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Nonlinear Control of Heat Transfer Dynamic Using Homotopy Perturbation Method

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Abstract. Nonlinear problems are more challenging and almost complex to be solved. A recently developed Homotopy Perturbation Method (HPM) is introduced. This method is used to represent the system as a less complicated (almost linear) model. To verify the effectiveness, HPM based model is compared with the original nonlinear dynamic in both open and closed loop PI controller. The simulation results reveal the ability of the proposed method.

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1. Introduction

In the two last decades with the rapid development of nonlinear systems, there has been appeared ever- increasing interest of scientists and engineers in the analytical techniques for nonlinear problems. The widely applied techniques i.e. perturbation method is of interest to be used in control systems [13,23]. Just recently, in order to develop this method and to eliminate the limitation of "small parameter" assumption i.e. the

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perturbation method has been proposed as a new technique, based on Homotopy in terminology. This method does not require small parameters in equations, using the interesting property of Homotopy. According to the method, a nonlinear problem is transformed into an infinite number of linear problems without using the perturbation techniques. Effectively, letting the small parameter float and converge to the unity, the problem is converted into a special perturbation problem with the small embedding parameter so the method is caught the name of the Homotopy Perturbation Method (HPM). The effectiveness of the new technique presented by HE [4-6, 9, 12]. This method can take full advantage of the traditional perturbation methods and Homotopy analysis method. It has successfully been applied to linear and nonlinear ordinary and partial differential equations, which almost describing a system dynamic incorporating the perturbation value (called Homotopy Perturbation Method, i.e. HPM). Duffing equation in [6], in the area of numerical and algebraic methods [1, 14, 22, 24], autonomous systems [23], system dynamic [3, 4, 9, 15], heat transfer [4, 9, 15], are such area, the method is applied. The idea, here, is how to apply this method in control engineering area. This paper is organized as follows:

Basic idea of HPM is studied Section 2. In Section 3, the heat transfer dynamics is considered as a case study and HPM solution of this equation is presented. The HPM based linearization is proposed in Section 4. In Section 5 a PID controller is applied on HPM based model and the results is discussed. Finally, the paper will be concluded in Section 6.

2. Basic Idea of Homotopy Perturbation Method

To illustrate the basic idea of the method briefly, the following nonlinear equation is considered:

$$A(x) + f(u) = 0, \; u \in \Re^m \tag{1}$$

Subject to boundary conditions:

$$\mathrm{B}(x, \partial x / \partial t) = 0, \; x \in \Re^n \tag{2}$$

whereAis a general differential operator,Bis a boundary operator, x is a known analytic, *n*dimensional function (here, state) and *u*is an *m*dimensional input (independent variable). The differential part $A(X)$ can be generally divided into two linear $L(X)$ and nonlinear $N(X)$ parts. Eq.(1) can therefore, be rewritten as:

$$\underbrace{L(x) + N(x)}_{A(x)} - f(u) = 0 \tag{3}$$

A Homotopy function $H(v, p)$ using an auxiliary variable $v(u, p)$ with $p \in [0,1]$ can be defined as:

$$H(v, p) = (1 - p)\left[L(v) - L(x_0)\right] + p\left[A(v) - f(u)\right] = 0, \; p \in [0,1] \tag{4}$$

P is called Homotopy parameter (inspired from "small parameter" in perturbation terminology). The idea behind using small parameter *p*is smart. By *p* equals *0.0*, Equation(4) is being completely linear whereas *p* equal to *1.0* the linear part in Equation(4) completely vanishes and (4) will be the same as (1). With a simple manipulation Equation (4) is reduced to the following Eq.(5):

$$H(v, p) = L(v) - L(x_0) + pL(x_0) + p[N(v) - f(u)] = 0, \quad p \in [0,1] \tag{5}$$

The initial guess x_0 needs to be a good initial approximation for equation (1) and satisfies the boundary conditions [17-19]. However, it is a property of the system and can be meaningfully found. A solution of (4) may be expressed [5, 18, 25] as:

(6)

$$\nu = p^0\nu_0 + p^1\nu_1 + p^2\nu_2 + ...$$

By substituting (6) in (5) and rearranging based on powers of p-terms, an infinite number of differential equations in terms of ν , is achieved. So such an attention should be made to avoid the secular terms to produce bounded-ness [23]. These sets of simple differential equation with proper initial conditions are then solved. Finally an approximate solution of Equation (1) can therefore be written by:

$$x = \lim_{p\to 1} \nu = \nu_0 + \nu_1 + \nu_2 + ... \quad (7)$$

The convergence the above method i.e. HPM, is discussed in [2, 5, 8, 10, 20, 21].

