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Extended Predictor-Corrector Methods for Solving Fuzzy Differential Equations under Generalized Differentiability

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Abstract. In this paper, the $(m+1)$-step Adams-Bashforth, Adams-Moulton, and Predictor-Corrector methods are used to solve first-order fuzzy linear ordinary differential equations. The concepts of fuzzy interpolation and generalized strongly differentiability are used, to obtain general algorithms. Each of these algorithms has advantages over current methods. Moreover, for each algorithm a convergence formula can be obtained . The convergence of these methods is proven in detail. Finally, these methods are illustrated using examples of initial value problems.

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1. Introduction

The topic of fuzzy differential equations (FDEs)has been attracting growing interest for some time, particularly regarding fuzzy control. As a result, the subfield has

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rapidly developed in recent years. At first, definitions of fuzzy derivatives and integrals were developed; then, the concept of FDEs was introduced, and sufficient conditions for the existence of unique solutions to these equations were established. Finally, algorithms for calculating approximate solutions were devised. Thus, in order to understand fuzzy differential equations and their associated numerical algorithms, introduction to fuzzy numbers and fuzzy calculus is necessary.

The concept of fuzzy sets which was originally introduced by Zadeh, led to the definition of the fuzzy number and its subsequent application to both fuzzy control [10] and approximate reasoning problems [38]. The basic arithmetic structure for fuzzy numbers was later developed by Mizumoto and Tanaka [26, 27], Nahmias [28], Dubios and Prade [11, 13] and Ralescu [32], all of whom defined a fuzzy number as a location of $r-$ levels $0 \leqslant r \leqslant 1$. Additional background material can be found in [5, 6, 9, 16–18, 20, 22, 35].

The fuzzy mapping function was introduced by Chang and Zadeh [10]. Later, Dubois and Prade presented an elementary fuzzy calculus based on the extension principle. Puri and Ralescu [30]then suggested two methods for the fuzzy derivative of fuzzy functions. The first method was based on the H-difference notation and was further investigated by Kaleva [21]. The second method was derived from the embedding technique and was analyzed by Goetschel and Voxman [18], both of whom used the method to develop various applications. Strongly generalized differentiability was introduced in [7] and studied in [8]. Indeed, the strongly generalized derivative is defined for a larger class of fuzzy-number-valued function than the set of functions for which the H-derivative is defined, and fuzzy differential equations may have solutions that have a decreasing length of their support, that is, decreasing uncertainty. Thus, we use strongly generalized differentiability concept in the present paper.

The concept of integrating a fuzzy function was first introduced by Dubois and Prade [12]. Alternative approaches were later suggested by Goetschel and Voxman [19], Kaleva [21], among others. While Goetschel and Voxman [18] and later Matloke used a Riemann integral type approach, Kaleva [21] defined the integral of a fuzzy function using Lebesgue integration.

The notion of fuzzy differential equation was initially introduced by Kandel and Byatt [23] and later applied to fuzzy processes and fuzzy dynamical systems. A Fuzzy Cauchy problems have been thoroughly researched by Kaleva [21], Seikkala [33], Ouyang and Wu [24], and Kloeden and Wu [36]. The numerical method used for solving fuzzy differential equations are introduced in [2, 14, 25]. In addition, several applications of FDEs to fuzzy control are presented in [31].

In this study, we concentrate on algorithms to solve FDEs that possess unique fuzzy solutions. In Section 2, some basic definitions and results are presented. In Section 3, the concept of fuzzy Newton finite differences is reviewed. The Adams-Bashforth method, Adams-Moulten method and their corresponding algorithms are introduced in Sections 4 and 5, respectively. We prove convergence of this methods in Section 6. In addition, we demonstrate these methods in Section 7. Finally, concluding remarks are presented in Section 8.

2. Preliminaries

We now recall some definitions that form the foundation of this paper. We denote R as the set of all real numbers.

The basic definition of fuzzy numbers is given in [34, 37] as follows:

A fuzzy number is a mapping $u : R \to [0, 1]$ with the following properties:

(a) u is upper semi-continuous,

(b) u is fuzzy convex, i.e., $u(\lambda x + (1-\lambda)y) \geqslant min\{u(x), u(y)\}$ for all $x, y \in R, \lambda \in [0,1]$,
(c) u is normal, i.e.,$\exists x_0 \in R$ for which $u(x_0) = 1$,
(d) supp $u = \{x \in R \mid u(x) > 0\}$ is the support of the u, and its closure $cl(supp\ u)$ is compact.
Let E be the set of all fuzzy number on R. The r-level set of a fuzzy number $u \in E$, $0 \leqslant r \leqslant 1$, denoted by $[u]_r$ is defined as

$$[u]_r = \begin{cases} \{x \in R \mid u(x) \geqslant r\} & if \quad 0 < r \leqslant 1 \\ cl(supp\ u) & if \quad r = 0 \end{cases}$$

It is clear that the r-level set of a fuzzy number is a closed and bounded interval $[\underline{u}(r), \overline{u}(r)]$, where $\underline{u}(r)$ denotes the left-hand endpoint of $[u]_r$ and $\overline{u}(r)$ denotes the right-hand $[u]_r$. R can thus be embedded in E, because, each $y \in R$ can be regarded as a fuzzy number, with $\widetilde{y}$ defined by

$$\widetilde{y}(t) = \begin{cases} 1 & if \quad t = y \\ o & if \quad t \neq y \end{cases}$$

It is well known that the following properties are true for all levels

$$[u \oplus v]_r = [u]_r + [v]_r, \qquad [k \odot u]_r = k[u]_r$$

From this, we see that a fuzzy number is determined by the endpoints of the intervals $[u]_r$. For arbitrary $u = [\underline{u}(r), \overline{u}(r)]$, $v = [\underline{v}(r), \overline{v}(r)]$ and $k > 0$, we define addition $u \oplus v$, subtraction $u \ominus v$ and scaler multiplication k as follows:

(a) Addition:

$$u \oplus v = [\underline{u}(r) + \underline{v}(r), \overline{u}(r) + \overline{v}(r)]$$

(b) Subtraction:

$$u \ominus v = [\underline{u}(r) - \overline{v}(r), \overline{u}(r) - \underline{v}(r)]$$

(c) Scalar multiplication:

$$ku = \begin{cases} (k\underline{u}(r), k\overline{u}(r)),\ k \geqslant 0, \\ \\ (k\overline{u}(r), k\underline{u}(r)),\ k < 0. \end{cases}$$

The Hausdorff distance between fuzzy numbers[5] is given by $D : E \times E \longrightarrow R \geqslant 0$,

$$D(u,v) = \sup_{r \in [0,1]} \max\{|\underline{u}(r) - \underline{v}(r)|, |\overline{u}(r) - \overline{v}(r)|\},$$

where $u = [\underline{u}(r), \overline{u}(r)]$,$v = [\underline{v}(r), \overline{v}(r)] \subset R$ are utilized. Thus, D is a metric in E and has the following properties :
$(i) D(u \oplus w, v \oplus w) = D(u,v), \quad \forall u, v, w \in E$,

$(ii) D(k \odot u, k \odot v) = |k| D(u, v), \quad \forall k \in R, u, v \in E,$
$(iii) D(u \oplus v, w \oplus e) \leqslant D(u, w) + D(v, e), \quad \forall u, v, w, e \in E,$
$(iv)(D, E)$ is a complete metric space.

Definition 2.1 *[5]Let E be a set of all fuzzy numbers. f is said to be a fuzzy-valued function if $f : R \to E$*

Definition 2.2 *[5] Let $f : R \to E$ be a fuzzy-valued function. If for an arbitrary, fixed $t_0 \in R$, $\epsilon > 0$, and $\delta > 0$ such that*

$$|t - t_0| < \delta \Rightarrow D(f(t), f(t_0)) < \epsilon,$$

the f is said to be continuous(See [14]).

Definition 2.3 *[5] A mapping $f : R \times E \to E$ is called continuous at point $(t_0, x_0) \in R \times E$ provided that for any fixed $r \in [0, 1]$ and arbitrary $\epsilon > 0$, there exists an $\delta(\epsilon, r)$ such that*

$$D([f(t, x)]_r, [f(t_0, x_0)]_r) < \epsilon$$

whenever $|t - t_0| < \delta(\epsilon, r)$ and $D([x]_r, [x_0]_r) < \delta(\epsilon, r)$ for all $t \in R$, $x \in E$ (See [34]).

Definition 2.4 *[5] A function $f : R \to E$ is Riemann-integrable on $[a, b]$ if there exists $I_R \in E$ with the property that: $\forall \epsilon > 0$, $\exists \delta > 0$, such that for any division of $[a, b]$, $d : a = x_0 < \ldots < x_n = b$, of norm $v(d) < \delta$ and for any point $\xi_i \in [x_i, x_{i+1}]$, $i = 0, \ldots n - 1$,*
$D(\sum_{i=0}^{n-1} f(\xi_i).(x_{i+1} - x_i), I_R) < \epsilon$.
We denote $I_R = \int_a^b f(x)$ as the fuzzy Riemann integral (see [15]).

