{\mathbf{K}} = [\bar{K}_{ij}]_{2^{k-1}M \times 2^{k-1}M}, \tag{39}$$

where

$$\bar{K}_{ij} = \begin{cases} K_{ij}, & |K_{ij}| \geqslant \varepsilon_0 \\ 0, & \text{otherwise.} \end{cases} \tag{40}$$

Obviously, $\bar{\mathbf{K}}$ is a sparse matrix. Now, we rewrite the integral equation (38) as follows

$$(\bar{\mathbf{K}} - \pi I)U = F. \tag{41}$$

Therefore, we can use (41) instead of (38).

There are two types of integrals to be evaluated in the system (38) or (41) as

I. the inner products $< f(t), \psi_{nm}(t) >$,
II. the double integrals $< \psi_i(t), < K(t, s), \psi_j(s) >>$.

To approximate these integrals, we use the composite q_N-point Gauss-Legendre rule with M non-uniform subdivisions relative to the coefficients $\{v_k\}$ and weights $\{w_k\}$ in interval $[-1, 1]$. Suppose that $g(x)$ defined on $(0, 1)$ satisfies

$$|g^{(2k)}(x)| \leqslant Cx^{-\epsilon-2k}, \quad for\ all\ x \in (0, 1), \tag{42}$$

for some $\epsilon \in (0, 1)$ and the constant C. Then, for any given integer $M > 0$, there holds

$$\int_0^1 g(x)\mathrm{d}x = \sum_{k=1}^{q_N} w_k \sum_{q=1}^{M} \frac{\Delta x_q}{2} g(\theta_k^q) + O(\frac{1}{M^{2q_N}}), \tag{43}$$

where

$$\theta_k^q = \frac{\Delta x_q}{2} v_k + \bar{x}_q,\ \Delta x_q = x_q - x_{q-1}\ and\ \bar{x}_q = \frac{x_q + x_{q-1}}{2},$$

with

$$x_q = \left(\frac{q}{M}\right)^s, \quad s = \frac{2q_N + 1}{1 - \epsilon}.$$

Based on the definitions of $k(t, s)$ and $f(t)$ in (11) and (13), respectively, it is clear that the logarithm-like singular integrals I and II are not convenient for numerical computations, since the singularity occurs along the diagonal. To deal with the singularity of $k(t, s)$ and $f(t)$ efficiently, we find that a change of variables for the integral is most helpful [11]. Let

$$t = y - x, \quad s = y + x \quad or \quad x = \frac{s - t}{2}, \quad \frac{s + t}{2}.$$

With this change of variables, the unit square is transformed into the diamond

$$\{(t,s) : |t| + |s-1| \leqslant 1\},$$

and so, the singularity shifts at the line $x = 0$. Since the condition (42) for any positive integer k and for any small positive number ϵ is now satisfied [11], we can use the composite q_N-point Gauss-Legendre rule with M non-uniform subdivisions for approximating the integrals I and II.

4. Numerical Experiment

Consider the boundary value problem for Laplace's equation [11]

$$\Delta u(\mathrm{x}) = 0, \qquad \mathrm{x} \in D = \left\{ (x_1, x_2) : x_1^2 + \frac{x_2^2}{4} < 1 \right\}, \tag{44}$$

with boundary condition

$$\frac{\partial u(\mathrm{x})}{\partial n_{\mathrm{x}}} + p(\mathrm{x})u(\mathrm{x}) = g(\mathrm{x}), \qquad \mathrm{x} \in \partial D = \left\{ (x_1, x_2) : x_1^2 + \frac{x_2^2}{4} = 1 \right\}. \tag{45}$$