3. HPM Based Nonlinear Temperature Control

The main idea is to propose HPM as an alternative method to handle a nonlinear system dynamic in the closed loop, under different control scheme. Since this method presents rather an approximation solution, a probable amount of uncertainties may be occurred. Hence, the robust analysis and control design may be needed to overcome the possible lack of the dedicated method. To show the significance of this scheme, a nonlinear heat transfer equation [3, 4, 9, 11,15] as a case study, is chosen and effectively tested.

3.1. Cooling of a lumped system with variable specific heat differential equation

Consider the cooling of a lumped system[3, 4, 9, 11,15] for a spherical ball (specification in table 1.) have volume *V*, surface area *A*, density ρ , specific heat *C* and initial temperature T_i (here $1200°K$). At time $t = 0$, the system is exposed to a convective environment at T_a temperature (here $300°K$) with convective heat transfer coefficient *h*. The specific heat *C* is described by:

$$C = C_a \left[1 + \beta(T - T_a)\right] \quad (8)$$

Where C_a is the specific heat at T_a temperature and β is constant. The cooling equation is described [3, 4, 9,15] as:

$$\rho VC\frac{dT}{dt} + hA(T - T_a) = 0, \qquad T(0) = T_i = 1200^O K \quad (9)$$

An alternative approach based on dimensionless parameters (small parameter) leads us to a perturbation type nonlinear differential equation [15]. Various solving methods are then suggested [13]. Whereas, due to need for applying an independent input, that type of representation has not much of interest. Therefore Homotopy technique is directly applied to Equation (8) for $t \geq 0$. Substituting Equation (8) into Equation (9), transform the equation to:

$$\rho VC_a\beta\dot{T}T + (\rho VC_a - \rho VC_a\beta T_a)\dot{T} + hAT - hAT_a = 0 \ , \quad \mathrm{T(0)=T_i} = 1200^O K \quad (10)$$

Where $\dot{T} = \frac{dT}{dt}$. By using the value of parameters in Table (1), Equation (9) in terms of coefficients C_j 's, can be rewritten by:

$$C_1\dot{T}T + C_2\dot{T} + C_3T - C_3T_a = 0, \quad \text{where} \quad T(0) = T_i = 1200^O K \quad (11)$$

Where:

$$C_1 = \rho VC_a\beta = 0.1036$$
$$C_2 = \rho VC_a - \rho VC_a\beta T = 78.6895$$
$$C_3 = hA = 1.7593$$

To make the equation as simple as, Equation (11) is divided to C_1.

$$\dot{T}T + C_4\dot{T} + C_5T - C_5T_a = 0, \quad \text{with } T(0) = T_i = 1200^O K \tag{12}$$

with:

$$C_4 = \frac{C_2}{C_1} = 759.32$$

$$C_5 = \frac{C_3}{C_1} = 16.98$$

With respect to Equation (3) the nonlinear, linear parts and the input term i.e. $f(u)$, can respectively, be written as:

$$N(x) = T\dot{T}, \quad L(x) = C_4\dot{T} + C_5T, \quad f(u) = C_5T_a \tag{13}$$

Now Equation (10) is solved by Homotopy Perturbation Method. Substituting Equation (10) in Equation (3), is deduced to:

$$\underbrace{C_4\dot{\nu} + C_5\nu}_{L(\nu)} - \underbrace{[C_4\dot{T}_0 + C_5T_0]}_{L(x_0)} + p\underbrace{[C_4\dot{T}_0 + C_5T_0]}_{L(x_0)} + p[\underbrace{\dot{\nu}\nu}_{N(\nu)} - \underbrace{C_5T_a}_{f(u)}] = 0, \tag{14}$$

