International Journal of
Mathematical Modelling & Computations
Vol. 01, No. 02, April 2011, 135- 147

Non-Polynomial Spline Solutions for Special Nonlinear Fourth-Order Boundary Value Problems

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Abstract. We present a sixth-order non-polynomial spline method for the solutions of two-point nonlinear boundary value problem $u^{(4)} + f(x, u) = 0, u(a) = \alpha_1, u''(a) = \alpha_2, u(b) = \beta_1, u''(b) = \beta_2$, in off step points. Numerical method of sixth-order with end conditions of the order 6 is derived. The convergence analysis of the method has been discussed. Numerical examples are presented to illustrate the applications of method, and to compare the computed results with other known methods.

Keywords: Non-polynomial spline, Boundary formula, Convergence analysis.

Index to information contained in this paper

1. Introduction

Consider the special nonlinear fourth-order boundary value problem given by

$$u^{(4)} + f(x, u) = 0, a < x < b, a, b, x \in \Re, \tag{1}$$

subjected to boundary conditions

$$u(a) = \alpha_1, u''(a) = \alpha_2, u(b) = \beta_1, u''(b) = \beta_2. \tag{2}$$

It is assumed that $f(x, u)$ is real and continuous on $[a, b]$, and $\alpha_i, \beta_i, i = 1, 2$ are finite real constants. The existence and uniqueness of the real valued function $u(x)$ which satisfies (1)-(2) has been given in [1]. E. H. Twizell in [20] derived a fourth-order finite difference method for the numerical solution of (1)-(2). C. P. Katti [11] has given a sixth order finite difference method for the two-point boundary value problem (1) with boundary conditions of first-order derivatives.

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In the case of linear differential equations (1), the class of fourth-order two-point boundary value problems have been solved by some authors see Usmani [22], [23], [24], and [25], Usmani et al. [26], Rashidinia et al. [16]-[18] and references therein. Numerical methods based on the finite difference of the various orders by which the solution of (1) are approximated over a finite set of grid points have been developed by Chawla et al. [5]-[6], Jain et al. [10]-[9] and references therein. Daele et al. [7] introduced a new second order method for solving the boundary value problems (1) based on non-polynomial spline function. Al-Said et al. [2]-[3] developed numerical methods for solutions of fourth-order obstacle problems with collocation, finite difference and spline techniques. S. S. Siddiqi and G. Akram [19] analyzed a system of fourth-order boundary value problems using non-polynomial spline functions. M.A. Ramadan et al [15] developed quintic non-polynomial spline solutions for fourth-order two-point boundary value problem. Siraj-ul-Islam et al. [21] developed numerical methods based on quartic non-polynomial splines for solution of of a system of third-order boundary-value problems. M.A.Khan et al. [13] developed and analyzed a class of methods based on non-polynomial sextic spline functions for the solution of a special fifth-order boundary-value problems. Khan et al.[12] have used parametric quintic spline function for the solution of a system of fourth-order boundary-value problems. Numerical methods for nonlinear fourth-order boundary value problems have been study by Mohamed Alihajji and Kamel Al-khaled [4]. Wazwaz [27] applied ADM for solving a special $2m$ order boundary value problem of the form $u^{(2m)}(x) = f(x, u), 0 < x < b$.
In this paper non-polynomial quintic spline relations have been derived using off-step points. We apply such non-polynomial quintic spline functions that have polynomial and trigonometric parts to develop new numerical method for obtaining smooth approximations to the solutions (1)-(2). Non-polynomial quintic spline formulation is derived in section 2. We develop the $O(h^{10})$ methods at end conditions in section 3. In section 4, convergence analysis is proved. Finally, in section 5, Numerical examples are given to illustrate the applications of the method.

2. Non-Polynomial Quintic Spline Functions

We introduce the set of grid points in the interval $[a, b]$

$$x_0 = a,\ \ x_{l-\frac{1}{2}} = a + (l - \frac{1}{2})h,\ \ h = \frac{b-a}{N},\ \ l = 1, 2, ..., N,\ \ x_N = b.$$

Non-polynomial quintic spline function $S_l(x)$ which interpolates $u(x)$ at the mesh points $x_{l-\frac{1}{2}}$, $l = 1, 2, ..., N$, depends on a parameter τ and reduces to ordinary quintic spline $S_l(x)$ in $[a, b]$ as $\tau \to 0$.
For each segment $[x_{l-\frac{1}{2}}, x_{l+\frac{1}{2}}]$, $l = 1, 2, ..., N-1$, the quintic spline $S_l(x)$, is defined as

$$S_l(x) = \sum_{i=0}^{3} a_{li}(x - x_l)^i + e_l \sin \tau(x - x_l) + f_l \cos \tau(x - x_l),\ l = 0, 1, 2, ..., N, \quad (3)$$

where $a_{li}, (i = 0, 1, 2, 3), e_l$ and f_l are constants and τ is free parameter.
Let u_l be an approximation to $u(x_l)$, obtained by the segment $S_l(x)$ of the mixed spline function passing through the points $(x_{l-\frac{1}{2}}, u_{l-\frac{1}{2}})$ and $(x_{l+\frac{1}{2}}, u_{l+\frac{1}{2}})$, to obtain the necessary conditions for the coefficients introduced in (3), we do not only require that $S_l(x)$ satisfies interpolatory conditions at $x_{l-\frac{1}{2}}, x_{l+\frac{1}{2}}$

but also the continuity of second and four derivatives at the common nodes (x_l, u_l).

