International Journal of
Mathematical Modelling & Computations
Vol. 01, No. 03, 2011, 171 - 174

ON an Extension of a Quadratic Transformation Formula due to Gauss

M. A. Rakha[a*], A. K. Rathie[b] , and P. Chopra [c]

[a] *Department of Mathematics and Statistics Faculty of Science - Suez Canal University - Ismailia (41522) - Egypt.*
[b] *Vedant College of Engineering and Technology, Rajasthan Technical University, Village: TULSI, Post-Jakhmund, Dist. BUNDI-323021, Rajasthan State India.*
[c] *Mathematics Department, Marudhar Engineering College, NH-11, Jaipur Road, Raisar, BIKANER-334 001, Rajasthan State, India.*

Abstract. The aim of this research note is to prove the following new transformation formula

$$(1-x)^{-2a}\,{}_3F_2\left[\begin{matrix} a,\, a+\frac{1}{2},\, d+1 \\ c+1, \quad d \end{matrix} ; -\frac{4x}{(1-x)^2}\right] = {}_4F_3\left[\begin{matrix} 2a,\, 2a-c,\, a-A+1,\, a+A+1 \\ c+1, \quad a-A, \quad a+A \end{matrix} ; -x\right]$$

for $d = c$, we get a known quadratic transformations due to Gauss. The result is derived with the help of the generalized Gauss's summation theorem available in the literature.

Keywords: Gauss hypergeometric function, ${}_2F_1$ Hypergeometric function, Contiguous function relation, Linear recurrence relation.

Index to information contained in this paper

1. Introduction and Results Required

We start with the following very interesting and useful quadratic transformation formula due to Gauss [4] viz.

*Corresponding author. Email: medhat_rakha@science.suez.edu.eg

http://www.ijm2c.com

$$(1-x)^{-2a}\,{}_2F_1\left[\begin{matrix}a,\, a+\frac{1}{2}, \\ c,\end{matrix}\,;-\frac{4x}{(1-x)^2}\right] = {}_2F_1\left[\begin{matrix}2a,\, 2a+1-c, \\ c,\end{matrix}\,;-x\right] \tag{1}$$

Bailey [2] derived this result by employing Gauss's summation theorem [3] viz.

$${}_2F_1\left[\begin{matrix}a,\, b \\ c\end{matrix}\,;1\right] = \frac{\Gamma(c)\Gamma(c-a-b)}{\Gamma(c-a)\Gamma(c-b)} \tag{2}$$

provided that $Re(c-a-b)>0$.

The aim of this research note is to continue what we did in [1], and provide the extension of (1) by the same technique developed by Bailey. For this the following results will be required in our present investigations:

- Binomial theorem

$$(1-x)^{-a} = \sum_{n=0}^{\infty}\frac{(a)_n}{n!}x^n \tag{3}$$

- Elementary identities

$$(2a+2n)_m = \frac{(2a)_{m+n}}{2^{2n}(a)_n(a+\frac{1}{2})_n} \tag{4}$$

$$(2a)_{m+n} = (2a+m)_n(2a)_m \tag{5}$$

$$(m-n)! = \frac{(-1)^n m!}{(-m)_n} \tag{6}$$

- Rainville [6, Lemma 10, p. 56 eq. (1)]

$$\sum_{n=0}^{\infty}\sum_{k=0}^{\infty}A(k,n) = \sum_{n=0}^{\infty}\sum_{k=0}^{n}A(k,n-k) \tag{7}$$

- Extension of Gauss's summation theorem [5]

$${}_3F_2\left[\begin{matrix}a,\, b,\, d+1 \\ c,\quad d\end{matrix}\,;1\right] = \frac{\Gamma(c)\Gamma(c-a-b-1)}{\Gamma(c-a)\Gamma(c-b)}\left[(c-a-b-1)+\frac{ab}{d}\right] \tag{8}$$

provided $Re(c-a-b)>1$.