H-derivatives refer to functions differentiated according to the Hukuhara method; it is well-known that the H-derivative for fuzzy mappings was initially introduced by Puri and Ralescu([30]). It is based on the notion of H-difference of sets, defined as follows:

Definition 2.5 *[5] Let $x, y \in E$. If there exists $z \in E$ such that $x = y \oplus z$, then z is called the H-difference of x and y, and it is denoted by $x -^h y$.*

In this paper, the sign "$-^h$" always stands for H-difference, also note that $x -^h y \neq x \ominus y$.
In this paper, we consider the following definition which was introduced by Bede and Gal in ([8]).

Definition 2.6 *Let $f : (a, b) \to E$ and $x_0 \in (a, b)$. f is strongly generalized differential at x_0 (Bede-Gal differential), if there exists an element $f'(x_0) \in E$, such that*
(i) for all $h > 0$ sufficiently small, $\exists f(x_0 + h) -^h f(x_0)$ and $\exists f(x_0) -^h f(x_0 - h)$ so that in the metric D
$\lim_{h \searrow 0} \frac{f(x_0+h) -^h f(x_0)}{h} = \lim_{h \searrow 0} \frac{f(x_0) -^h f(x_0-h)}{h} = f'(x_0)$
or
(ii) for all $h > 0$ sufficiently small, $\exists f(x_0) -^h f(x_0 + h)$ and $\exists f(x_0 - h) -^h f(x_0)$ so that in the metric D
$\lim_{h \searrow 0} \frac{f(x_0) -^h f(x_0+h)}{-h} = \lim_{h \searrow 0} \frac{f(x_0-h) -^h f(x_0)}{-h} = f'(x_0)$
or

(iii) for all $h > 0$ sufficiently small, $\exists f(x_0 + h) -^h f(x_0)$ and $\exists f(x_0 - h) -^h f(x_0)$ so that in the metric D
$\lim_{h\searrow 0} \frac{f(x_0+h)-^h f(x_0)}{h} = \lim_{h\searrow 0} \frac{f(x_0-h)-^h f(x_0)}{-h} = f^{'}(x_0)$
or
(iv) for all $h > 0$ sufficiently small, $\exists f(x_0) -^h f(x_0 + h)$ and $\exists f(x_0) -^h f(x_0 - h)$ so that in the metric D
$\lim_{h\searrow 0} \frac{f(x_0)-^h f(x_0+h)}{-h} = \lim_{h\searrow 0} \frac{f(x_0)-^h f(x_0-h)}{h} = f^{'}(x_0)$
The denominators of h and $-h$ denote multiplication by $\frac{1}{h}$ and $\frac{-1}{h}$, respectively.

To account for the special case when f is a fuzzy-valued function, we have the following theorem.

THEOREM 2.7 *Let $f : R \to E$ be a function and denote $[f(t)]_r = [\underline{f}(t, r), \overline{f}(t, r)]$, for each $r \in [0, 1]$. Then*
(1)If is f differentiable according to Definition 2.6 (i), then $\underline{f}(t, r)$ and $\overline{f}(t, r)$ are differentiable functions and
$[f^{'}(t)]_r = [\underline{f}^{'}(t, r), \overline{f}^{'}(t, r)]$.
(2)If is f differentiable according to Definition 2.6 (ii), then $\underline{f}(t, r)$ and $\overline{f}(t, r)$ are differentiable functions and
$[f^{'}(t)]_r = [\overline{f}^{'}(t, r), \underline{f}^{'}(t, r)]$.

THEOREM 2.8 *[4] Let $f : R \to E$ and let $d : \{a = b_0 < b_1 < \ldots < b_n = b\}$ be a division of the interval $[a, b]$ such that f is (i)- or (ii)-differentiable according to Definition 2.6 on each intervals $[b_{i-1}, b_i]$, $i = 1, ..., n$, with the same kind of differentiability on each subinterval. Then*
$\int_a^b f^{'}(x)dx = \sum_{i\in I}(f(b_i) - f(b_{i-1})) \oplus (-1) \odot \sum_{i\notin I}(f(b_{k-1}) - f(b_k))$,
where
$I = \{i \in \{1, ..., n\}$ *such that f is (i)-differentiable on $(b_{i-1}, b_i)\}$(See e.g. [8]).*

LEMMA 2.9 *For $x_0 \in R$, the fuzzy differential equation $y^{'} = f(x, y)$, where $y(x_0) = y_0 \in E$ and $f : R \times E \longrightarrow E$ is continuous, is equivalent to one of the following integral equations:*
$y(x) = y_0 \oplus \int_{x_0}^x f(t, y(t))dt, \qquad \forall x \in [x_0, x_1]$
or
$y(0) = y(x) \oplus (-1) \odot \int_{x_0}^x f(t, y(t))dt, \qquad \forall x \in [x_0, x_1]$
on some interval $(x_0, x_1) \subset R$, depending on which type of differentiability is considered, according to Definition 2.6, that is, (i) or (ii), respectively(See [8]).

The equivalence between the two equations means that any solution to one equation is a solution to the other as well.

Remark 1 In the case of strongly generalized differentiability, we have two different integral equations for the fuzzy differential equation $y^{'} = f(x, y)$, while in the case of H-differentiability, we have only one. The second integral equation in Lemma (2.9) can be written in the form $y(x) = y_0 -^h (-1) \odot \int_{x_0}^x f(t, y(t))dt$(See [8]).

The following theorems concern the existence of solutions to fuzzy initial-value problems under generalized differentiability (see [8]).

THEOREM 2.10 *Note the following conditions.*

- *(a) Let $R_0 = [x_0, x_0 + p] \times \overline{B}(y_0, q)$, $p, q > 0$, $y_0 \in E$, where $\overline{B}(y_0, q) = \{y \in E : D(y, y_0) \leqslant q\}$, denote a closed ball in E and let $f : R_0 \longrightarrow E$ be a continuous function such that $D(\widetilde{0}, f(x, y)) = \|f(x, y)\| \leqslant M$ for all $(x, y) \in R_0$*
- *(b) Let $g : [x_0, x_0 + p] \times [0, q] \longrightarrow E$, such that $g(x, 0) \equiv 0$, $0 \leqslant g(x, u) \leqslant M_1$.*

$\forall \in [x_0, x_0+p]$, $0 \leqslant u \leqslant q$, $g(x,u)$ *is non-decreasing in* u, *and* g *is such that the initial-value problem* $u^{'}(x) = g(x, u(x)), u(x_0) = 0$ *only has the solution* $u(x) \equiv 0$ *on* $[x_0, x_0 + p]$.

(c) *If* $D(f(x,y), f(x,z)) \leqslant g(x, D(y,z))$, $\forall (x,y),(x,z) \in R_0$ *and* $D(y,z) \leqslant q$.

(d) *There exists* $d > 0$ *such that for* $x \in [x_0, x_0 + d]$, *the sequence* $\overline{y}_n : [x_0, x_0 + d] \longrightarrow E$ *given by* $\overline{y}_0(x) = y_0$, $\overline{y}_{n+1}(x) = y_0 -^h (-1) \odot \int_{x_0}^{x} f(t, \overline{y}_n)dt$ *and is defined for any* $n \in N$.

If the conditions (a) through (d) hold, then the fuzzy initial-value problem

$$\begin{cases} y' = f(x,y), \\ y(x_0) = y_0 \end{cases}$$

has two solutions, (i)-differentiable and the other one (ii)-differentiable , according to Definition 2.6:

$y, \overline{y} : [x_0, x_0 + r] \longrightarrow B(y_0, q)$ *where* $r = \min\{p, \frac{q}{M}, \frac{q}{M_1}, d\}$ *and the successive iterations*

$$y_0(x) = y_0, y_{n+1}(x) = y_0 \oplus \int_{x_0}^{x} f(t, y_n(t))dt, \tag{1}$$

and

$$\overline{y}_0(x) = y_0, \overline{y}_{n+1}(x) = y_0 -^h (-1) \odot \int_{x_0}^{x} f(t, \overline{y}_n(t))dt, \tag{2}$$

converge to the two solutions, respectively.

THEOREM 2.11 *Note the following conditions.*

(a) *Let* $R_0 = [x_0, x_0 + p] \times \overline{B}(y_0, q)$, $p, q > 0$, $y_0 \in E$, *where* $\overline{B}(y_0, q) = \{y \in E : D(y, y_0) \leqslant q\}$, *denote a closed ball in* E *and let* $f : R_0 \longrightarrow E$ *be a continuous function such that* $D(\widetilde{0}, f(x,y)) = \|f(x,y)\| \leqslant M$ *for all* $(x,y) \in R_0$

(b) *Let* $g : [x_0, x_0 + p] \times [0, q] \longrightarrow E$, *such that* $g(x, 0) \equiv 0$, $0 \leqslant g(x,u) \leqslant M_1$. $\forall \in [x_0, x_0+p]$, $0 \leqslant u \leqslant q$, $g(x,u)$ *is non-decreasing in* u, *and* g *is such that the initial-value problem* $u^{'}(x) = g(x, u(x)), u(x_0) = 0$ *only has the solution* $u(x) \equiv 0$ *on* $[x_0, x_0 + p]$.

(c) *If* $D(f(x,y), f(x,z)) \leqslant g(x, D(y,z))$, $\forall (x,y),(x,z) \in R_0$ *and* $D(y,z) \leqslant q$.