Based on the discussions in Section 1 and using the parameterization on ∂D by

$$\mathrm{r}(t) = (\cos(2\pi t), 2\sin(2\pi t)), \qquad 0 \leqslant t \leqslant 1,$$

we can reduce the boundary value problem (44) with boundary condition (45) to the logarithmic singular Fredholm integral equation of the second kind given in the form

$$u(t) - \int_0^1 K(t,s)u(s)\mathrm{d}s = f(t), \qquad 0 \leqslant t \leqslant 1, \tag{46}$$

for the unknown $u(t) = u(\cos(2\pi t), 2\sin(2\pi t))$, with the kernel given by

$$\begin{aligned} K(s,t) = {} & 2p(\cos(2\pi s), 2\sin(2\pi s))\sqrt{1+\cos^2(2\pi s)} \\ & \ln\left(2|\sin(\pi(t-s))|\sqrt{1+3\cos^2(\pi(t+s))}\right) + \frac{2}{1+3\cos^2(\pi(t+s))}, \end{aligned}$$

and the right-hand side by

$$\begin{aligned} f(t) = -2 \int_0^1 & g(\cos(2\pi s), 2\sin(2\pi s))\sqrt{1+3\cos^2(\pi(t+s))} \\ & \times \ln\left(2|\sin(\pi(t-s))|\sqrt{1+3\cos^2(\pi(t+s))}\right)\mathrm{d}s. \end{aligned}$$

In this example, we choose [11]

$$p(\mathrm{x}) = 1, \qquad \text{and} \qquad g(\mathrm{x}) = \frac{2x_1}{\sqrt{1+3x_1^2}}, \qquad \mathrm{x} \in \partial D,$$

Table 1. Some numerical results

x	Exact solution	Approximate solution $k = 5, M = 1, \varepsilon_0 = 10^{-5}$	Approximate solution $k = 7, M = 2, \varepsilon_0 = 10^{-4}$	Method in [15] $J = 6, M = 128$
0.0	2.000000000	2.012793632	1.999845675	1.997598453
0.1	1.809016994	1.801758813	1.809845719	1.817502729
0.2	1.309016994	1.313681740	1.309368174	1.313650253
0.3	0.690983006	0.699015322	0.690215035	0.709715322
0.4	0.190983005	0.196792468	0.190315246	0.196792468
0.5	0.000000000	0.004976471	0.000225436	0.001501546
0.6	0.190983005	0.182415187	0.190895428	0.198530388
0.7	0.690983006	0.698631822	0.690863182	0.663110146
0.8	1.309016994	1.290284679	1.309849641	1.290284678
0.9	1.809016994	1.803207533	1.809847554	1.803207532
1.0	2.000000000	2.029639157	2.000287293	2.005675432

Table 2. Some error estimates

k	M	$\parallel u_{ex} - \hat{u} \parallel_\infty$	$\parallel u_{ex} - \hat{u} \parallel_2$
4	2	9.12×10^{-2}	7.21×10^{-2}
4	3	6.91×10^{-2}	3.15×10^{-2}
5	2	4.31×10^{-2}	3.72×10^{-2}
5	3	3.34×10^{-2}	1.68×10^{-2}
6	2	2.15×10^{-2}	1.81×10^{-2}
6	3	1.64×10^{-2}	9.23×10^{-3}
7	2	9.21×10^{-3}	8.34×10^{-3}
7	3	7.92×10^{-3}	4.65×10^{-3}

so that the boundary value problem (44) with boundary condition (45) has the unique exact solution

$$u_{ex}(\mathrm{x}) = u(x_1, x_2) = 1 + x_1, \qquad \mathrm{x} \in D.$$

Table 4 shows the numerical results for this example with $k = 5, M = 1, \varepsilon_0 = 10^{-5}$ and $k = 5, M = 1, \varepsilon_0 = 10^{-4}$ and meanwhile, results are compared with those of [15]. The approximate solution for $k = 4-7, M = 2, \varepsilon_0 = 10^{-5}$ is graphically shown in Figure 1 which agrees with exact solution. It is seen the numerical results are improved, as parameter k increases. Table 4 represents the error estimate for the result obtained of $\parallel . \parallel_\infty$ and $\parallel . \parallel_2$. The following norms are used for the errors of the approximation which are defined by

$$\parallel u_{ex} - \hat{u} \parallel_\infty = max\{|u_{ex}(x) - \hat{u}(x)|, \quad 0 \leqslant x \leqslant 1\},$$