Again using υfrom (6) as $\nu = \nu_0 + p\nu_1 + p^2\nu_2 + ...$ in (14) results:

$$\begin{aligned} &C_4(\nu_0 + p\nu_1 + p^2\nu_2 + ...)^\bullet + C_5(\nu_0 + p\nu_1 + p^2\nu_2 + ...) \\ &-[C_4 \times 1200 \times (-0.0224) \times e^{-0.0224t} + C_5 \times (-0.0224) \times e^{-0.0224t}] \\ &+p[C_4 \times 1200 \times (-0.0224) \times e^{-0.0224t} + C_5 \times (-0.0224) \times e^{-0.0224t}] \\ &+p\{(\nu_0 + p\nu_1 + p^2\nu_2 + ...)^\bullet \times (\nu_0 + p\nu_1 + p^2\nu_2 + ...) - C_5T_a\} = 0 \end{aligned} \tag{15}$$

The above equation is rearranged in ascending powers of p as:

$$\begin{aligned} p^0: \; &C_4\dot{\nu}_0 + C_5\nu_0 - [C_4 \times 1200 \times (-0.0224) \times e^{-0.0224t} \\ &+C_5 \times (-0.0224) \times e^{-0.0224t}] = 0, \quad \text{with} \quad \nu_0(0)=1200 \end{aligned} \tag{16}$$

$$\begin{aligned} p^1: \; &C_4\dot{\nu}_1 + C_5\nu_1 - [C_4 \times 1200(-0.0224e^{-0.0224t}) \\ &+C_5 \times (-0.0224e^{-0.0224t})] + (\dot{\nu}_0\nu_0 - C_5T_a) = 0, \quad \text{with} \quad \nu_1(0)=0 \end{aligned} \tag{17}$$

$$p^2: \; C_4\dot{\nu}_2 + C_5\nu_2 + (\dot{\nu}_0\nu_1 + \dot{\nu}_1\nu_0) = 0, \quad \text{with} \quad \nu_2(0) = 0 \tag{18}$$

Appropriate initial conditions must be chosen such that satisfy the initial and boundary conditions [12, 17- 19]. Equations (16), (17) and (18) with the initial conditions are then respectively solved as follows:

$$\nu_0(t) = 1200e^{-0.0224t} \tag{19}$$

$$\nu_1(t) = T_a + (1893.23 - T_a)e^{-0.0224t} - 1893.23e^{-0.0448t} \tag{20}$$

Assuming $T_a = 300^O K$,

$$\nu_1(t) = 300 + 1593.23e^{-0.0224t} - 1893.23e^{-0.0448t} \tag{21}$$

and finally

$$\nu_2(t) = 543.07e^{-0.0224t} - 5027.37e^{-0.0448t} + 4484.3e^{-0.0672t} \tag{22}$$

In order not to be exhaustive, T_a is replaced with $T_a = 300$, where it is needed. From the point of view of the convergence, higher term of ν is preferred. However, as it will be shown shortly, this case is not very crucial. So, the 2nd and 3rd order of ν is here chosen and the properties are investigated.

3.2. Implementation of HPM 2nd order approximation

In order to solve Equation (12), considering two terms for ν in Equation(6) by

$$T(t) \cong \lim_{p \to 1}(\nu_0 + p\nu_1) \tag{23}$$

similar to the case which is done by [4, 9, 15]. The results of solving Equation(16) and (17) are as follows:

$$T(t) = \lim_{p\to 1}\{1200e^{-0.0224t} + p(300 + 1593.23e^{-0.0224t} - 1893.23e^{-0.0448t})\} \tag{24}$$

When the limit is applied, we have:

$$\begin{aligned} T(t) &= 1200e^{-0.0224t} + 300 + 1593.23e^{-0.0224t} - 1893.23e^{-0.0448t} \\ &= 300 + 2793.23e^{-0.0224t} + -1893.23e^{-0.0448t} \end{aligned} \tag{25}$$

3.3. Implementation of HPM 3rd order approximation

Truncating Equation (6) until 3 terms to solving the differential Equation (12) yields, the differential Equations (16)-(18).

$$T(t) \cong \lim_{p \to 1}(\nu_0 + p\nu_1 + p^2\nu_2) \tag{26}$$

This leads us to:

$$\begin{aligned} T(t) = \lim_{p\to 1}\{&1200e^{-0.0224t} + p(300 + 1593.23e^{-0.0224t} - 1893.23e^{-0.0448t}) \\ &+ p^2(543.07e^{-0.0224t} - 5027.37e^{-0.0448t} + 4484.3e^{-0.0672t})\} \end{aligned} \tag{27}$$