(d) *There exists* $d > 0$ *such that for* $x \in [x_0, x_0 + d]$, *the sequence* $\overline{y}_n : [x_0, x_0 + d] \longrightarrow E$ *given by* $\overline{y}_0(x) = y_0$, $\overline{y}_{n+1}(x) = y_0 -^h (-1) \odot \int_{x_0}^{x} f(t, \overline{y}_n)dt$ *and is defined for any* $n \in N$.

If the conditions (a) through (d) hold, then the fuzzy initial-value problem

$$\begin{cases} y' = f(x,y), \\ y(x_0) = y_0 \end{cases}$$

has two solutions, (i)-differentiable and the other one (ii)-differentiable , according to Definition 2.6:

$y, \overline{y} : [x_0, x_0 + r] \longrightarrow B(y_0, q)$ *where* $r = \min\{p, \frac{q}{M}, \frac{q}{M_1}, d\}$ *and the successive iterations*

$$y_0(x) = y_0, y_{n+1}(x) = y_0 \oplus \int_{x_0}^{x} f(t, y_n(t))dt, \tag{3}$$

and

$$\overline{y}_0(x) = y_0, \overline{y}_{n+1}(x) = y_0 -^h (-1) \odot \int_{x_0}^{x} f(t, \overline{y}_n(t))dt, \tag{4}$$

converge to the two solutions, respectively.

According to Theorem 2.11, we restrict our attention to functions that are (i)- or (ii)-differentiable on their domain except for a finite number of points (See [8]). The following results address fuzzy differential equations with triangular data. We recall that for $a < b < c$ and $a, b, c \in R$, the triangular fuzzy number $u = (a, b, c)$ determined by a, b, c is given such that $\underline{u}(r) = a + (b - c)r$ and $\overline{u}(r) = c - (c - b)r$ are the endpoints of the r-level sets, for all $r \in [0, 1]$. Here $\underline{u}(r) = \overline{u}(r) = b$ and it is denoted by $[u]_1$. The set of triangular fuzzy numbers will be denoted by E. The following Lemma 2.12gives a sufficient condition for the existence of the H-difference between two triangular fuzzy numbers.

LEMMA 2.12 *(See [8]) Let $u, v \in E$ be such that $u(1) - \underline{u}(0) > 0$, $\overline{u}(0) - u(1) > 0$, and $len(v) = (\overline{v}(0) - \underline{v}(0)) \leqslant \min\{u(1) - \underline{u}(0), \overline{u}(0) - u(1)\}$. Then the H-difference $u \ominus v$ exists.*

The following corollary gives a simple sufficient condition for the existence of fuzzy differential equations under strongly generalized differentiability.

COROLLARY 2.13 *Let $f : R_0 \longrightarrow E$ where $R_0 = [x_0, x_0 + p] \times (\overline{B}(y_0, q) \cap E)$, and $y_0 \in E$ such that $y(0,1) - \underline{y}(0,0)$ and $\overline{y}(0,0) - y(0,1)$. Let $m = \min\{y(0,1) - \underline{y}(0,0), \overline{y}(0,0) - y(0,1)\}$. Under assumptions (a)-(c) of Theorem (2.11), the fuzzy initial-value problem*

$$\begin{cases} y' = f(x, y), \\ y(x_0) = y_0 \end{cases}$$

has two solutions $\underline{y}, \overline{y} : [x_0, x_0 + r] \longrightarrow B(y_0, q)$ where $r = \min\{p, \frac{q}{M}, \frac{q}{M}, \frac{m}{2M}\}$, and the successive iterations in (3) and (4) converge to these two solutions, respectively.

3. Fuzzy Newton Finite Differences

It often occurs that interpolation data are not sets of real numbers but are ranges of values whose distribution within the range may not be probabilistic but known possibilistically or qualitatively. For this reason, Allahviranloo and Hajjari in [3] suggest fuzzy Newton finite difference formula as follows:

For a given partition $\Delta : \{a = x_0 < x_1 < \ldots < x_n = b\}$, where every x_i has an associated fuzzy number $f_i \in E$, we find a polynomial $p : R \to E$, such that $p(x_i) = f_i$ for all $i = 0, 1, \ldots, n$. Representing $p(x)$ using $r-$cuts, we can write

$$[p(x)]_r = [\underline{p}(x;r), \overline{p}(x;r)] = \{z \in R : z = p(x;r), p(x_i;r) \in [f_i]_r, r \in [0,1]\}.$$

The fuzzy polynomial such as the Newton forward-difference formula is written in the $r-$cuts form as

$$[p(x)]_r = \sum_{\nu=j}^{j+t} \binom{\theta}{\nu} [\Delta^\nu f_j]_r, \ r \in [0,1], \ 0 \leqslant j \leqslant t \leqslant j+t \leqslant n, \tag{5}$$

where $\theta = \frac{x - x_0}{h}, h = \frac{x_n - x_0}{n}$. The upper limit $\overline{p}(x;r)$ of the interval $[p(x)]_r$ is the solution of the optimization problem

$$\max \ p(x;r), \quad s.t. \ \underline{f_i}(r) \leqslant p(x_i;r) \leqslant \overline{f_i}(r)$$

and, similarly, the lower limit $\underline{p}(x;r)$ of the interval $[p(x)]_r$ is the solution of the optimization problem

$$\min\ p(x;r), \quad s.t.\ \ \underline{f_i}(r) \leqslant p(x_i;r) \leqslant \overline{f_i}(r).$$

From the numerical stability of mentioned algorithm, we have $0 \leq \theta < 1$. Then for the even n:

$$\begin{cases} \binom{\theta}{\nu} < 0, & 2k+2 = \nu \geqslant 2, \\ & \qquad\qquad k = 0, 1, \ldots, \frac{n}{2} - 1 \\ \binom{\theta}{\nu} > 0, & \nu = 0, \quad 2k+1 = \nu \geqslant 2, \end{cases} \tag{6}$$

and for the odd n:

$$\begin{cases} \binom{\theta}{\nu} < 0, & 2k+2 = \nu \geqslant 2, \\ & \qquad\qquad k = 0, 1, \ldots, [\frac{n}{2}] - 1 \\ \binom{\theta}{\nu} > 0, & \nu = 0, \quad 2k+1 = \nu \geqslant 2, \end{cases} \tag{7}$$

The $r-$level sets of $\Delta^\nu f_j$ in (5) are as follow

$$[\Delta^\nu f_j]_r = [\underline{\Delta^\nu f_j(r)}\ ,\ \overline{\Delta^\nu f_j(r)}],\ r \in [0,1],\ 0 \leqslant \nu \leqslant n. \tag{8}$$

From Eqs. (6, 7) and algebraic operations of fuzzy intervals, for the even $n = j + t$:

$$\underline{p}(x;r) = \underline{f_j}(r) + \sum_{\nu=0}^{\frac{n}{2}-1} \binom{\theta}{2\nu+1} \underline{\Delta^{2\nu+1} f_j}(r) + \sum_{\nu=1}^{\frac{n}{2}} \binom{\theta}{2\nu} \overline{\Delta^{2\nu} f_j}(r),$$

$$\overline{p}(x;r) = \overline{f_j}(r) + \sum_{\nu=0}^{\frac{n}{2}-1} \binom{\theta}{2\nu+1} \overline{\Delta^{2\nu+1} f_j}(r) + \sum_{\nu=1}^{\frac{n}{2}} \binom{\theta}{2\nu} \underline{\Delta^{2\nu} f_j}(r),$$

and for the odd $n = j + t$:

$$\underline{p}(x;r) = \underline{f_j}(r) + \sum_{\nu=1}^{[\frac{n}{2}]} \binom{\theta}{2\nu+1} \underline{\Delta^{2\nu+1} f_j}(r) + \sum_{\nu=1}^{[\frac{n}{2}]} \binom{\theta}{2\nu} \overline{\Delta^{2\nu} f_j}(r),$$

$$\overline{p}(x;r) = \overline{f_j}(r) + \sum_{\nu=1}^{[\frac{n}{2}]} \binom{\theta}{2\nu+1} \overline{\Delta^{2\nu+1} f_j}(r) + \sum_{\nu=1}^{[\frac{n}{2}]} \binom{\theta}{2\nu} \underline{\Delta^{2\nu} f_j}(r),$$

where

$$\underline{\Delta^{2\nu} f_j}(r) = \sum_{m=0}^{\nu} \binom{2\nu}{2m} \underline{f_{j+2m}}(r) - \sum_{m=0}^{\nu-1} \binom{2\nu}{2m+1} \overline{f_{j+2m+1}}(r),$$

$$\overline{\Delta^{2\nu} f_j}(r) = \sum_{m=0}^{\nu} \binom{2\nu}{2m} \overline{f_{j+2m}}(r) - \sum_{m=0}^{\nu-1} \binom{2\nu}{2m+1} \underline{f_{j+2m+1}}(r), \quad 0 \leqslant \nu \leqslant \frac{n}{2},$$

and

$$\underline{\Delta^{2\nu+1} f_j}(r) = \sum_{m=0}^{[\frac{2\nu+1}{2}]} \binom{2\nu+1}{2m+1} \underline{f_{j+2m+1}}(r) - \sum_{m=0}^{[\frac{2\nu+1}{2}]} \binom{2\nu+1}{2m} \overline{f_{j+2m}}(r),$$

$$\overline{\Delta^{2\nu+1} f_j}(r) = \sum_{m=0}^{[\frac{2\nu+1}{2}]} \binom{2\nu+1}{2m+1} \overline{f_{j+2m+1}}(r) - \sum_{m=0}^{[\frac{2\nu+1}{2}]} \binom{2\nu+1}{2m} \underline{f_{j+2m}}(r), \quad 0 \leqslant \nu \leqslant [\frac{n}{2}].$$

For more information see [3].