To derive expression for the coefficients, we first denote:

$$S_l(x_{l\pm\frac{1}{2}}) = u_{l\pm\frac{1}{2}}, \quad S_l''(x_{l\pm\frac{1}{2}}) = M_{l\pm\frac{1}{2}}, \quad S_l^{(4)}(x_{l\pm\frac{1}{2}}) = F_{l\pm\frac{1}{2}}. \tag{4}$$

From algebraic manipulation we get the following expression:

$$\begin{aligned}
a_{l0} &= -\frac{(8+\theta^2)[F_{l-\frac{1}{2}} + F_{l+\frac{1}{2}}] + \tau^2\theta^2[M_{l-\frac{1}{2}} + M_{l+\frac{1}{2}}] - 8\tau^4[u_{l-\frac{1}{2}} + u_{l+\frac{1}{2}}]}{16\tau^4},\\
a_{l1} &= \frac{(24+\theta^2)[F_{l-\frac{1}{2}} - F_{l+\frac{1}{2}}] + \tau^2\theta^2[M_{l-\frac{1}{2}} - M_{l+\frac{1}{2}}] - 24\tau^4[u_{l-\frac{1}{2}} - u_{l+\frac{1}{2}}]}{24\theta\tau^3},\\
a_{l2} &= \frac{[F_{l+\frac{1}{2}} - F_{l-\frac{1}{2}}] + \tau^2[M_{l+\frac{1}{2}} - M_{l-\frac{1}{2}}]}{4\tau^2},\\
a_{l3} &= \frac{[F_{l+\frac{1}{2}} - F_{l-\frac{1}{2}}] + \tau^2[M_{l+\frac{1}{2}} - M_{l-\frac{1}{2}}]}{6h\tau^2},\\
e_l &= \frac{F_{l+\frac{1}{2}} - F_{l-\frac{1}{2}}}{2\tau^4\sin(\frac{\theta}{2})},\\
f_l &= \frac{F_{l+\frac{1}{2}} - F_{l-\frac{1}{2}}}{2\tau^4\cos(\frac{\theta}{2})},
\end{aligned}$$

where $\theta = \tau h$ and $l = 1, 2, ..., N-1$.

Using the continuity of first and third derivatives at (x_l, u_l), that are $S_{l-1}'(x_l) = S_l'(x_l)$ and $S_{l-1}'''(x_l) = S_l'''(x_l)$, we obtain the following relations:

$$M_{l-\frac{3}{2}} + 22M_{l-\frac{1}{2}} + M_{l+\frac{1}{2}} = \frac{24}{h^2}(u_{l-\frac{3}{2}} - 2u_{l-\frac{1}{2}} + u_{l+\frac{1}{2}}) - h^2[(\frac{24}{\theta^4} - \frac{12}{\theta^3\sin(\frac{\theta}{2})}$$

$$+\frac{1}{\theta^2})F_{l-\frac{3}{2}} + 2(-\frac{24}{\theta^4} + \frac{12\cos(\theta)}{\theta^3\sin(\frac{\theta}{2})} + \frac{11}{\theta^2})F_{l-\frac{1}{2}} + (\frac{24}{\theta^4} - \frac{12}{\theta^3\sin(\frac{\theta}{2})} + \frac{1}{\theta^2})F_{l+\frac{1}{2}}, \tag{5}$$

$$M_{l-\frac{3}{2}} - 2M_{l-\frac{1}{2}} + M_{l+\frac{1}{2}} = h^2[\frac{1-\cos(\frac{\theta}{2})}{2\theta^2\cos(\frac{\theta}{2})}F_{l-\frac{3}{2}} + 2(\frac{\cos(\frac{\theta}{2}) - \cos(\theta)}{2\theta^2\cos(\frac{\theta}{2})})F_{l-\frac{1}{2}}]$$

$$+\frac{1-\cos(\frac{\theta}{2})}{2\theta^2\cos(\frac{\theta}{2})}F_{l+\frac{1}{2}}. \tag{6}$$

From (5) and (6) we have

$$\begin{aligned} h^2 M_{l-\frac{1}{2}} =&(u_{l-\frac{3}{2}} - 2u_{l-\frac{1}{2}} + u_{l+\frac{1}{2}}) + (-\frac{1}{\tau^4} - \frac{\theta(-24+\theta^2)}{48\tau^4 \sin(\frac{\theta}{2})})F_{l-\frac{3}{2}} + (\frac{2}{\tau^4} - \frac{\theta^2}{\tau^4} \\ &+ \frac{\theta(-24+\theta^2)\cos(\theta)}{24\tau^4 \sin(\frac{\theta}{2})})F_{l-\frac{1}{2}} + (-\frac{1}{\tau^4} - \frac{\theta(-24+\theta^2)}{48\tau^4 \sin(\frac{\theta}{2})})F_{l+\frac{1}{2}}. \end{aligned} \tag{7}$$