When the limit takes effect, Equation (27) transforms to:

$$\begin{aligned} T(t) = &1200e^{-0.0224t} + 300 + 1593.23e^{-0.0224t} - 1893.23e^{-0.0448t} \\ &+ 543.07e^{-0.0224t} - 5027.37e^{-0.0448t} + 4484.3e^{-0.0672t} \end{aligned} \tag{28}$$

and therefore:

$$T(t) = 300 + 3363.3e^{-0.0224t} - 6920.6e^{-0.0448t} + 4484.3e^{-0.0672t} \tag{29}$$

4. Control Problem, A simulation Approach

Equation (3) is of the control form, considering $f(u) = T_a$ as an independent variable, i.e. input. So, the control problem is to design a control strategy such that the ball (object) temperature can be adjusted via an independent input e.g. constant T_a . The rate of temperature variation and changes cannot be tuned, unless either a controller (such as a PID) is used to shape the dynamical behavior, or change of the physical situation. To signify the possibility of using HPM in control problem, equation (9) (and of course (10)) is numerically simulated according to Figure 1.

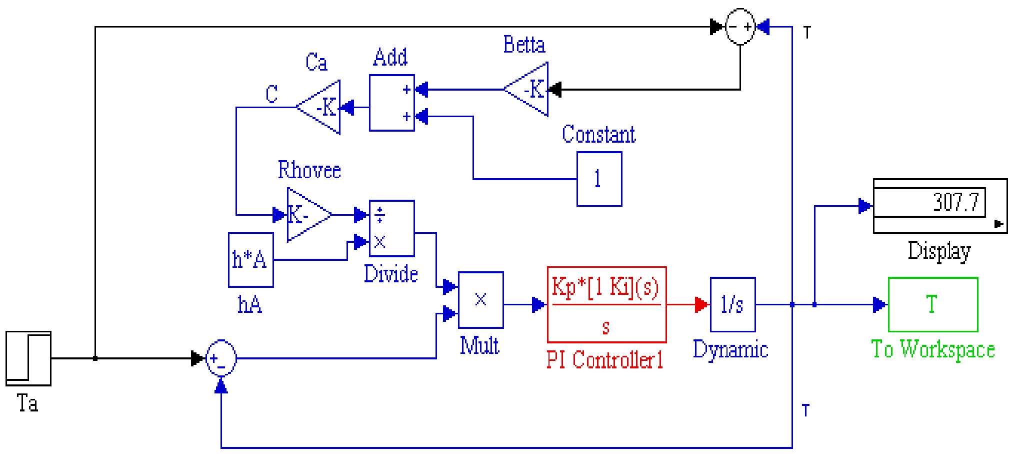

Figure 1: Simulation block diagram of the heat transfer dynamic under simulink

In Figure (1) PI controller in not activated yet. It is kept for the next controlling stage.
The Procedure is as follows:
Equation (29) is a time domain function and implicitly be affected by input, so it describes an open loop time response of the dynamic. In other word, it is a linear time dependent series, representing a nonlinear system behavior. Hence, the linear system theory and of course, the design procedure can be applied with a possible amount of uncertainties and discrepancies. Primarily the Laplace Transform equivalent is achieved as:

$$T(s) = \frac{300}{s} + \frac{3336.3}{s+0.0224} - \frac{6920.6}{s+0.0448} + \frac{4484.3}{s+0.0672} \tag{30}$$

A division to input Laplace transform i.e. $\frac{T_a}{s}$ releases the implicit dependency of the above function from the input and derives the system function $M(s)$ which is equal to:

$$M(s) = \frac{T(s)}{T_a(s)} = \frac{4s^3 + 0.317s^2 + 0.0193s + .000067}{s^3 + 0.1344s^2 + 0.0055s + 0.000067} \tag{31}$$

Since a closed loop control is of interest, an inner (open) loop transfer function i.e. $G(s) = \frac{M(s)}{1-M(s)}$ is derived as follows:

$$G(s) = \frac{4s^3 + 0.3175s^2 + 0.01927s + 0.000067}{-s(3s^2 + 0.183s + 0.01375)} \tag{32}$$

Similarly, acting the same procedure described for 2nd order HPM, leads us to have the following open loop transfer function:

$$G(s) = \frac{4s^2 + 0.343s + 0.001}{-s(3s^2 + 0.276)} \tag{33}$$

5. Results and Discussion

The same input i.e. $T_a = 300°K$ is applied into two parallel systems shown in Fig.2, simultaneously.