PROPOSITION 3.1 *(see [3]). The* $[p(x)]_r = [\underline{p}(x;r), \overline{p}(x;r)]$ *are the* $r-$*level sets of fuzzy-valued polynomial.*

THEOREM 3.2 *(see [3]). Let* $p(x)$ *be the interpolation of fuzzy function* f *for interpolation points and suppose that the* $\underline{f^{(n+1)}}$ *and* $\overline{f^{(n+1)}}$ *exist and bounded on* $[x_0, x_n]$ *then*

$$\lim_{h\to 0} \underline{p}(x,h;r) = \underline{f}(x;r),$$
$$\lim_{h\to 0} \overline{p}(x,h;r) = \overline{f}(x;r).$$

THEOREM 3.3 *(see [3]). If* $|x_i| \neq |x_j|,\ 0 \leq i \neq j \leq n$ *then the fuzzy Newton finite difference interpolation problem is unique.*

4. Adams-Bashforth Methods

For solving the fuzzy initial-value problem by Adams-Bashforth $m+1$-step Method, let $t_n = t_0 + nh$, $h = \frac{T-t_0}{N}$, $1 \leqslant n \leqslant N$ and the fuzzy initial values be $y(t_{n-m}) = \alpha(n-m)$, ..., $y(t_n) = \alpha(n)$, where the fuzzy numbers are in r$-$cut form.
According to Lemma (2.9) and Theorem (2.7) we have:

$$y(t_{n+1}) = y(t_n) \oplus \int_{t_n}^{t_{n+1}} f(t, y(t))dt$$

or

$$y(t_{n+1}) = y(t_n) -^h (-1) \odot \int_{t_n}^{t_{n+1}} f(t, y(t))dt.$$

If on an interval we expect a solution with increasing support, then we find a (i)-differentiable solution. If on an interval we expect a solution with decreasing support, then we find a (ii)-differentiable solution. Such that

$$[y(t_{n+1})]_r = [\underline{y}_{n+1}(r), \overline{y}_{n+1}(r)],$$

$$[y(t_n)]_r = [\underline{y}_n(r), \overline{y}_n(r)],$$

and

$$\begin{cases} \underline{y}_{n+1}(r) = \underline{y}_n(r) + \int_{t_n}^{t_{n+1}} \underline{f}(t,r)dt \\ \overline{y}_{n+1}(r) = \overline{y}_n(r) + \int_{t_n}^{t_{n+1}} \overline{f}(t,r)dt. \end{cases}$$

or

$$\begin{cases} \underline{y}_{n+1}(r) = \underline{y}_n(r) + \int_{t_n}^{t_{n+1}} \overline{f}(t,r)dt \\ \overline{y}_{n+1}(r) = \overline{y}_n(r) + \int_{t_n}^{t_{n+1}} \underline{f}(t,r)dt. \end{cases}$$

Also let

$$\underline{f}(t,r) \simeq \underline{p}(t,r)$$

$$\overline{f}(t,r) \simeq \overline{p}(t,r)$$

by using the fuzzy Newton finite differences presented in Section 3, such that $t = t_n + \theta h$, $dt = hd\theta$. The following results will be obtained:

$$\begin{aligned} \underline{p}(t;r) &= \textstyle\sum_{k=0}^{m}(-1)^k\binom{-\theta}{k}\underline{\Delta^k f_{n-k}}(r) \\ \overline{p}(t;r) &= \textstyle\sum_{k=0}^{m}(-1)^k\binom{-\theta}{k}\overline{\Delta^k f_{n-k}}(r). \end{aligned} \tag{9}$$

For $k = 2\nu$

$$\begin{aligned} \underline{\Delta^k f_{n-k}}(r) &= \textstyle\sum_{m=0}^{\nu}\binom{2\nu}{2m}\underline{f_{n-k+2m}}(r) - \sum_{m=0}^{\nu-1}\binom{2\nu}{2m+1}\overline{f_{n-k+2m+1}}(r), \\ \overline{\Delta^k f_{n-k}}(r) &= \textstyle\sum_{m=0}^{\nu}\binom{2\nu}{2m}\overline{f_{n-k+2m}}(r) - \sum_{m=0}^{\nu-1}\binom{2\nu}{2m+1}\underline{f_{n-k+2m+1}}(r), \end{aligned} \tag{10}$$

and for $k = 2\nu + 1$

$$\begin{aligned} \underline{\Delta^k f_{n-k}}(r) &= \textstyle\sum_{m=0}^{[\frac{2\nu+1}{2}]}\binom{2\nu+1}{2m+1}\underline{f_{n-k+2m+1}}(r) - \sum_{m=0}^{[\frac{2\nu+1}{2}]}\binom{2\nu+1}{2m}\overline{f_{n-k+2m}}(r), \\ \overline{\Delta^{2\nu+1} f_j}(r) &= \textstyle\sum_{m=0}^{[\frac{2\nu+1}{2}]}\binom{2\nu+1}{2m+1}\overline{f_{n-k+2m+1}}(r) - \sum_{m=0}^{[\frac{2\nu+1}{2}]}\binom{2\nu+1}{2m}\underline{f_{n-k+2m}}(r). \end{aligned} \tag{11}$$

Now, from Eqs. (9) it follows that:

$$\int_{t_n}^{t_{n+1}} \underline{p}(t;r)dt = h\int_0^1\left(\textstyle\sum_{k=0}^{m}(-1)^k\binom{-\theta}{k}\underline{\Delta^k f_{n-k}}(r)\right)d\theta$$

$$\int_{t_n}^{t_{n+1}} \overline{p}(t;r)dt = h\int_0^1\left(\textstyle\sum_{k=0}^{m}(-1)^k\binom{-\theta}{k}\overline{\Delta^k f_{n-k}}(r)\right)d\theta$$

since f is continuous and $(-1)^k\binom{-\theta}{k}$ is integrable and positive, so

$$\int_{t_n}^{t_{n+1}} \underline{p}(t;r)dt = h\textstyle\sum_{k=0}^{m}\underline{\Delta^k f_{n-k}}(r)\int_0^1(-1)^k\binom{-\theta}{k}d\theta$$

$$\int_{t_n}^{t_{n+1}} \overline{p}(t;r)dt = h\textstyle\sum_{k=0}^{m}\overline{\Delta^k f_{n-k}}(r)\int_0^1(-1)^k\binom{-\theta}{k}d\theta.$$

If we consider $\gamma_k = \int_0^1(-1)^k\binom{-\theta}{k}d\theta$, then:

$$\begin{cases} \underline{y}_{n+1}(r) = \underline{y}_n(r) + h\sum_{k=0}^{m}\underline{\Delta^k f_{n-k}}(r)\gamma_k \\ \overline{y}_{n+1}(r) = \overline{y}_n(r) + h\sum_{k=0}^{m}\overline{\Delta^k f_{n-k}}(r)\gamma_k \end{cases} \tag{12}$$

or

$$\begin{cases} \underline{y}_{n+1}(r) = \underline{y}_n(r) + h\sum_{k=0}^{m} \overline{\Delta^k f_{n-k}}(r)\gamma_k \\ \overline{y}_{n+1}(r) = \overline{y}_n(r) + h\sum_{k=0}^{m} \underline{\Delta^k f_{n-k}}(r)\gamma_k \end{cases} \quad (13)$$

4.1 *Adams-Bashforth Four-Step Method*

From Eqs. (12, 13) and $m = 3$, the following results will be obtained.

$$\begin{cases} \underline{y}_{n+1}(r) = \underline{y}_n(r) + \frac{h}{24}[55\underline{f_n}(r) - 59\overline{f_{n-1}}(r) + 37\underline{f_{n-2}}(r) - 9\overline{f_{n-3}}(r)] \\ \overline{y}_{n+1}(r) = \overline{y}_n(r) + \frac{h}{24}[55\overline{f_n}(r) - 59\underline{f_{n-1}}(r) + 39\overline{f_{n-2}}(r) - 9\underline{f_{n-3}}(r)] \end{cases} \quad (14)$$

or

$$\begin{cases} \underline{y}_{n+1}(r) = \underline{y}_n(r) + \frac{h}{24}[55\overline{f_n}(r) - 59\underline{f_{n-1}}(r) + 39\overline{f_{n-2}}(r) - 9\underline{f_{n-3}}(r)] \\ \overline{y}_{n+1}(r) = \overline{y}_n(r) + \frac{h}{24}[55\underline{f_n}(r) - 59\overline{f_{n-1}}(r) + 37\underline{f_{n-2}}(r) - 9\overline{f_{n-3}}(r)] \end{cases} \quad (15)$$