The elimination of M_l's using (6) and (7) gives

$$\begin{aligned} u_{l-\frac{5}{2}} - 4u_{l-\frac{3}{2}} + 6u_{l-\frac{1}{2}} - 4u_{l+\frac{1}{2}} + u_{l+\frac{3}{2}} =& h^4[\alpha(F_{l-\frac{5}{2}} + F_{l+\frac{3}{2}}) + \beta(F_{l-\frac{3}{2}} + F_{l+\frac{1}{2}}) \\ &+ \gamma F_{l-\frac{1}{2}}], \quad l = 3(1)N-2, \end{aligned} \tag{8}$$

where

$$\begin{aligned} \alpha &= \frac{1}{\theta^4} - \frac{1}{2\theta^3 \sin(\frac{\theta}{2})} + \frac{1}{48\theta \sin(\frac{\theta}{2})}, \\ \beta &= -\frac{4}{\theta^4} + \frac{1+\cos(\theta)}{\theta^3 \sin(\frac{\theta}{2})} + \frac{11-\cos(\theta)}{24\theta \sin(\frac{\theta}{2})}, \\ \gamma &= \frac{6}{\theta^4} - \frac{1+2\cos(\theta)}{\theta^3 \sin(\frac{\theta}{2})} + \frac{1-22\cos(\theta)}{24\theta \sin(\frac{\theta}{2})}. \end{aligned}$$

When $\tau \to 0$, that $\theta \to 0$, then $(\alpha, \beta, \gamma) \to \frac{1}{1920}(1, 236, 1446)$, and the relations defined by (8) reduce into quintic polynomial spline function. Now by using the spline relation (8) and discretize the given system (1) at the grid points x_l. We obtain $(N-4)$ nonlinear equation in the (N) unknowns $u_{l-\frac{1}{2}}$, $l = 1, 2, ..., N$ as

$$\begin{aligned} &(u_{l-\frac{5}{2}} + \alpha h^4 f(x_{l-\frac{5}{2}}, u_{l-\frac{5}{2}})) + (-4u_{l-\frac{3}{2}} + \beta h^4 f(x_{l-\frac{3}{2}}, u_{l-\frac{3}{2}})) + \\ &(6u_{l-\frac{1}{2}} + \gamma h^4 f(x_{l-\frac{1}{2}}, u_{l-\frac{1}{2}})) + (-4u_{l+\frac{1}{2}} + \beta h^4 f(x_{l+\frac{1}{2}}, u_{l+\frac{1}{2}})) + \\ &(u_{l+\frac{3}{2}} + \alpha h^4 f(x_{l+\frac{3}{2}}, u_{l+\frac{3}{2}})) + t_l = 0, \qquad l = 3(1)N-2. \end{aligned} \tag{9}$$

Taylor expansion of the local truncation errors $t_l, l = 3, ..., N-2$, associated with our method are given by

$$\begin{aligned} t_l =& (1 - 2\alpha - 2\beta - \gamma)\, h^4 u_l^{(4)} + \left(\frac{-1}{2} + \alpha + \beta + \frac{1}{2}\gamma\right) h^5 u_l^{(5)} \\ &+ \left(\frac{7}{24} - \frac{17}{4}\alpha - \frac{5}{4}\beta - \frac{1}{8}\gamma\right) h^6 u_l^{(6)} + \left(-\frac{5}{48} + \frac{49}{24}\alpha + \frac{13}{24}\beta + \frac{\gamma}{48}\right) h^7 u_l^{(7)} + \\ &\left(\frac{23}{640} - \frac{353\alpha}{192} - \frac{41\beta}{192} - \frac{\gamma}{384}\right) h^8 u_l^{(8)} + \left(-\frac{23}{2304} + \frac{1441\alpha}{1920} + \frac{121\beta}{1920} + \frac{\gamma}{3840}\right) h^9 u_l^{(9)} \\ &+ \left(\frac{2497}{967680} - \frac{8177\alpha}{23040} - \frac{73\beta}{4608} - \frac{\gamma}{46080}\right) h^{10} u_l^{(10)} + O(h^{11}). \end{aligned} \tag{10}$$

For different choices of parameters α, β and γ we can obtain classes of the methods such as:

Second-Order Method

For $\alpha = \frac{9}{240}, \beta = \frac{1}{24000}$ and $\gamma = 1 - 2\alpha - 2\beta$ gives $t_l = \frac{133}{8000}h^6 u_l^{(6)} + O(h^7), \quad l = 3, ..., N-2.$

Fourth-Order Method

For $\alpha = \frac{1}{24}, \beta = 0$ and $\gamma = 1-2\alpha-2\beta$ gives $t_l = \frac{-31}{720}h^8 u_l^{(8)} + O(h^9), \quad l = 3, ..., N-2.$

Sixth-Order Method

For $\alpha = \frac{-1}{720}, \beta = \frac{31}{180}$ and $\gamma = 1 - 2\alpha - 2\beta$ gives $t_l = \frac{1}{3024}h^{10} u_l^{(10)} + O(h^{11}), \quad l = 3, ..., N-2.$