and

$$\parallel u_{ex} - \hat{u} \parallel_2 = \left(\int_0^1 |u_{ex}(x) - \hat{u}(x)|^2 \mathrm{d}x \right)^{\frac{1}{2}},$$

where $\hat{u}(x)$ is the approximate solution of the exact solution $u_{ex}(x)$. In our compactions, we have used the 10-points composite non-uniform Gauss-Legendre (CNGL) quadrature rule with 10 subdivisions to approximate the integration numerically. The final linear algebraic systems are solved directly by using "LinearSolve" command from "LinearAlgebra" package in Maple 15 software with the Digits environment variable assigned to be 20. All calculations are run on a Pentium 4 PC Laptop with 2.50 GHz of CPU and 4 GB of RAM.

5. Conclusion

Boundary integral equations with logarithmic singular kernels are usually difficult to solve analytically. Therefore, the study of these types of integral equations

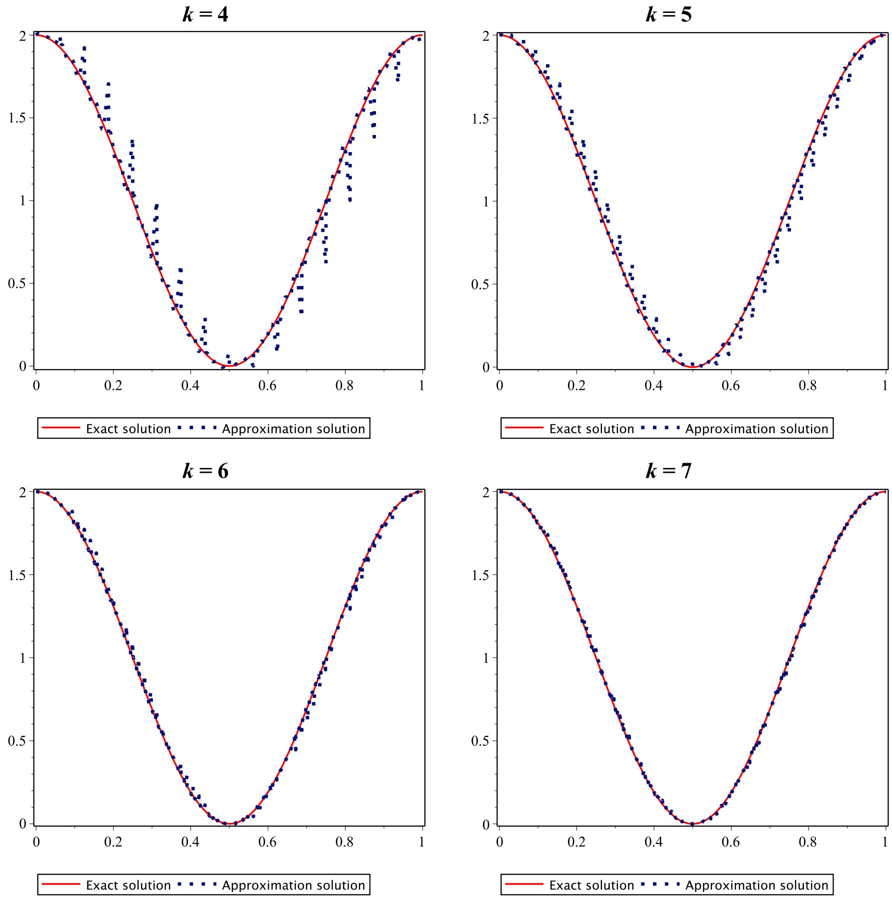

Figure 1. Approximate solutions with $k = 4 - 7, \;\; M = 2, \;\; \varepsilon_0 = 10^{-5}$

and numerical methods for solving them are very useful in application. The main purpose of this article is to describe an efficient and accurate scheme for solving boundary integral equations of the second kind with logarithmic singular kernels using the CAS wavelets. The properties of CAS wavelets are used to reduce the problem to the solution of algebraic equations. To obtain better results, use of the larger parameter k is recommended. The convergence accuracy of this method was examined in a numerical example.

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