4.2 *Adams-Bashforth Three-Step Method*

From Eqs.(12, 13) and $m = 2$, $\underline{y}_{n+1}(r)$, $\overline{y}_{n+1}(r)$ will be computed as follows:

$$\begin{cases} \underline{y}_{n+1}(r) = \underline{y}_n(r) + \frac{h}{12}[23\underline{f_n}(r) - 16\overline{f_{n-1}}(r) + 5\underline{f_{n-2}}(r)] \\ \overline{y}_{n+1}(r) = \overline{y}_n(r) + \frac{h}{12}[23\overline{f_n}(r) - 16\underline{f_{n-1}}(r) + 5\overline{f_{n-2}}(r) \end{cases} \quad (16)$$

or

$$\begin{cases} \underline{y}_{n+1}(r) = \underline{y}_n(r) + \frac{h}{12}[23\overline{f_n}(r) - 16\underline{f_{n-1}}(r) + 5\overline{f_{n-2}}(r) \\ \overline{y}_{n+1}(r) = \overline{y}_n(r) + \frac{h}{12}[23\underline{f_n}(r) - 16\overline{f_{n-1}}(r) + 5\underline{f_{n-2}}(r)] \end{cases} \quad (17)$$

4.3 *Adams-Bashforth $(m + 1)$-step Method Algorithm*

To approximate the solution of the following fuzzy initial value problem, the positive integer N is chosen.

$$\begin{cases} y'(t) = f(t, y(t)), & a \leqslant t \leqslant b, \\ y(a) = \alpha. \end{cases} \quad (18)$$

Step 1. Consider $h = \frac{b-a}{N}$, $t_0 = a$, $\underline{w}_0 = \underline{\alpha}$, $\overline{w}_0 = \overline{\alpha}$
Step 2. Let $i = 0$.
Step 3. Let $t_{i+1} = a + (i + 1)h$, then by using the Runge-Kutta method compute $\underline{w}_{i+1}, \overline{w}_{i+1}$
Step 4. $i = i + 1$
Step 5. If $i \leqslant m - 1$ go to step 3.

Step 6. Let $j = M$
Step 7. For $k = 0, 1, ..., m$ compute:
$\gamma_k, \underline{\Delta^k f_{j-k}}(r), \overline{\Delta^k f_{j-k}}(r), \underline{w}_{j+1}$ and $\overline{w}_{j+1}$
Step 8. Let $t_{j+1} = t_0 + (j+1)h$
Step 9. $j = j + 1$
Step 10. If $j \leqslant N - 1$ go to step 7.
Step 11. The algorithm will terminate and $[\underline{w}(b,r), \overline{w}(b,r)]$ approximates the real values $[\underline{y}(b,r), \overline{y}(b,r)]$.

5. Adams-Moulton Methods

Solving Eqs. (18) by the Adams-Moulton $(m+1)$-step method is similar to that by the Adams-Bashforth $(m+1)$-step method with an extra point t_{n+1}. Now, if we consider $\gamma'_k = \int_0^1 (-1)^k \binom{1-\theta}{k} d\theta$, then

$$\begin{cases} \underline{y}_{n+1}(r) = \underline{y}_n(r) + h\sum_{k=0}^{m+1} \underline{\Delta^k f_{n-k+1}}(r)\gamma'_k \\ \overline{y}_{n+1}(r) = \overline{y}_n(r) + h\sum_{k=0}^{m+1} \overline{\Delta^k f_{n-k+1}}(r)\gamma'_k \end{cases} \tag{19}$$

or

$$\begin{cases} \underline{y}_{n+1}(r) = \underline{y}_n(r) + h\sum_{k=0}^{m+1} \overline{\Delta^k f_{n-k+1}}(r)\gamma'_k \\ \overline{y}_{n+1}(r) = \overline{y}_n(r) + h\sum_{k=0}^{m+1} \underline{\Delta^k f_{n-k+1}}(r)\gamma'_k \end{cases} \tag{20}$$

For $k = 2\nu$

$$\begin{aligned} \underline{\Delta^k f_{n-k+1}}(r) &= \sum_{m=0}^{\nu} \binom{k}{2m} \underline{f_{n-k+1+2m}}(r) - \sum_{m=0}^{\nu-1} \binom{k}{2m+1} \overline{f_{n-k+2m+2}}(r), \\ \overline{\Delta^k f_{n-k+1}}(r) &= \sum_{m=0}^{\nu} \binom{k}{2m} \overline{f_{n-k+2m+1}}(r) - \sum_{m=0}^{\nu-1} \binom{k}{2m+1} \underline{f_{n-k+2m+2}}(r), \end{aligned} \tag{21}$$

and for $k = 2\nu + 1$

$$\begin{aligned} \underline{\Delta^k f_{n-k+1}}(r) &= \sum_{m=0}^{[\frac{k}{2}]} \binom{k}{2m+1} \underline{f_{n-k+2m+2}}(r) - \sum_{m=0}^{[\frac{k}{2}]} \binom{2\nu+1}{2m} \overline{f_{n-k+2m+1}}(r), \\ \overline{\Delta^k f_{n-k+1}}(r) &= \sum_{m=0}^{[\frac{k}{2}]} \binom{k}{2m+1} \overline{f_{n-k+2m+2}}(r) - \sum_{m=0}^{[\frac{k}{2}]} \binom{k}{2m} \underline{f_{n-k+2m+1}}(r). \end{aligned} \tag{22}$$

5.1 *Adams-Moulton Three-Step Method*

From Eqs. (19, 20) and $m = 2$, $\underline{y}_{n+1}(r)$, $\overline{y}_{n+1}(r)$ will be computed as follows:

$$\underline{y}_{n+1}(r) = \underline{y}_n(r) + h\underline{\Delta^0 f_{n+1}}(r)\gamma'_0 + \underline{\Delta^1 f_n}(r)\gamma'_1 + \underline{\Delta^2 f_{n-1}}(r)\gamma'_2 + \underline{\Delta^3 f_{n-2}}(r)\gamma'_3$$

$$\overline{y}_{n+1}(r) = \overline{y}_n(r) + h\overline{\Delta^0 f_{n+1}}(r)\gamma'_0 + \overline{\Delta^1 f_n}(r)\gamma'_1 + \overline{\Delta^2 f_{n-1}}(r)\gamma'_2 + \overline{\Delta^3 f_{n-2}}(r)\gamma'_3$$

or

$$\underline{y}_{n+1}(r) = \underline{y}_n(r) + h\overline{\Delta^0 f_{n+1}}(r)\gamma'_0 + \overline{\Delta^1 f_n}(r)\gamma'_1 + \overline{\Delta^2 f_{n-1}}(r)\gamma'_2 + \overline{\Delta^3 f_{n-2}}(r)\gamma'_3$$

$$\underline{y}_{n+1}(r) = \underline{y}_n(r) + h\underline{\Delta^0 f_{n+1}}(r)\gamma'_0 + \underline{\Delta^1 f_n}(r)\gamma'_1 + \underline{\Delta^2 f_{n-1}}(r)\gamma'_2 + \underline{\Delta^3 f_{n-2}}(r)\gamma'_3$$

It is obvious that by replacing γ'_i, $i = 1, 2, 3$, the following results will be obtained.

$$\underline{y}_{n+1}(r) = \underline{y}_n(r) + \frac{h}{24}[9\underline{f_{n+1}}(r) + 19\underline{f_n}(r) - 5\overline{f_{n-1}}(r) + \underline{f_{n-2}}(r)]$$

$$\overline{y}_{n+1}(r) = \overline{y}_n(r) + \frac{h}{24}[9\overline{f_{n+1}}(r) + 19\overline{fn}(r) - 5\underline{f_{n-1}}(r) + \overline{f_{n-2}}(r)]$$

or

$$\underline{y}_{n+1}(r) = \overline{y}_n(r) + \frac{h}{24}[9\overline{f_{n+1}}(r) + 19\overline{f_n}(r) - 5\underline{f_{n-1}}(r) + \overline{f_{n-2}}(r)]$$

$$\underline{y}_{n+1}(r) = \underline{y}_n(r) + \frac{h}{24}[9\underline{f_{n+1}}(r) + 19\underline{fn}(r) - 5\overline{f_{n-1}}(r) + \underline{f_{n-2}}(r)]$$

$$\underline{y}_{n+1}(r) = \underline{\alpha}_{n+1}(r), \overline{y}_{n+1}(r) = \overline{\alpha}_{n+1}(r), \ldots, \underline{y}_{n-2}(r) = \underline{\alpha}_{n-2}(r), \overline{y}_{n-2}(r) = \overline{\alpha}_{n-2}(r)$$

5.2 *Adams-Moulton Two-Step Method*

From Eqs. (19-22) and $m = 1$, the following results will be obtained.

$$\underline{y}_{n+1}(r) = \underline{y}_n(r) + \frac{h}{12}[5\underline{f}_{n+1}(r) + 8\underline{f_n}(r) - \overline{f_{n-1}}(r)]$$

$$\overline{y}_{n+1}(r) = \overline{y}_n(r) + \frac{h}{12}[5\overline{f_{n+1}}(r) + 8\overline{f_n}(r) - \underline{f_{n-1}}(r)]$$

or

$$\underline{y}_{n+1}(r) = \underline{y}_n(r) + \frac{h}{12}[5\overline{f}_{n+1}(r) + 8\overline{f_n}(r) - \underline{f_{n-1}}(r)]$$

$$\overline{y}_{n+1}(r) = \overline{y}_n(r) + \frac{h}{12}[5\underline{f_{n+1}}(r) + 8\underline{f_n}(r) - \overline{f_{n-1}}(r)]$$

5.3 *Predictor-Corrector* $(m+1)$*-step Method Algorithm*

To approximate the solution of the fuzzy initial value problem in Eqs. (18), the positive integer N is chosen.