3. End Condition

To obtain the unique solution of the nonlinear system (9) we need four more equations to be associated with the system. By using Taylor series and method of undetermined coefficients the boundary formulas associated with boundary conditions for the sixth-order method can be determine as follows. In order to obtain the six-order boundary formula we define the following identities:

$$c_0 u_0 + \sum_{j=1}^{3} c_j u_{j-\frac{1}{2}} + \lambda h^2 u_0^{''} + \rho_0 h^4 u_0^{(4)} + h^4 \sum_{j=1}^{5} \rho_j u_{j-\frac{1}{2}}^{(4)} + t_1'^{*} = 0, \tag{11}$$

$$c_0' u_0 + \sum_{j=1}^{4} c_j' u_{j-\frac{1}{2}} + \lambda' h^2 u_0^{''} + h^4 \sum_{j=1}^{6} \rho_j' u_{j-\frac{1}{2}}^{(4)} + t_2'^{*} = 0, \tag{12}$$

$$c_0'^{*} u_N + \sum_{j=1}^{4} c_j'^{*} u_{N-j+\frac{1}{2}} + \lambda'^{*} h^2 u_N^{''} + h^4 \sum_{j=1}^{6} \rho_j'^{*} u_{N-j+\frac{1}{2}}^{(4)} + t_{N-1}'^{*} = 0, \tag{13}$$

$$c_0^{*} u_N + \sum_{j=1}^{3} c_j^{*} u_{N-j+\frac{1}{2}} + \lambda^{*} h^2 u_N^{''} + h^4 \rho_0^{*} u_N^{(4)} + h^4 \sum_{j=1}^{5} \rho_j^{*} u_{N-j+\frac{1}{2}}^{(4)} + t_N'^{*} = 0. \tag{14}$$

In order that $t_1'^{*}$, $t_2'^{*}$, $t_{N-1}'^{*}$ and $t_N'^{*}$ are $O(h^{10})$ we find that

$$(c_0, c_1, c_2, c_3, \lambda) = (c_0^{*}, c_1^{*}, c_2^{*}, c_3^{*}, \lambda^{*}) = (-6, 10, -5, 1, \frac{5}{4}),$$

$$(\rho_0, \rho_1, \rho_2, \rho_3, \rho_4, \rho_5) = (\frac{8041}{7620480}, \frac{-16255}{43008}, \frac{-293953}{1161216}, \frac{2999}{71680}, \frac{-3763}{301056}, \frac{3035}{1741824}),$$

$$(\rho_0^{*}, \rho_1^{*}, \rho_2^{*}, \rho_3^{*}, \rho_4^{*}, \rho_5^{*}) = (\frac{8041}{7620480}, \frac{-16255}{43008}, \frac{-293953}{1161216}, \frac{2999}{71680}, \frac{-3763}{301056}, \frac{3035}{1741824}),$$

$$(c_0', c_1', c_2', c_3', c_4', \lambda') = (c_0'^{*}, c_1'^{*}, c_2'^{*}, c_3'^{*}, c_4'^{*}, \lambda'^{*}) = (2, -5, 6, -4, 1, \frac{1}{4}),$$

$$(\rho_1', \rho_2', \rho_3', \rho_4', \rho_5', \rho_6') = (\frac{-132283}{884736}, \frac{-7263443}{10321920}, \frac{-1775513}{15482880}, \frac{-632117}{15482880},, \frac{172129}{10321920},, \frac{-85373}{30965760}),$$

$$(\rho_1'^{*}, \rho_2'^{*}, \rho_3'^{*}, \rho_4'^{*}, \rho_5'^{*}, \rho_6'^{*}) = (\frac{-132283}{884736}, \frac{-7263443}{10321920}, \frac{-1775513}{15482880}, \frac{-632117}{15482880},, \frac{172129}{10321920},, \frac{-85373}{30965760}),$$

$$t_l'^* = \begin{cases} \frac{336383}{371589120} h^{10} u_0^{(10)} + O(h^{11}), & l = 1, \\ -\frac{360041}{154828800} h^{10} u_0^{(10)} + O(h^{11}), & l = 2, \\ -\frac{360041}{154828800} h^{10} u_N^{(10)} + O(h^{11}), & l = N-1, \\ \frac{336383}{371589120} h^{10} u_N^{(10)} + O(h^{11}), & l = N. \end{cases} \tag{15}$$

4. Convergence Analysis

In this section, we investigate the convergence analysis of the method. The equations (9) along with boundary condition (11)-(14) yields nine diagonal nonlinear system of equations, and may be written in matrix form as

$$A_0 U^{(1)} + h^4 B \mathbf{f}^{(1)}(U^{(1)}) = R^{(1)}, \tag{16}$$

in (16), A_0 and B are square matrices of order N and are given by

$$A_0 = \begin{pmatrix} 10 & -5 & 1 & & & & & \\ -5 & 6 & -4 & 1 & & & & \\ 1 & -4 & 6 & -4 & 1 & & & \\ & \ddots & \ddots & \ddots & \ddots & \ddots & & \\ & & \ddots & \ddots & \ddots & \ddots & \ddots & \\ & & & \ddots & \ddots & \ddots & \ddots & \ddots \\ & & & & 1 & -4 & 6 & -4 & 1 \\ & & & & & 1 & -4 & 6 & -5 \\ & & & & & & 1 & -5 & 10 \end{pmatrix} \tag{17}$$