2. New Transformation Formula

The extension of the Gauss's transformation formula (1) to be established is

$$(1-x)^{-2a}\,{}_3F_2\left[\begin{matrix}a,\,a+\frac{1}{2},\,d+1\\ c+1,\quad d\end{matrix};-\frac{4x}{(1-x)^2}\right] \tag{9}$$
$$= {}_4F_3\left[\begin{matrix}2a,\,2a-c,\,a-A+1,\,a+A+1\\ c+1,\quad a-A,\quad a+A\end{matrix};-x\right]$$

2.1 *Derivation*

In order to derive (9), we proceed as follows. Start with the left-hand side of (9), we have

$$\text{L.H.S.} = (1-x)^{-2a}\,{}_3F_2\left[\begin{matrix}a,\,a+\frac{1}{2},\,d+1\\ c+1,\quad d\end{matrix};-\frac{4x}{(1-x)^2}\right]$$

express ${}_3F_2$ as a series, we have

$$= \sum_{n=0}^{\infty}\frac{(a)_n(a+\frac{1}{2})_n(d+1)_n(-1)^n2^{2n}x^n}{(c+1)_n(d)_n n!}(1-x)^{-(2a+2n)}$$

using (3), we have

$$= \sum_{m=0}^{\infty}\sum_{n=0}^{\infty}\frac{(2a+2n)_m}{m!}\frac{(a)_n(a+\frac{1}{2})_n(d+1)_n(-1)^n2^{2n}}{(c+1)_n(d)_n n!}x^{n+m}$$

using (4), we have

$$= \sum_{m=0}^{\infty}\sum_{n=0}^{\infty}\frac{(2a)_{2n+m}(d+1)_n(-1)^n}{(c+1)_n(d)_n n!m!}x^{n+m}$$

changing m by $m-n$ and using (7)

$$= \sum_{m=0}^{\infty}\sum_{n=0}^{m}\frac{(2a)_{m+n}(d+1)_n(-1)^n}{(c+1)_n(d)_n n!(m-n)!}x^{m}$$

using (5) and (6), we have

$$= \sum_{m=0}^{\infty}\frac{(2a)_m}{m!}x^m\sum_{n=0}^{m}\frac{(-m)_n(2a+m)_m(d+1)_n}{(c+1)_n(d)_n n!}$$

summing up the inner series

$$= \sum_{m=0}^{\infty}\frac{(2a)_m}{m!}x^m\,{}_3F_2\left[\begin{matrix}-m,\,2a+m,\,d+1\\ c+1,\quad d\end{matrix};1\right],$$

using (8) by taking $a = -m$, $b = 2a + m$, $d = d$ and $c = c + 1$,
we get after much simplification

$$= \sum_{m=0}^{\infty} \frac{(a-A)(a+A)}{d(2a-c)} \frac{(-1)^m (2a)_m (2a-c)_m (a-A+1)_m (a+A+1)_m x^m}{(c+1)_m (a-A)_m (a+A)_m m!}$$

finally summing up the series, we get

$$= {}_4F_3 \left[\begin{matrix} 2a, 2a-c, a-A+1, a+A+1 \\ c+1, \quad a-A, \quad a+A \end{matrix} ; -x \right].$$

$$= \text{R.H.S.}$$

This completes the proof of (9).■

2.2 *Special Case*

In (9), if we take $d = c$, we get the Gauss's quadratic transformation formula (1). Hence (9) can be regarded as an extension of (1).

References

[1] Rakha M. A., Rathie N., Chopra P., On an extension of a quadratic transformation formula due to Kummer, Math, Commun., **14** (2009) 207-209.
[2] Bailey W. N., Products of generalized hypergeometric series, Proc. London Math. Soc, **2** (1928) 242 - 254.
[3] Bailey W. N., Generalized hypergeometric series, Cambridge University Press, Cambridge (1935).
[4] Gauss C. F., Disquisitiones generales circa seriem infinitam ..., Comm. soc. reg. sci. Gött. rec., **II** (1876) 123-162.
[5] Prudnikov A. P., Brychkov Iu. A., Marichev O. I., More Special Functions, Integrals and Series, Gordon and Breach, New York, **3** (1990).
[6] Rainville E. D., Special functions, The Macmillan Company, New York, (1960).