Step 1. Let $h = \frac{b-a}{N}$, $t_0 = a$, $\underline{w}_0(r) = \underline{\alpha_0}(r)$, ..., $\underline{w}_m(r) = \underline{\alpha_m}(r)$ $\overline{w}_0(r) = \overline{\alpha_0}(r)$, ..., $\overline{w}_m(r) = \overline{\alpha_m}(r)$

Step 2. Let $n = m$.

Step 3. For $k = 0, 1, ..., m$ compute:

$\underline{\Delta^k f_{n-k}}(r)$, $\overline{\Delta^k f_{n-k}}(r)$ and γ_k, then

$\underline{w}^0_{n+1}(r) = \underline{w}_n(r) + h\sum_{k=0}^{m} \underline{\Delta^k f_{n-k}}(r)\gamma_k$

$\overline{w}^0_{n+1}(r) = \overline{w}_n(r) + h\sum_{k=0}^{m} \overline{\Delta^k f_{n-k}}(r)\gamma_k$

Step 4. Let $t_{n+1} = t_0 + (n+1)h$

Step 5. For $k = 0, 1, ..., m$ compute:

$\underline{\Delta^k f_{n-k+1}}(r)$, $\overline{\Delta^k f_{n-k+1}}(r)$ and γ'_k , then

$\underline{w}_{n+1}(r) = \underline{w}_n(r) + h\sum_{k=0}^{m} \underline{\Delta^k f_{n-k+1}}(r)\gamma'_k$

$\overline{w}_{n+1}(r) = \overline{w}_n(r) + h\sum_{k=0}^{m} \overline{\Delta^k f_{n-k+1}}(r)\gamma_k$

Step 6. $n = n + 1$

Step 7. If $n \leqslant N - 1$ go to step 3.

Step 8. The algorithm will terminate and $[\underline{w}(b, r), \overline{w}(b, r)]$ approximates the real values $[\underline{y}(b, r), \overline{y}(b, r)]$.

6. Convergence

Without less of generality, we investigated convergence of proposed algorithm in case of (i)-differentiable, case (ii)-differentiable is proved in a similar way. To integrate the system given in Eqs. (12), the interval $[a, b]$ will be replaced by a set of discrete equally spaced grid points $a = t_o < t_1 < t_2 < ... < t_N = b$ at which the exact solution $[\underline{Y}(t, r), \overline{Y}(t, r)]$ is approximated by some $[\underline{y}(t, r), \overline{y}(t, r)]$. The exact and approximate solutions at t_n, $0 \leqslant n \leqslant N$, are denoted by $Y_n(r) = [\underline{Y}(t_n, r), \overline{Y}(t_n, r)]$, and $y_n(r) = [\underline{y}(t_n, r), \overline{y}(t_n, r)]$, respectively. The grid points at which the solution is calculated are from Eqs. (12)and the polygon curves

$$\underline{y}(t, h, r) = [t_0, \underline{y}_0(r)], [t_1, \underline{y}_1(r)], ..., [t_N, \underline{y}_N(r)],$$

$$\overline{y}(t, h, r) = [t_0, \overline{y}_0(r)], [t_1, \overline{y}_1(r)], ..., [t_N, \overline{y}_N(r)],$$

are Adams-Bashforth approximates to $\underline{Y}(t,r)$ and $\overline{Y}(t,r)$, respectively, over the interval $a = t_o < t < t_N = b$. It is sufficient to show

$$\lim_{h\longrightarrow 0} \underline{y}(t,h,r) = \underline{Y}(t,r), \lim_{h\longrightarrow 0} \overline{y}(t,h,r) = \overline{Y}(t,r).$$

From Eqs.(12) we obtain:

$$\begin{aligned} \underline{y}_{n+1} &= \underline{y}_n + h[\underline{\Delta^0 f_n}(r)\gamma_0 + ... + \underline{\Delta^m f_{n-m}}(r)\gamma_m] \\ \overline{y}_{n+1} &= \overline{y}_n + h[\overline{\Delta^0 f_n}(r)\gamma_0 + ... + \overline{\Delta^m f_{n-m}}(r)\gamma_m], \end{aligned} \tag{23}$$

then

$$\underline{y}_{n+1} = \underline{y}_n + h[\beta_0 \underline{f_n}(r) + ... + \beta_m \underline{f_{n-m}}(r) - \beta_1 \overline{f_{n-1}}(r) - \beta_3 \overline{f_{n-3}}(r) - ... - \beta_{m-1} \overline{f_{n-m+1}}(r)]$$

$$\overline{y}_{n+1} = \overline{y}_n + h[\beta_1 \overline{f_{n-1}}(r) + ... + \beta_3 \overline{f_{n-3}}(r) + ... + \beta_{m-1} \overline{f_{n-m+1}}(r) - \beta_0 \underline{f_n}(r) - \beta_2 \underline{f_{n-2}}(r)$$
$$-... - \beta_m \underline{f_{n-m}}(r)]. \tag{24}$$

For exact values, the following results will be obtained:

$$\underline{Y}_{n+1} = \underline{Y}_n + h[\beta_0 \underline{f}(t_n, \underline{Y}_n, \overline{Y}_n, r) + ... + \beta_m \underline{f}(t_{n-m}, \underline{Y}_{n-m}, \overline{Y}_{n-m}, r) - \beta_1 \overline{f}(t_{n-1}, \underline{Y}_{n-1}, \overline{Y}_{n-1}, r)$$
$$-... - \beta_{m-1} \overline{f}(t_{n-m+1}, \underline{Y}_{n-m+1}, \overline{Y}_{n-m+1}, r)] + \gamma_{m+1} h^{m+2} \underline{Y}^{m+2}(\xi),$$

$$\overline{Y}_{n+1} = \overline{Y}_n + h[\beta_1 \overline{f}(t_{n-1}, \underline{Y}_{n-1}, \overline{Y}_{n-1}, r) + ... + \beta_3 \overline{f}(t_{n-3}, \underline{Y}_{n-3}, \overline{Y}_{n-3}, r) + ... + \beta_{m-1} \overline{f}(t_{n-m+1},$$
$$\underline{Y}_{n-m+1}, \overline{Y}_{n-m+1}, r) - \beta_0 \underline{f}(t_n, \underline{Y}_n, \overline{Y}_n, r) - ... - \beta_m \underline{f}(t_{n-m}, \underline{Y}_{n-m}, \overline{Y}_{n-m}, r)] + \gamma'_{m+1} h^{m+2} \overline{Y}^{m+2}(\xi) \tag{25}$$

provided that $\underline{Y}, \overline{Y} \in C^{m+2}[t_0, T]$. Let

$$w_{n+1}(r) = \underline{Y}_{n+1}(r) - \underline{y}_{n+1}(r),$$

$$v_{n+1}(r) = \overline{Y}_{n+1}(r) - \overline{y}_{n+1}(r),$$

then

$$w_{n+1}(r) = w_n(r) + h[\beta_0(\underline{f}_n(r) - \underline{f}(t_n, \underline{Y}_n, \overline{Y}_n, r)) + ... + \beta_m(\underline{f}_{n-m}(r) - \underline{f}(t_{n-m}, \underline{Y}_{n-m}, \overline{Y}_{n-m}, r))$$

$$-... - \beta_{m-1}(\overline{f}_{n-m+1}(r) - \overline{f}(t_{n-m+1}, \underline{Y}_{n-m+1}, \overline{Y}_{n-m+1}, r))] + \gamma_{m+1} h^{m+2} \underline{Y}^{m+2}(\xi)$$

$$v_{n+1} = v_n + h[\beta_1(\overline{f}_{n-1}(r) - \overline{f}(t_{n-1}, \underline{Y}_{n-1}, \overline{Y}_{n-1}, r)) + ... + \beta_3(\underline{f}_{n-3}(r) - \underline{f}(t_{n-3}, \underline{Y}_{n-3}, \overline{Y}_{n-3}, r)) +$$

$$\beta_{m-1}(\overline{f}_{n-m+1}(r) - \overline{f}(t_{n-m+1}, \underline{Y}_{n-m+1}, \overline{Y}_{n-m+1}, r)) - \beta_0(\underline{f}_n(r) - \underline{f}(t_n, \underline{Y}_n, \overline{Y}_n, r)) - ...$$

$$-\beta_m(\underline{f}_{n-m}(r) - \underline{f}(t_{n-m}, \underline{Y}_{n-m}, \overline{Y}_{n-m}, r))]$$

$$+\gamma'_{m+1} h^{m+2} \overline{Y}^{m+2}(\xi). \tag{26}$$

Suppose l_i, $i = 1, 2, ..., m$, are Lipschitz constants. If $l = \max\{l_1, ..., l_m\}$ then

$$|w_{n+1}(r)| \leqslant |w_n(r)| + hl[|\beta_0|(|w_n(r)| + |v_n(r)|) + ... + |\beta_m|(|w_{n-m}(r)| + |v_{n-m}(r)|)$$
$$+... + |\beta_{m-1}|(|w_{n-m+1}(r)| + |v_{n-m+1}(r)|)] + |\gamma_{m+1}| h^{m+2} \underline{M}$$

$$|v_{n+1}(r)| \leqslant |v_n(r)| + hl[|\beta_0|(|w_n(r)| + |v_n(r)|) + ... + |\beta_m|(|w_{n-m}(r)| + |v_{n-m}(r)|)$$

$$+... + |\beta_{m-1}|(|w_{n-m+1}(r)| + |v_{n-m+1}(r)|)] + |\gamma'_{m+1}| h^{m+2} \overline{M}. \tag{27}$$

where

$$\overline{M} = \max |\overline{Y}^{m+2}(\xi)|$$

$$\underline{M} = \max |\underline{Y}^{m+2}(\xi)|.$$

Now let $|u_{n+1}(r)| = |w_{n+1}(r)| + |v_{n+1}(r)|$ then:

$$\begin{aligned} |u_{n+1}(r)| &\leqslant |u_n(r)| + hl[2|\beta_0|(|u_n(r)|) + ... + 2|\beta_m|(|u_{n-m}(r)|) \\ &+... + 2|\beta_{m-1}|(|u_{n-m+1}(r)|)] + |\gamma_{m+1}| h^{m+2}(\underline{M} + \overline{M}). \end{aligned} \tag{28}$$