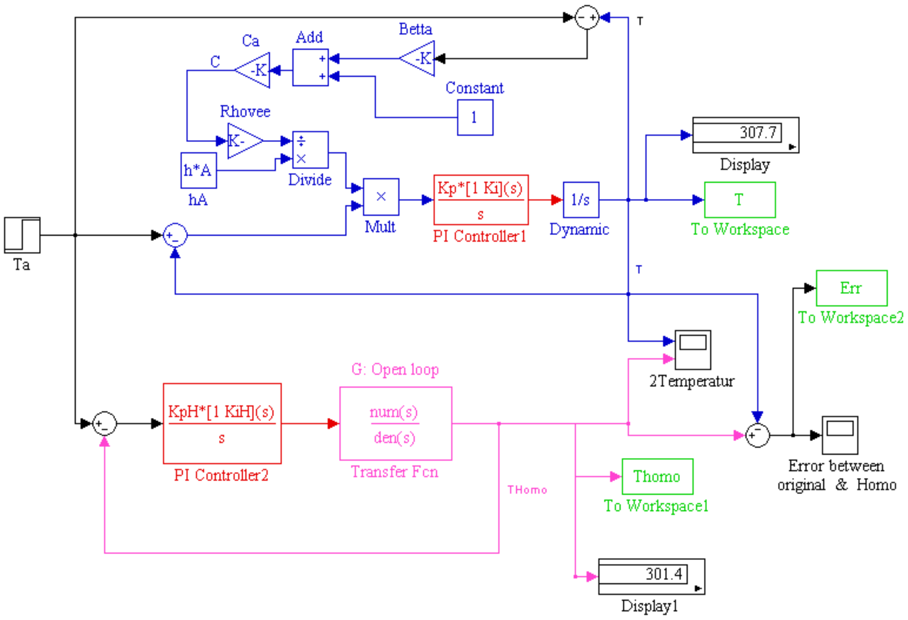

Figure 2: The heat transfer dynamic together with the HPM based model

5.1. A 2^{nd} order approximation of ν

The approximated model is considered as equation (33). It is used inside a temperature control loop according to Figure (2). The responses and the relevant open loop error (The controller is not working yet) are shown in Fig (3) and (4) respectively.

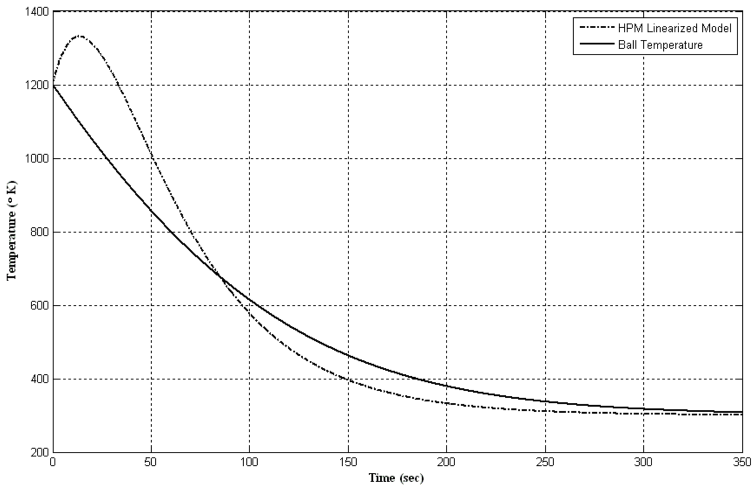

Figure 3: The ball temperature and the estimated behavior according to HPM, considering a 2^{nd} order approximation of ν