P is monotone three diagonal matrix defined by

$$p_{ij} = \begin{cases} 3 & i = j = 1, N, \\ 2 & i = j = 2, 3, ..., N-1, \\ -1 & |i-j| = 1, \\ 0 & \text{otherwise.} \end{cases} \tag{18}$$

and the matrix B in case of sixth-order method defined by

$$B = \begin{pmatrix} \frac{-16255}{43008} & \frac{-293953}{1161216} & \frac{2999}{71680} & \frac{-3763}{301056} & \frac{3035}{1741824} & & & & & \\ \frac{-132283}{884736} & \frac{-7263443}{10321920} & \frac{-1775513}{15482880} & \frac{632117}{15482880} & \frac{172129}{10321920} & \frac{-85373}{30965760} & & & & \\ \alpha & \beta & \gamma & \beta & \alpha & 0 & & & & \\ & \ddots & \ddots & \ddots & \ddots & \ddots & & & & \\ & & \ddots & \ddots & \ddots & \ddots & \ddots & & & \\ & & & \ddots & \ddots & \ddots & \ddots & \ddots & & \\ & & & & 0 & \alpha & \beta & \gamma & \beta & \alpha \\ & & & & \frac{-85373}{30965760} & \frac{172129}{10321920} & \frac{632117}{15482880} & \frac{-1775513}{15482880} & \frac{-7263443}{10321920} & \frac{-132283}{884736} \\ & & & & & \frac{3035}{1741824} & \frac{-3763}{301056} & \frac{2999}{71680} & \frac{-293953}{1161216} & \frac{-16255}{43008} \end{pmatrix} \tag{19}$$

Theorem 4.1 *Let Λ be the $N \times N$ matrix*

$$\lambda_{ij} = \begin{cases} \xi & i = j = 1, N, \\ x & i = j = 2, 3, ..., N-1, \\ -1 & |i-j| = 1, \\ 0 & \textit{otherwise}, \end{cases} \tag{20}$$

The sufficient condition for $\Lambda^{-1} \geqslant 0$ *is that* $\xi > r$ *where* $r = \frac{x-\sqrt{x^2-4}}{2}, (x \geqslant 2)$.

Proof Suppose that Λ is nonsingular and we denote $Q = (q_{ij})_{N\times N}$ be the inverse of $\bigwedge$. The identity $\bigwedge Q = I$ is expressed by the equations

$$\begin{cases} \xi q_{1,j} - q_{2,j} = \delta_{1,j}, & (i) \\ -q_{i-1,j} + xq_{i,j} - q_{i+1,j} = \delta_{i,j}, \quad i = 2,3,...,N-1, & (ii) \\ \xi q_{N,j} - q_{N-1,j} = \delta_{N,j}, & (iii) \end{cases} \tag{21}$$

for each $j = 1, 2, ..., N$, where $\delta_{i,j}$ is the Kronecker delta. By using [14] the elements $q_{i,j}$ can be written as:

$$- e_{i-1} + xe_i - e_{i+1} = 0, \quad i = 2,3,...,N-1, \quad e_1 = 0, \quad \text{and} \quad e_N = 1, \tag{22}$$

$$- t_{i-1,j} + xt_{i,j} - t_{i+1,j} = \delta_{i,j},\ i = 2,3,...,N-1,\ t_{1,j} = 0,\ \text{and}\ t_{N,j} = o, \tag{23}$$

for each $j = 2, 3, ..., N-1$, and $t_{i,1} = t_{i,N} = 0$. The solutions to (22) and (23) are given in [8] and [14]. We have

$$q_{i,j} = t_{i,j} + \psi_j e_{N-i+1} + \chi_j e_i, \tag{24}$$

where ψ_j and χ_j must be determined. We known that the expression $q_{i,j}$ satisfy the equations (21)(ii) in general. It satisfies equations (21)(i) and (21)(iii). By substituting equation (24) into (21)(i) and (21)(iii) we get

$$(\xi - e_{N-1})\psi_j - e_2\chi_j = t_{2,j} + \delta_{1,j}, \tag{25}$$

$$(-e_2)\psi_j + (\xi - e_{N-1})\chi_j = t_{N-1,j} + \delta_{N,j}. \tag{26}$$

By using (25) and (26) we get:

$$\begin{pmatrix} \xi - e_{N-1} & -e_2 \\ -e_2 & \xi - e_{N-1} \end{pmatrix} \begin{pmatrix} \psi_j \\ \chi_j \end{pmatrix} = \begin{pmatrix} t_{2,j} + \delta_{1,j} \\ t_{N-1,j} + \delta_{N,j} \end{pmatrix}, \tag{27}$$

if equation (27) has a unique solution, then the inverse of matrix $\bigwedge$ is $Q = (q_{i,j})_{N\times N}$. The sufficient condition on ξ and x so that $\bigwedge^{-1} \geqslant 0$ can be obtained as follows.
We assume that $x \geqslant 2$ so that $t_{i,j} \geqslant 0$ for $i = 1, 2, ..., N, j = 1, 2, ..., N$. From expression (24), the ψ_j and χ_j should be nonnegative, thus

$$\begin{pmatrix} \xi - e_{N-1} & -e_2 \\ -e_2 & \xi - e_{N-1} \end{pmatrix}^{-1} \geqslant 0, \tag{28}$$

by simplified (28) we get

$$\begin{pmatrix} \xi - r & \\ & \xi - r \end{pmatrix}^{-1} > 0, \tag{29}$$

where $r = \frac{x-\sqrt{x^2-4}}{2}$. If N is sufficiently large, then condition (29) implies condition (28). Thus $x \geqslant 2$ and condition (29) implies that $\bigwedge^{-1} \geqslant 0$ for N sufficiently large if $\xi > r$.