Form (28) the differential equation (29) will be constructed as follows:

$$\begin{aligned} &|k_{n+1}(r)| = |k_n(r)| + hl[2|\beta_0|(|k_n(r)|) + ... + 2|\beta_m|(|k_{n-m}(r)|) \\ &+... + 2|\beta_{m-1}|(|k_{n-m+1}(r)|)] + |\gamma_{m+1}| h^{m+2}(\underline{M} + \overline{M}) \\ &k_i(r) \geqslant \delta \qquad i = 0, 1, ..., m \end{aligned} \tag{29}$$

such that k_i, $i = 0, ..., m$, are the roots of differential equation (29).
Now, we prove that $|u_i(r)| \leqslant \delta \leqslant k_i(r)$ for all $i = m+1, ..., s$.

From (8.22) we have:

$$|u_{m+1}(r)| \leqslant |u_m(r)| + 2hl[|\beta_0|u_m(r) + ... + |\beta_m|u_0(r)] + |\gamma_{m+1}|h^{m+2}(\underline{M} + \overline{M})$$
$$\leqslant |k_m(r)| + 2hl[|\beta_0|k_m(r) + ... + |\beta_m|k_0(r)] + |\gamma_{m+1}|h^{m+2}(\underline{M} + \overline{M}) = k_{m+1}(r) \tag{30}$$

so

$$u_{m+1}(r) \leqslant k_{m+1}(r), \qquad r \in [0, 1].$$

To obtain a solution of Eq. (29), first let $k_i(r) = -c(r)$. From Eq. (31) it follows that:

$$-c(r) = -c(r)+2hl[|\beta_0|(-c(r))+...+|\beta_m|(-c(r))+...+|\beta_{m-1}|(-c(r))]+|c_{m+1}|h^{m+2}(\underline{M}+\overline{M})$$

$$c(r) = \frac{|c_{m+1}|h^{m+1}(\underline{M} + \overline{M})}{2l\sum_{i=0}^{m}\beta_i}, \quad 0 \leqslant r \leqslant 1.$$

The homogeneous solution will be computed as follows:
For all r, $k_n = x^n$, let

$$x^{n+1} = x^n + 2hl[|\beta_0|x^n + ... + |\beta_1|x^{n-1} + |\beta_m|x^{n-m}].$$

If $m = n$, we have

$$H(x) = x^{m+1} - x^m - 2hl[|\beta_0|x^m + ... + |\beta_{m-1}|x^1 + |\beta_m|x^0]. \tag{31}$$

Now, we will show Eq. (31) has a root $x > 1$ as follows:

$$H(1) = 1 - 1 - 2hl\sum_{i=0}^{m}\beta_i = -2hl\sum_{i=0}^{m}\beta_i < 0$$

Let

$$x = 1 + 2hl\sum_{i=0}^{m}\beta_i > 1.$$

From Eq. (31) we have

$$x^{-m}H(x) = x - 1 - 2hl[|\beta_0| + ... + |\beta_{m-1}|x^{1-m} + |\beta_m|x^{-m}$$

$$\geqslant 2hl\textstyle\sum_{i=0}^{m}|\beta_i| - 2hl\sum_{i=0}^{m}|\beta_i| = 0.$$

Thus $H(x) \geqslant 0$ then there exists $1 < z^\star(r) < 1+2hl\sum_{i=0}^{m}|\beta_i|$ such that $H(z^\star) = 0$.

Since $2hl\sum_{i=0}^{m}|\beta_i| > -1$, then $1 + 2hl\sum_{i=0}^{m}|\beta_i| < e^{2hl\sum_{i=0}^{m}|\beta_i|}$

$z^\star(r) < 1 + 2hl\sum_{i=0}^{m}|\beta_i| < e^{2hl\sum_{i=0}^{m}|\beta_i|}$

$(z^\star(r))^n < e^{2hnl\sum_{i=0}^{m}|\beta_i|} = e^{2(x_n-a)l\sum_{i=0}^{m}|\beta_i|}$.

Then the solution of Eq.(29) will be obtained as follows such that $h = \frac{x_n - x_0}{N}$.
Let $k_n = ((z^\star(r))^n - 1)c(r)$.

$$k_n(r) \leqslant \frac{|c_{m+1}|h^{m+1}(\underline{M}+\overline{M})}{2l\sum_{i=0}^{m}\beta_i} e^{2(x_n-a)l\sum_{i=0}^{m}(|\beta_i|-1)u_n(r)}$$

$$\leqslant \frac{|c_{m+1}|h^{m+1}(\underline{M}+\overline{M})}{2l\sum_{i=0}^{m}\beta_i} e^{2(x_n-a)l\sum_{i=0}^{m}(|\beta_i|-1)|v_n|,|w_n|}$$

$$\leqslant \frac{|c_{m+1}|h^{m+1}(\underline{M}+\overline{M})}{2l\sum_{i=0}^{m}\beta_i} e^{2(x_n-a)l\sum_{i=0}^{m}|\beta_i|}$$

are obtained. If $h \longrightarrow 0$ then $v_n \longrightarrow 0$, $w_n \to 0$, which completes the proof of the following theorem.

THEOREM 6.1 *For an arbitrary fixed* $r: \quad 0 \leqslant r \leqslant 1$, *the Adams-Bashforth* $(m+1)-$*step approximates of Eq.(12) converge to the exact solution* $\underline{Y}(t,r), \overline{Y}(t,r)$ *for* $\underline{Y}, \overline{Y} \in C^{m+1}[t_0, T]$.

Remark 1 The convergence order of the Adams-Bashforth $(m+1)-$step method is $O(h^{m+1})$.

7. Numerical Examples

In this section, some fuzzy initial value problems will solved by suggested methods. The results which are obtained are more exact than the results in [1, 2, 29, 39] that are obtained by two-step method.

Example 1. Consider the initial value problem

$$y'(t) = y(t) + t + 1,$$

with initial condition

$$y(0) = (0.96 + 0.04r, 1.01 - 0.01r)$$

$$y(0.01) = (0.01 + (0.96 + 0.04r)e^{-0.01}, 0.01 + (1.01 - 0.01r)e^{-0.01}),$$

$$y(0.02) = (0.02 + (0.96 + 0.04r)e^{-0.02}, 0.02 + (1.01 - 0.01r)e^{-0.02}).$$

The exact solution at $t = 0.1$ with increasing support, i.e. (i)-differentiable, is given by

$$Y_1(0.1, r) = (0.1 + (0.96 + 0.04r)e^{-0.1}, 0.1 + (1.01 - 0.01r)e^{-0.1}), \qquad 0 \leqslant r \leqslant 1.$$

the result of the Adams-Bashforth two-step method and the Predictor-Corrector three-step method with $N = 10$ has been shown in table (1) and table (2).

Also, our proposed solutions are plotted simultaneously to compare with exact solutions. For more detail see Figures 1. and 2.

$Table(1). \ The\ result\ of\ the\ Adams-Bashforth\ two-step\ method\ with\ N = 10$

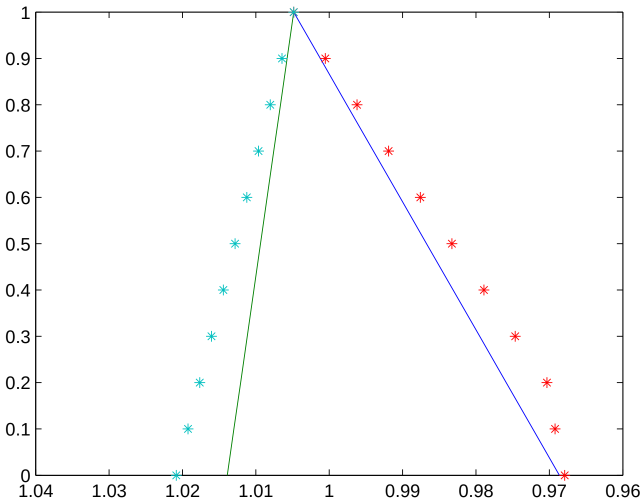

Figure 1. Compare The result of the Adams-Bashforth two-step method with N=10 and exact solution.