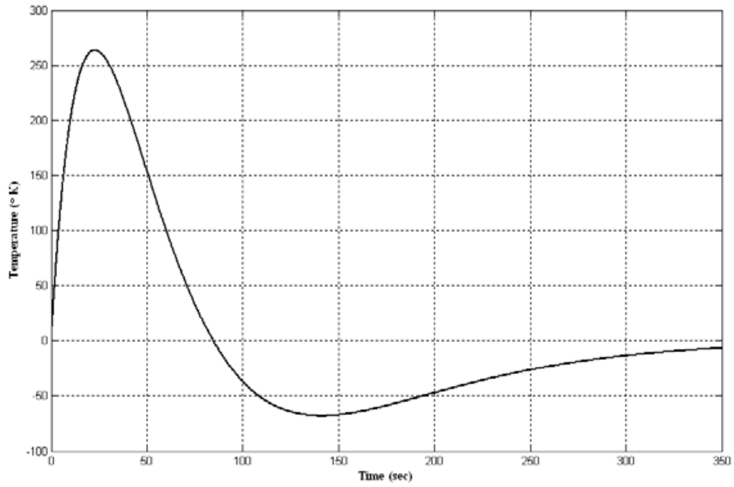

Figure 4:The error between the ball temperature and HPM estimated

To be able to compare the outcome, a normalized cost function, as an error index, in terms of the squared error with respect the actual squared value in a certain time (350 seconds) is defined. The normalized cost function is a few more than 2 %, i.e. 2.4191 %. To verify the significance of the HPM linearization method, the model is located inside the loop and controlled via a simple and classic PI controller. Due to unavoidable approximation, the index increases and reaches to 18.04 %. The results can be seen in Fig (5), (6).

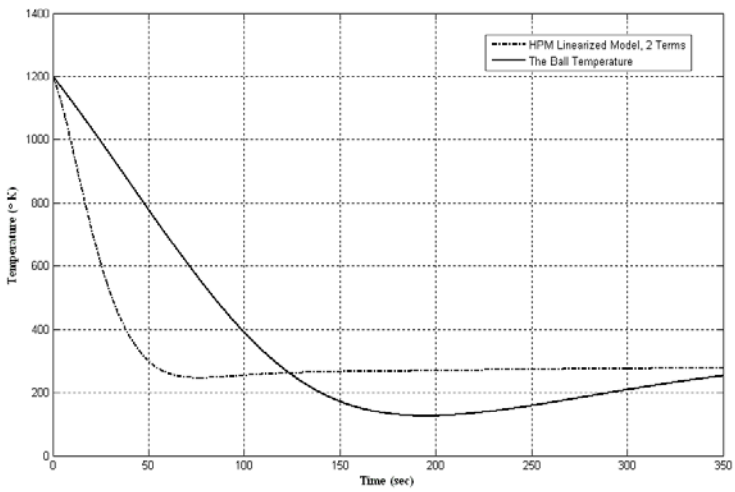

Figure 5: The closed loop ball temperature and the HPM estimated when a 2nd order approximation ν is considered

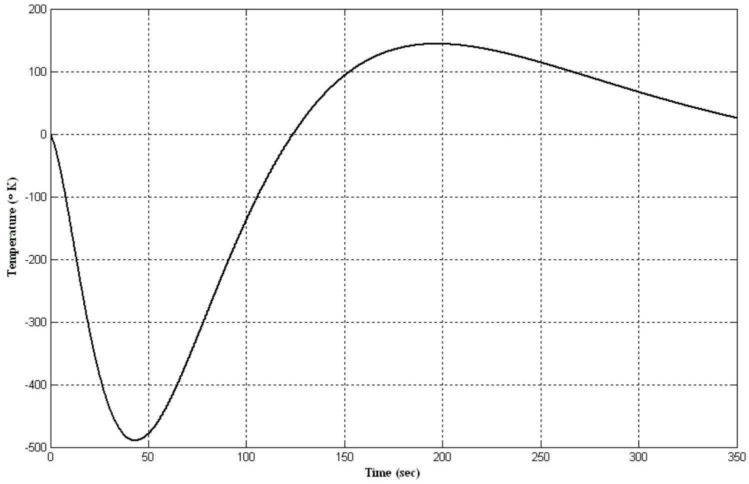

Figure 6: The error between the ball temperature and the HPM estimated

5.2. A 3rd order approximation of ν

The model is described by equation (32) and used as a plant in Figure (2).The output temperature i.e. step responses are then plotted in Figure 7.

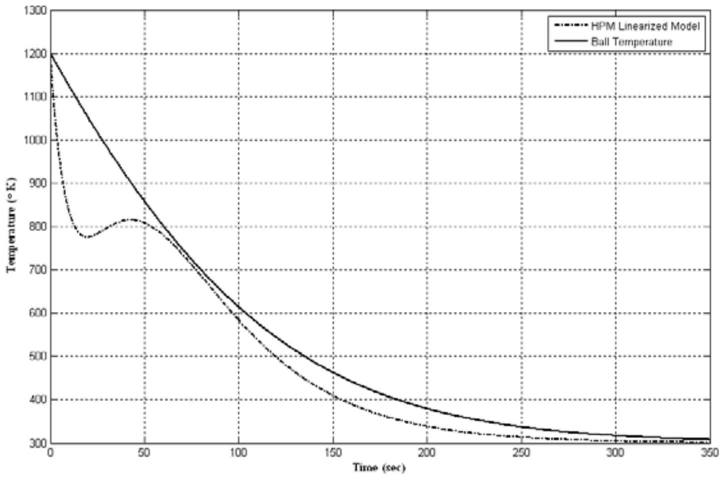

Figure 7: The ball and the HPM linearized model temperature, considering a 3rd order approximation of ν