Consequently the matrix P which is special case of the matrix $\bigwedge$ for $\xi = 3$ and $x = 2$ is monotone matrix. Thus the matrix $A_0 = P^2$, is a monotone matrix. By using Lemma 1, in [24] the symmetric matrix A_0, is irreducible and monotone and

$$\|A_0^{-1}\| \leqslant \frac{[5(b-a)^4 + 10(b-a)^2h^2 + 9h^4]}{384h^4}. \tag{30}$$

The N components vectors $\mathbf{f}^{(1)}$ and $R^{(1)}$, are given by

$$\mathbf{f}^{(1)} = (f^{(1)}_{\frac{1}{2}}, ..., f^{(1)}_{N-\frac{1}{2}})^t, \tag{31}$$

where $\mathbf{f}^{(1)}_l(U^{(1)}) = f(x_l, u^{(1)}_l), l = \frac{1}{2}, \frac{3}{2}, ..., N - \frac{1}{2}$, and

$$R^{(1)} = \begin{pmatrix} 6\alpha_1 - \frac{5}{4}h^2\alpha_2 - h^4(\frac{8041}{7620480})f(x_0, \alpha_1) \\ -2\alpha_1 - \frac{h^2}{4}\alpha_2 \\ 0 \\ \vdots \\ 0 \\ -2\beta - \frac{1}{4}h^2\beta_2 \\ 6\beta_1 - \frac{5}{4}h^2\beta_2 - h^4(\frac{8041}{7620480})f(x_N, \beta_1) \end{pmatrix}. \tag{32}$$

$$A_0\overline{U}^{(1)} + h^4B\mathbf{f}^{(1)}(\overline{U}^{(1)}) = R^{(1)} + t^{(1)}, \tag{33}$$

where the vector $\overline{U}^{(1)} = u(x_{l-\frac{1}{2}}), l = 1, 2, ..., N$, is the exact solution and $t^{(1)} = [t^{(1)}_{\frac{1}{2}}, t^{(1)}_{\frac{3}{2}}, ..., t^{(1)}_{N-\frac{1}{2}}]^T$, is the local truncation errors.
From (16) and (33) we have:

$$[A]E^{(1)} = [A_0 + h^4BF_k(U^{(1)})]E^{(1)} = t^{(1)}, \tag{34}$$

where

$$E^{(1)} = \overline{U}^{(1)} - U^{(1)} = [e^{(1)}_{\frac{1}{2}}, e^{(1)}_{\frac{3}{2}}, ..., e^{(1)}_{N-\frac{1}{2}}]^T, \tag{35}$$

$$\mathbf{f}^{(1)}(\overline{U}^{(1)}) - \mathbf{f}^{(1)}(U^{(1)}) = F_k(U^{(1)})E^{(1)},$$

and $F_k(U^{(1)}) = \mathrm{diag}\{\frac{\partial f^{(1)}_l}{\partial u^{(1)}_l}\}, l = \frac{1}{2}, \frac{3}{2}, ..., N - \frac{1}{2}$, is a diagonal matrix of order N.
■

Lemma 4.2 *If M is a square matrix of order N and $\|M\| < 1$, then $(I + M)^{-1}$ exists and $\|(I + M)^{-1}\| \leqslant \frac{1}{(1-\|M\|)}$.*

Lemma 4.3 *The matrix $[A_0 + h^4BF_k(U^{(1)})]$ in (34) is nonsingular, provided $Y < \frac{1981808640}{5306209\lambda}$, where $\lambda = [5(b-a)^4 + 10(b-a)^2h^2 + 9h^4]$ and $Y = max|\frac{\partial f^{(1)}_l}{\partial u^{(1)}_l}|, l = \frac{1}{2}, \frac{3}{2}, ..., N - \frac{1}{2}$. (The norm referred to is the L_∞ norm).*

Proof We know that $[A_0+h^4BF_k(U^{(1)})] = A_0[I+h^4A_0^{-1}BF_k(U^{(1)})]$. But the inverse of A_0 exists we have to show that the inverse of $[I + h^4A_0^{-1}BF_k(U^{(1)})]$ exists. By using lemma 4.2, we have

$$h^4\|A_0^{-1}BF_k(U^{(1)})\| \leqslant h^4\|A_0^{-1}\|\|B\|\|F_k(U^{(1)})\| < 1, \tag{36}$$

by using (19) we obtain $\|B\| \leqslant \frac{5306209}{5160960}$ and also we have $\|F_k(U^{(1)})\| \leqslant Y = \max|\frac{\partial f_l^{(1)}}{\partial u_l^{(1)}}|, l = \frac{1}{2}, \frac{3}{2}, ..., N - \frac{1}{2}$, and then by using (30) and (36) we obtain

$$Y < \frac{1981808640}{5306209\lambda}.$$

As a consequence of Lemmas 4.3 and 4.2 the nonlinear system (16) has a unique solution if $Y < \frac{1981808640}{5306209\lambda}$. ■

To show that the matrix $[A = A_0 + h^4BF_k(U^{(1)})]$ is monotone.
we will prove the following theorem:

THEOREM 4.4 *If $Y < \frac{1981808640}{5306209\lambda}$, then the matrix A given by (34) is monotone where $\lambda = 5(b-a)^4 + 10(b-a)^2h^2 + 9h^4$.*