r	$\underline{y}_1$	$\underline{Y}_1$	ERROR	$\overline{y}_1$	$\overline{Y}_1$	ERROR
0.0	0.961710	0.968644	0.006934	1.020827	1.013886	0.006941
0.1	0.969225	0.972263	0.006241	1.019228	1.012981	0.006247
0.2	0.970336	0.975883	0.005547	1.017630	1.012076	0.005554
0.3	0.974649	0.979502	0.004853	1.016031	1.011171	0.004860
0.4	0.978920	0.983121	0.004159	1.014432	1.010266	0.004166
0.5	0.983275	0.986741	0.003465	1.012834	1.009362	0.003472
0.6	0.987588	0.990360	0.002772	1.011235	1.008457	0.002778
0.7	0.991901	0.993979	0.002078	1.009637	1.007552	0.002085
0.8	0.996214	0.997599	0.001384	1.008038	1.006647	0.001391
0.9	1.000528	1.001218	0.000690	1.006439	1.005742	0.000697
1.0	1.004841	1.004837	0.000004	1.004841	1.004837	0.000004

r	$\underline{y}_1$	$\underline{Y}_1$	ERROR	$\overline{y}_1$	$\overline{Y}_1$	ERROR
0.0	0.964346	0.968644	0.004297	1.018183	1.013886	0.004297
0.1	0.968395	0.972263	0.003868	1.016849	1.012981	0.003868
0.2	0.972445	0.975883	0.003438	1.015514	1.012076	0.003438
0.3	0.976494	0.979502	0.003008	1.014179	1.011171	0.003008
0.4	0.980543	0.983121	0.002578	1.012845	1.010266	0.002578
0.5	0.984592	0.986741	0.002149	1.011510	1.009362	0.002149
0.6	0.988641	0.990360	0.001719	1.010176	1.008457	0.001719
0.7	0.992690	0.993979	0.001289	1.008841	1.007552	0.001289
0.8	0.996739	0.997599	0.000859	1.007506	1.006647	0.000859
0.9	1.000788	1.001218	0.000430	1.006172	1.005742	0.0004297
1.0	1.004837	1.004837	0.000000	1.004837	1.004837	0.000000

$Table(2). The\ result\ of\ the\ Predictor-Corrector\ three-step\ method\ with\ N = 10$

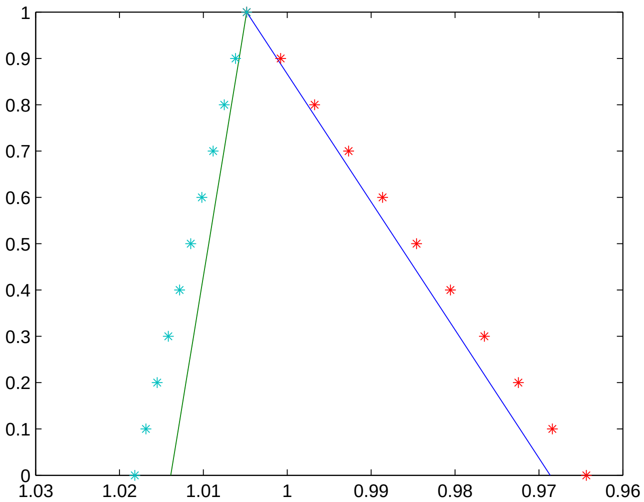

Figure 2. Compare The result of the Predictor-Corrector three-step method with N=10 and exact solution.

Example 2. Consider the initial value problem

$$y'(t) = -y(t),$$

$$y(0) = (0.96 + 0.04r, 1.01 - 0.01r)$$

$$y(0.01) = ((0.96 + 0.04r)e^{-0.01}, (1.01 - 0.01r)e^{-0.01}),$$

$$y(0.02) = ((0.96 + 0.04r)e^{-0.02}, (1.01 - 0.01r)e^{-0.02}),$$

The exact solution at $t = 0.1$ with decreasing support, i.e. (ii)-differentiable is given by

$$Y_2(0.1, r) = ((0.96 + 0.04r)e^{-0.1}, (1.01 - 0.01r)e^{-0.1}, \qquad 0 \leqslant r \leqslant 1$$

Adams-Bashforth two-step method and the Predictor-Corrector three-step method with $N = 10$ has been shown in table (3)and table (4). Also, our proposed solutions are plotted simultaneously to compare with exact solutions. For more detail see Figures 3. and 4.

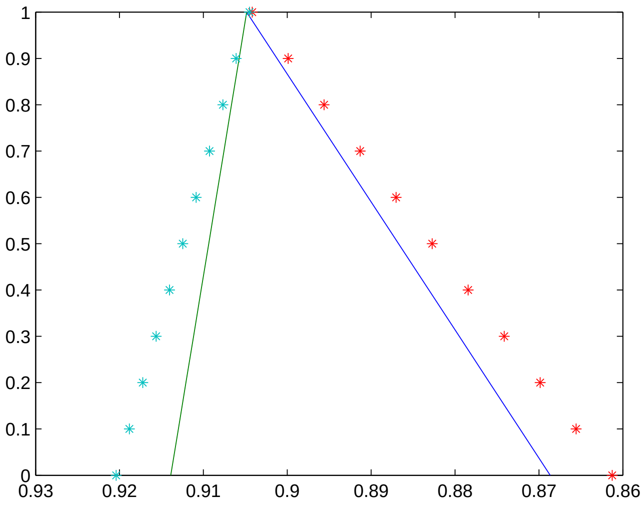

Figure 3. Compare The result of the Adams-Bashforth two-step method with N=10 and exact solution.

r	$\underline{y}_2$	$\underline{Y}_2$	ERROR	$\overline{y}_2$	$\overline{Y}_2$	ERROR
0.0	0.861268	0.868644	0.007376	0.920408	0.913886	0.006522
0.1	0.865559	0.872263	0.006704	0.918817	0.912981	0.005836
0.2	0.869851	0.875883	0.006031	0.917226	0.912076	0.005150
0.3	0.874143	0.879502	0.005359	0.915635	0.911171	0.004464
0.4	0.878434	0.883121	0.004687	0.914044	0.910266	0.003777
0.5	0.882726	0.886741	0.004014	0.912453	0.909362	0.003091
0.6	0.887018	0.890360	0.003342	0.910862	0.908457	0.002405
0.7	0.891310	0.893979	0.002670	0.909271	0.907552	0.001719
0.8	0.895601	0.897599	0.001997	0.907680	0.906647	0.001033
0.9	0.899893	0.901218	0.001325	0.906089	0.905742	0.000347
1.0	0.904185	0.904837	0.000653	0.904498	0.904837	0.000339

$Table(3).\ Adams - Bashforth\ two - step\ method\ with\ N = 10$

r	$\underline{y}_2$	$\underline{Y}_2$	ERROR	$\overline{y}_2$	$\overline{Y}_2$	ERROR
0.0	0.864346	0.868644	0.004298	0.918183	0.913886	0.004297
0.1	0.868395	0.872263	0.003868	0.916849	0.912981	0.003868
0.2	0.872445	0.875883	0.003438	0.915514	0.912076	0.003438
0.3	0.876494	0.879502	0.003008	0.914179	0.911171	0.003008
0.4	0.880543	0.883121	0.002578	0.912845	0.910266	0.002579
0.5	0.884592	0.886741	0.002149	0.911510	0.909362	0.002148
0.6	0.888641	0.890360	0.001719	0.910176	0.908457	0.002624
0.7	0.892690	0.893979	0.001289	0.908841	0.907552	0.002194
0.8	0.896739	0.897599	0.00086	0.907506	0.906647	0.000859
0.9	0.900788	0.901218	0.00043	0.906172	0.905742	0.00043
1.0	0.904887	0.904837	0.00005	0.904837	0.904837	0.00000

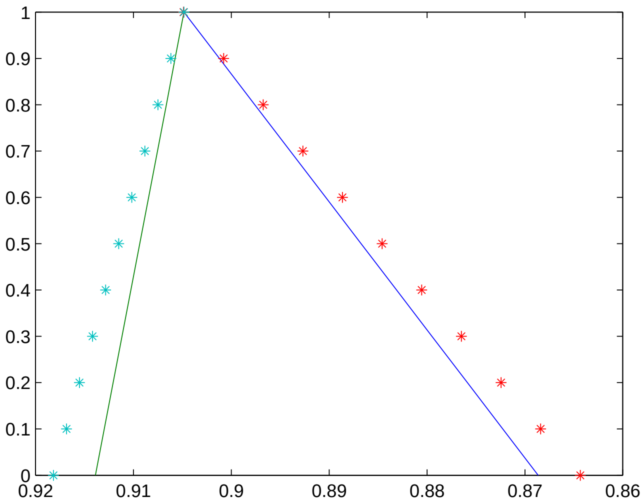

Figure 4. Compare The result of the Predictor-Corrector three-step method with N=10 and exact solution.

$$Table(4).\ The\ Predictor-Corrector\ three-step\ method\ with\ N=10$$

8. Conclusion

Note that the convergence order of the Euler method in the [25] is $O(h)$. It is shown that in proposed method, a higher order of convergence is obtained. In this work for Adams-Bashforth $(m+1)$-step method and Adams-Moulton $(m+1)$-step method are considered as predictor and corrector. It has shown a predictor-corrector method from convergence order $O(h^{m+1})$.In this paper examples are solved by four-step method . With comparing, we observe the results which are obtained by this method are more exact than the results in [1, 2, 29, 39] that are obtained by two-step method.

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