In spite of the difference especially at the beginning which estimates a rapid change of the temperature, it chases the actual temperature behavior soon after. The relevant error can also be seen in the Figure 8.

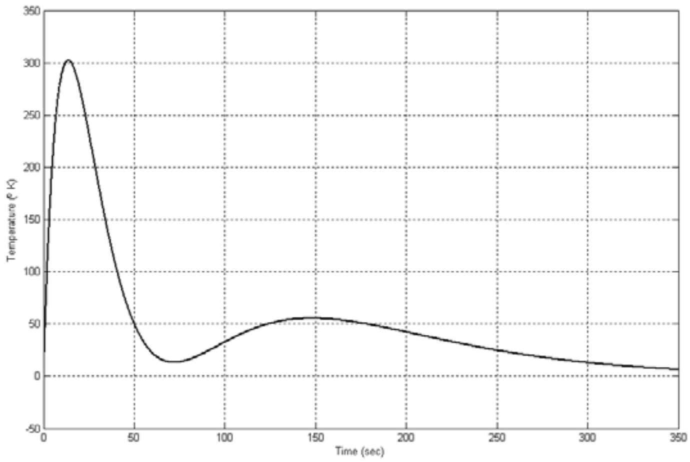

Figure 8:The temperature error between the HPM linearized model and the nonlinear model

The appropriate cost is less than 2 percent which seems improvement over the similar case i.e. the 2nd order approximation. The responses error between two systems when the controller is used is plotted in Figure 9 and Figure 10.

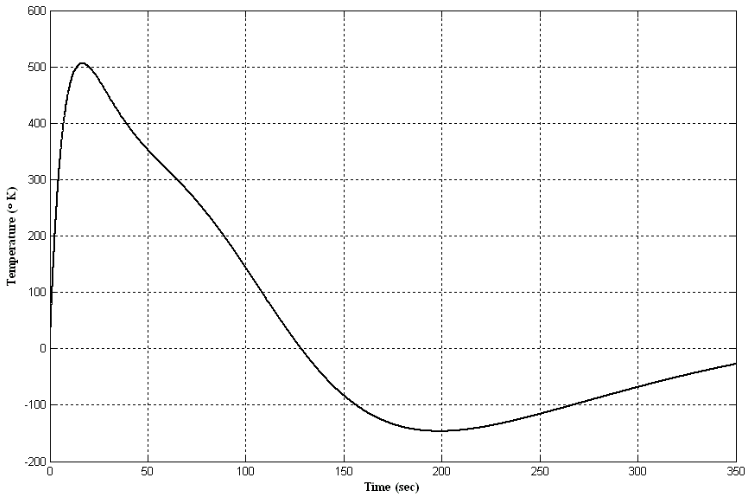

Figure 9:The temperature error between the linearized model by homotopy perturbation method and the nonlinear model in a closed loop PI controlled (a 3rd order approximation of V is considered).

Although, with respect to the Figure 8, the error shows a bit growing, the index is small (18 %). Still the overall performance in terms the dynamic specifications, i.e. transient behavior and the steady state error is satisfactory. The unestimated dynamic may have been appeared in this situation. One may use different scheme to compare according to the other indices such that chosen in [16].

Meanwhile, the error index is helpful to have an overall estimation of uncertainties. This can be used to overcome the discrepancy by designing a robust controller. Another possible way of treatment the error and/or decreasing the difference may be achieved by letting two controllers are designed separately. This is a normal situation, especially when the dynamic is not known. Hence the controller will be designed only by itself, acquiring the information via an estimated dynamic. However the controller will be showing its effectiveness by increasing the simulation times to 700 seconds. Therefore the result is showing the effectiveness of the algorithm (shown in Figures10 and 11).