Proof : From (34) we have

$$A = A_0 + h^4BF_k(U^{(1)}) = P^2 + h^4BF_k(U^{(1)}),$$

hence $AP^{-2} = I + h^4BF_k(U^{(1)})P^{-2}$, so that

$$\begin{aligned}P^2A^{-1} &= (I + h^4BF_k(U^{(1)})P^{-2})^{-1} = \\ &= I - (h^4BF_k(U^{(1)})P^{-2}) + (h^4BF_k(U^{(1)})P^{-2})^2 - (h^4BF_k(U^{(1)})P^{-2})^3 + \dots \\ &= [I - (h^4BF_k(U^{(1)})P^{-2})][I + (h^4BF_k(U^{(1)})P^{-2})^2 + (h^4BF_k(U^{(1)})P^{-2})^4 + \dots].\end{aligned}$$

Also if

$$\rho(h^4BF_k(U^{(1)})P^{-2}) < 1$$

then, the two infinite series convergence. Let

$$\|F_k(U^{(1)})\| \leqslant Y = \max|\frac{\partial f_l^{(1)}}{\partial u_l^{(1)}}|, l = \frac{1}{2}, \frac{3}{2}, ..., N - \frac{1}{2},$$

then

$$A^{-1} = [P^{-2} - P^{-2}h^4BF_k(U^{(1)}P^{-2})][I + (h^4BF_k(U^{(1)})P^{-2})^2 + (h^4BF_k(U^{(1)})P^{-2})^4 + ...],$$

where the infinite series is nonnegative. Hence to show that A is monotone, it sufficient to show that $[P^{-2} - P^{-2}h^4BF_k(U^{(1)})P^{-2}] > 0$. Here we have

$$P^{-2} > P^{-2}h^4BF_k(U^{(1)})P^{-2} \Rightarrow I > h^4BF_k(U^{(1)})P^{-2} \Rightarrow$$

$$\|h^4 A_0^{-1} B F_k(U^{(1)})\| \leqslant h^4 \|A_0^{-1}\| \|B\| \|F_k(U^{(1)})\| < 1. \tag{37}$$

By substituting $\|B\|, \|P^{-2}\|$ and $\|F_k(U^{(1)})\|$ into (37) we get

$$Y < \frac{1981808640}{5306209\lambda}$$

■

THEOREM 4.5 *Let $u(x_{l-\frac{1}{2}})$ be the exact solution of the boundary value problem (1) with boundary conditions (2) and we assume $u_{l-\frac{1}{2}}$, $l = 1, 2, ..., N$ be the numerical solution obtained by solving the nonlinear system (16). Then we have*

$$\|E^{(1)}\| \equiv O(h^6), (\textit{provided } Y < \tfrac{1981808640}{5306209\lambda}, \alpha = \tfrac{-1}{720}, \beta = \tfrac{31}{180}, \gamma = \tfrac{79}{120}).$$

Proof We can write the error equation (34) in the following form

$$E^{(1)} = (A_0 + h^4 B F_k(U^{(1)}))^{-1} t^{(1)} = (I + h^4 A_0^{-1} B F_k(U^{(1)}))^{-1} A_0^{-1} t^{(1)},$$
$$\|E^{(1)}\| \leqslant \|(I + h^4 A_0^{-1} B F_k(U^{(1)}))^{-1}\| \|A_0^{-1}\| \|t^{(1)}\| \|,$$

It follows that

$$\|E^{(1)}\| \leqslant \frac{\|A_0^{-1}\| \|t^{(1)}\|}{1 - h^4 \|A_0^{-1}\| \|B\| \|F_k(U^{(1)})\|}, \tag{38}$$

provided that $h^4 \|A_0^{-1}\| \|B\| \|F_k(U^{(1)})\| < 1$. Also we have

$$\|t^{(1)}\| \leqslant \frac{360041}{154828800} h^{10} M_{10}, \quad (\alpha = \frac{-1}{720}, \beta = \frac{31}{180}, \gamma = \frac{79}{120}), \tag{39}$$

where $M_{10} = \max|u^{(10)}(\xi)|, a \leqslant \xi \leqslant b$.
Substituting $\|A_0^{-1}\|, \|F_k(U^{(1)})\|$, $\|B\|$ and $\|t^{(1)}\|$ from above relations in (38) and simplifying we obtain

$$\|E^{(1)}\| \leqslant \frac{1858157199360\lambda h^6 M_{10}}{306841053560832000 - 821553972019200\lambda Y} \equiv O(h^6), \tag{40}$$

where $\lambda = 5(b-a)^4 + 10(b-a)^2 h^2 + 9h^4$. It is a sixth-order convergent method provided

$$Y < \frac{1981808640}{5306209\lambda} \tag{41}$$

■

Consequently it follows that the numerical method under consideration is sixth-order convergent process.

5. Numerical Results

In this section we present the results obtained by applying the numerical method discussed in pervious sections to the following two-point boundary-value problems.

Example 5.1 We consider the differential equation

$$u^{(4)} + u^2 = -8x\cos(x) - 13\sin(x) + x^2\sin(x) + (x^2-1)^2(\sin(x))^2,$$

$$0 < x < 1, \tag{42}$$

with the boundary conditions:

$$u(0) = u(1) = 0, u''(0) = 0, \quad u''(1) = 2\sin(1) + 4\cos(1). \tag{43}$$

The analytical solution is $u(x) = (x^2-1)\sin(x)$.