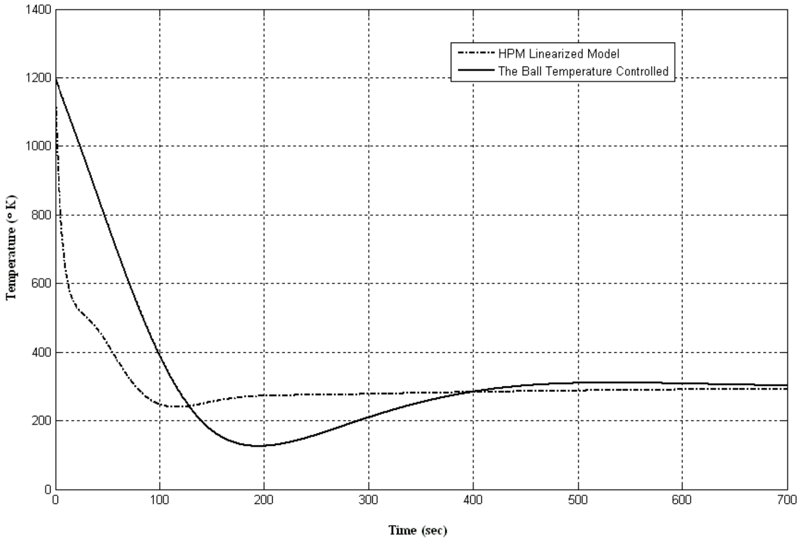

Figure 10:The ball and the HPM linearized model temperature in the closed loop control

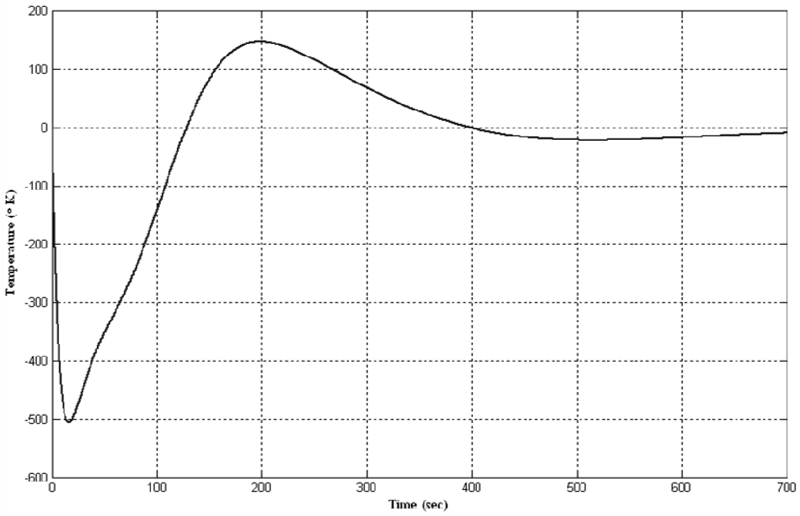

Figure 11: The error between the closed loop ball temperature and the estimated one in 700 seconds

More promising with the last case, the normalized cost is getting better and is reaching to 13.2%.

6. Conclusion

The Homotopy Perturbation Method (HPM) is introduced as a linearizing technique as a novel idea. This method is used to approximate the nonlinear dynamic with a linearized model providing high level of accuracy. The significance of the proposed method is shown by applying the HPM estimated model inside the closed loop considering a conventional controller. Since a nonlinear dynamic is inherently modeled by an infinite number of time series functions, it must be practically truncated. Indeed a robust controller may cope with the occurreduncertainties. However it is numerically shown that the HPM and especially with a 3rd order ν, provides a satisfactory outcome.

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Appendix:

Table 1: The specification of the Ball [11]

The Abbreviation	Showing	Unit	Value
m	mass of the ball	kg	0.26138
C	Specific heat of the ball	J/(kg-°K)	
C_a	initial Specific heat of the ball	J/(kg-°K)	420
A	surface area if the ball	m^2	5.02654×10^{-3}
ε	Emitance (Max)		0.85
ρ	density of the ball		7800
h	the convective cooling coefficient	W/(s-m^2-°K)	350
β			9.44×10^{-4}

Table 2: The PI Controller Coefficients when it is needed

The Controller Location	Proportional Gain, K_p	Integrator Gain, K_i
PI Controller in Nonlinear system	1	0.01
PI Controller in HPM Linearized system	1	0.01