Example 5.2 Consider the differential equation

$$u^{(4)} - 6e^{4u} = -\frac{12}{(1+x)^4}, 0 < x < 1, \tag{44}$$

with the boundary conditions:

$$u(0) = 0, u(1) = \ln(2), u''(0) = -1, \quad u''(0) = \frac{-1}{4}. \tag{45}$$

The analytical solution is $u(x) = \ln(1+x)$.

Example 5.3 We consider the differential equation

$$\begin{aligned} u^{(4)} - 5u^3 = 96x\cos(x) - 16x\left(-1+x^2\right)\cos(x) + 24\sin(x) - 48x^2\sin(x) \\ - 24\left(-1+x^2\right)\sin(x) + \left(-1+x^2\right)^2\sin(x) - 5\left(-1+x^2\right)^6\sin(x)^3, \\ 0 < x < 1, \end{aligned} \tag{46}$$

with the boundary conditions:

$$u(0) = u(1) = 0, u''(0) = 0, \quad u''(1) = 8\sin(1). \tag{47}$$

The analytical solution is $u(x) = (x^2-1)^2\sin(x)$.
We solved examples 5.1, 5.2 and 5.3 by using non-polynomial quintic spline method with step lengths $h = 2^{-m}, m = 2, 3, 4, 5, 6, 7$ for $\alpha = \frac{-1}{720}, \beta = \frac{31}{180}, \gamma = \frac{79}{120}$. The maximum absolute errors in solutions for our method are listed in table 1.

Table 1. Maximum absolute errors in solution.

m	Example 5.1	Example 5.2	Example 5.3
3	5.22793×10^{-10}	3.38556×10^{-7}	3.56386×10^{-8}
4	8.07062×10^{-12}	1.18519×10^{-9}	5.78254×10^{-10}
5	1.85186×10^{-13}	3.83505×10^{-11}	1.30425×10^{-11}
6	3.61431×10^{-15}	1.01539×10^{-12}	2.18424×10^{-13}
7	8.01699×10^{-15}	1.81576×10^{-14}	2.61800×10^{-16}

Example 5.4 Consider the differential equation

$$u^{(4)} + xu = -(8 + 7x + x^3)e^x, \quad 0 < x < 1, \tag{48}$$

with the boundary conditions:

$$u(0) = u(1) = 0, u''(0) = 0, \quad u''(1) = -4e. \tag{49}$$

The analytical solution for this boundary value problem is $u(x) = x(1-x)e^x$.

Example 5.5 Consider the following problem ,

$$u^{(4)} - u = -8x\cos(x) - 12\sin(x), \quad 0 < x < 1, \tag{50}$$

with the boundary conditions:

$$u(0) = u(1) = 0, u''(0) = 0, \quad u''(1) = 4\cos(1) + 2\sin(1). \tag{51}$$

The exact solution is given by $u(x) = (x^2-1)\sin(x)$. Examples 5.4 and 5.5 solved by using non-polynomial quintic spline method with step lengths $h = 2^{-m}, m = 3, 4, 5$ for $\alpha = \frac{-1}{720}, \beta = \frac{31}{180}, \gamma = \frac{79}{120}$. The maximum absolute errors in solutions for our method are listed in table 2.

Table 2. Maximum absolute errors in solution.

m	Example 5.4	Example 5.5
3	1.691902×10^{-9}	5.248385×10^{-10}
4	2.23407×10^{-11}	8.071303×10^{-12}
5	2.020517×10^{-13}	1.990957×10^{-14}

Table 3. Maximum absolute errors in solution Example 5.2

m	Second-order in [20]	Fourth-order in [20]	Method A in [1]	Method B in [1]
3	1.9×10^{-4}	3.7×10^{-6}	1.4×10^{-5}	1.4×10^{-5}
4	4.6×10^{-5}	2.9×10^{-7}	8.3×10^{-7}	8.3×10^{-7}
5	1.1×10^{-5}	1.9×10^{-8}	5.4×10^{-8}	5.4×10^{-8}

Table 4. Maximum absolute errors in solution Example 5.4

m	Sixth-order in [17]	Sixth-order in [10]	Sixth-order in[23]	Sixth-order in [23]
3	2.47×10^{-9}	1.91×10^{-7}	2.66×10^{-6}	3.86×10^{-7}
4	3.93×10^{-11}	3.12×10^{-9}	4.68×10^{-8}	6.59×10^{-9}
5	3.25×10^{-13}	4.98×10^{-11}	7.72×10^{-10}	1.05×10^{-10}
6	-	-	8.01×10^{-12}	9.81×10^{-12}

Table 5. Maximum absolute errors in solution Example 5.4 and 5.5 in [15].

m	Example 5.4 Fourth-order (1)	Example 5.4 Fourth-order (2)	Example 5.5 Fourth-order (1)	Example 5.5 Fourth-order (2)
3	1.91×10^{-7}	2.09×10^{-7}	5.96×10^{-8}	6.48×10^{-8}
4	3.12×10^{-9}	7.92×10^{-9}	9.10×10^{-10}	2.29×10^{-9}
5	5.02×10^{-11}	1.27×10^{-9}	1.42×10^{-12}	3.66×10^{-10}

6. Conclusion

We have developed new non-polynomial quintic spline method at mid point for finding the numerical solution of nonlinear fourth-order boundary value problems. The boundary condition corresponding to the sixth-order method is developed. The convergence analysis of our presented method is discussed based on monotonicity of the coefficient matrix.

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